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Introduction

In this book we describe how to use quaternions and spinors to study conformal immersions of Riemann surfaces into $\boldsymbol{R}^3$. The theory is global.

The principal new idea is to use quaternionic calculus. The classical approach to surface theory is based on vector calculus, moving frames, and complex analysis. When applied to the study of generic problems these tools often lead to complicated nonlinear differential equations. Further, apparently insurmountable complications arise at the singular loci of vector fields and moving frames. We use quaternionic calculus to obtain simpler differential equations and to cut through the confusion caused by singularities. At the same time the quaternionic approach to studying surfaces naturally incorporates the topological invariants of the immersion, in particular, its regular homotopy type.

Our main interest is in conformal immersions. This stems primary from our interest in the questions: what are the minimal sets of invariants needed to identify a surface? How does one construct a surface with particular properties, for example, shape or prescribed Gauss map? The bulk of classical work on surfaces in space forms concerns isometric immersions. This often leads to interesting but distracting problems concerning the possibility of isometric embedding a surface with a prescribed Riemannian metric. Thus one risks proving vacuous rigidity results for metrics which are not realizable. In contrast, every conformal structure can be realized by a conformal immersion. Furthermore in many applications the conformal structure comes up naturally, while this is not the case with isometric immersions. We obtain results on isometric immersions as more refined cases of the conformal theory. Indeed, we began to develop the theory in order to tackle a metric geometry problem posed by Bonnet [**KPP**].

The first part of the book develops the necessary quaternionic calculus on surfaces, its application to surface theory and the study of regular homotopy classes of immersions, conformal immersions, spinor transforms, and the connection between extrinsic and intrinsic conformal geometry. The integrability conditions for spinor transforms lead naturally to Dirac spinors and their application to conformal immersions. In the second part of the book we present a complete spinor calculus on a Riemann surface, the definition of a conformal Dirac operator, and a generalized Weierstrass representation valid for all surfaces. On a Riemann surface one can interpret spinors as the square roots of conformal $\boldsymbol{R}^3$-valued one-forms. In particular, spinors encode the conformal immersions of the tangent plane of a Riemann surface into $\boldsymbol{R}^3$. This approach provides a tool to take smooth square roots of geometric objects like vectors and forms, and to uncover new invariants. This theory suggests new existence and rigidity paradigms for immersions, and new insights into classical existence and rigidity problems. A significant advantage of the new approach is that it leads to nonsingular linear differential equations.

The idea to study conformal immersions via quaternionic and spinor calculus is akin to well established ideas in particle physics and quantum mechanics. The group of nonzero quaternions H^* is the universal cover of the relevant gauge group, $SO(3) \times \boldsymbol{R}^+$. Thus it is not surprising that a quaternionic calculus is well adapted to the study of conformal immersions. In contrast, complex calculus is an efficient tool to study geometries whose gauge group is $\boldsymbol{C}^*$. The problem of reformulating low dimensional geometry in terms of quaternions was posed by W. Hamilton. The connection between spinors and surface immersions has been established at least since the 1960's. (See [**JT, HH, Pin85**]). As far as we can tell, Dennis Sullivan was the first to exploit this connection to obtain convenient representation of surface immersions using spinors [**Sul89**]. His result is for minimal surfaces. It appears that there were several other unpublished attempts to employ spinors to generalize the Weierstrass representation of minimal surfaces and to obtain a Weierstrass type representation of constant mean curvature surfaces (for example, Abresch). Several related papers and preprints appeared [**Bob93, KS93, Ko, KT95, Ric95**].

