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July 29 – August 2, 2002
Vilnius University
Vilnius, Lithuania

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Preface

Algebraic geometry and geometric modeling both deal with curves and surfaces generated by polynomial equations. Algebraic geometry investigates the theoretical properties of polynomial curves and surfaces; geometric modeling uses polynomial, piecewise polynomial, and rational curves and surfaces to build computer models of mechanical components and assemblies for industrial design and manufacture.

Although geometric modeling and algebraic geometry feature different problems and emphasize different concerns, geometric modeling has routinely borrowed theoretical insights and computational techniques from algebraic geometry. Using classical resultants to solve systems of polynomial equations or to find the implicit equations of rational curves and surfaces, computing intersections, analyzing singularities, and modeling shapes with implicit polynomial equations all have their origins in classical algebraic geometry.

During the first half of the twentieth century, algebraic geometry moved away from computational methods in favor of deeper and deeper abstractions – schemes, sheaves, and sheaf cohomology were all the rage – but in the second half of the twentieth century as computers became cheap and ubiquitous this fashion reversed in favor of less abstract, more algorithmic techniques. The trend today towards algorithmic algebraic geometry lends itself quite naturally to applications in geometric modeling.

Geometric modeling was initiated in the automobile industry during the 1960's with the work of Bezier, de Casteljau, Gordon, and Coons on computer aided surface design [**BFK84**]. Paradoxically, the earliest mathematical roots of geometric modeling are not in geometry, but rather in approximation theory and numerical analysis. It was not until 1983 that classical computational methods from algebraic geometry were first introduced into geometric modeling by Tom Sederberg in his Ph.D. thesis [**Sed83**], where he applies Dixon's resultant to find the implicit equation for rational surfaces. Since Sederberg's dissertation, applications in geometric modeling of the classical resultants of Dixon, Macaulay, Bezout, and Sylvester have been investigated by many researchers, including Sederberg and Anderson [**SAG84**], Tiller and Montaudouin [**TM84**], Manocha and Canny [**MC92a, MC92b**], Chionh and Goldman [**CGM91, CG92**], and Bajaj and Abhyankar [**BA89**].

Applications of algebraic geometry in geometric modeling are now quite common. To give a sense of the broad impact of algebraic geometry on geometric modeling, we list below a few representative samples of the extensive research and numerous researchers in geometric modeling who have used or are currently using algebraic geometry.

- Schwartz and his associates (1983) analyze all the intersections of quadric surfaces for inclusion in solid modeling systems based on classical methods from algebraic geometry [**OSS83**].
- Warren (1986) studies geometric continuity between algebraic curves and surfaces and then applies his results to find blends between algebraic patches in his Ph.D. thesis [**War86**].
- Garrity and Warren (1989) build data structures for the intersection curves between algebraic surfaces using techniques adapted from algebraic geometry [**GW89**].
- Farouki and his collaborators (1989) investigate degenerate intersections of quadric surfaces using the Segre characteristic of a quadratic form [**FNO89**].
- Hoffmann (1989) uses Groebner bases to solve polynomial equations, find intersections of algebraic surfaces, and locate singularities in his book on solid modeling [**Hof96**].
- Bajaj and his students (1990) developed the software toolkit GANITH, based on computational algebraic geometry, to assist with geometric modeling applications [**BR90**].
- Winkler and his collaborators (1991) developed CASA, a computer algebra package for constructive algebraic geometry, that implements the classical analysis of singularities in order to robustly render algebraic curves and surfaces [**WGKW91, WHH00**].
- Sederberg and Chen (1995) and Zhang, Chionh, and Goldman (2000) explore the use of syzygies for the efficient implicitization of rational surfaces with base points [**SC95, ZCG00**].

Algebraic geometers have also contributed directly to geometric modeling.

- Abhyankar and Bajaj (1987-1989) collaborated on a series of papers investigating rational parametrizations of curves and surfaces