The theory in this monograph grew from the work of the GANG seminar at the University of Massachusetts, Amherst, during 1995-1996. The main speakers were the authors. Additional talks were given by Fran Burstall, Udo Hetrich-Jeromin, Martin Killian, Jorg Richter, Nick Schmidt, and Iskander Taimanov. Since 1996 the theory was developed further and continues to be developed in a series of lectures, and papers [**Pin96, FP, GK6, KPP, GK4, FP2, GK3, GK5**]. The purpose of this monograph is to give a self contained presentation of the part of the theory developed to study Bonnet's problem, Christoffel's problem, shape class immersions, and the surface reconstruction applications discussed in [**KK, KK3, KK2**]. Early results were announced in [**GK96b**]. During the preparation of this book the Dirac spinor ideas have been generalized to the theory of quaternionic holomorphic bundles [**PP**]. This theory has implications to the study of Willmore surfaces, the energy of harmonic 2-tori and to Dirac eigenvalue estimates over compact surfaces [**BFLPP, FLPP**].

The first author thanks Rosa Maria Chavez, Maria Elisa Galvao, Roberto Costa, and Fabiano Brito, and the geometry group at University of Sao Paulo, Brazil for the opportunity to give a month long series of lectures and to prepare a set of notes on quaternionic calculus and geometry [**GK6**], and to Lucio Rodriguez and Manfredo DoCarmo, and Ivan and Renato Tribuzy for their hospitality and useful discussions. We are grateful to Robert Bryant, Dennis DeTurck, Bob Hardt, Gary Jensen, Steven Semmes, and Mike Wolf for their interest and insights.

Basic Conventions

Following W. Hamilton we identify Euclidean four-space with the space of quaternions $\boldsymbol{H} := \{\rho + xi + yj + zk | (\rho, x, y, z) \in \boldsymbol{R}^4\}$, and Euclidean three-space as the subspace of imaginary quaternions $\mathrm{im}(\boldsymbol{H}) = \{xi + yj + zk | (x, y, z) \in \boldsymbol{R}^3\}$. Thus throughout this book $\boldsymbol{R}^4 = \boldsymbol{H}$ and $\boldsymbol{R}^3 = \mathrm{im}(\boldsymbol{H})$. The quaternionic multiplication gives a unified approach to the scalar product $< \ | \ >$, and the cross product $\times$ of vectors in Euclidean three-space. Indeed, for every two vectors $a, b \in \mathrm{im}(\boldsymbol{H})$ we have

$$ab = - < a|b > + a \times b. \tag{0.0.1}$$

As usual the double covering map from $\boldsymbol{H}^*$ onto the group formed by scalings and rotations is the map assigning to every nonzero quaternion q the transformation

$v \in \boldsymbol{R}^3 \to \bar{q}vq \in \boldsymbol{R}^3$. In particular, we will identify the universal cover Spin(3) of $SO(3)$ with the unit quaternions S^3. The differential df of an immersion f of M into $\boldsymbol{R}^3 = \mathrm{im}(\boldsymbol{H})$ is a $\boldsymbol{H}$-valued 1-form on the abstract surface M. Starting from these basic ideas one can reformulate surface theory in terms of quaternionic-valued objects. We are concerned primarily with oriented surfaces, so M will denote an oriented connected surface. Unless we explicitly specify otherwise, we will also assume that M is a Riemann surface, i.e., that there is a chosen conformal structure on M.

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Many problems in pure and applied mathematics boil down to determining the shape of a surface in space or constructing surfaces with prescribed geometric properties. These problems range from classical problems in geometry, elasticity, and capillarity to problems in computer vision, medical imaging, and graphics. There has been a sustained effort to understand these questions, but many problems remain open or only partially solved.

This book describes how to use quaternions and spinors to study conformal immersions of Riemann surfaces into $\mathbb{R}^3$. The first part develops the necessary quaternionic calculus on surfaces, its application to surface theory and the study of conformal immersions and spinor transforms. The integrability conditions for spinor transforms lead naturally to Dirac spinors and their application to conformal immersions. The second part presents a complete spinor calculus on a Riemann surface, the definition of a conformal Dirac operator, and a generalized Weierstrass representation valid for all surfaces. This theory is used to investigate first, to what extent a surface is determined by its tangent plane distribution, and second, to what extent curvature determines the shape.

The book is geared toward graduate students and researchers interested in differential geometry and geometric analysis and their applications in computer vision and computer graphics.

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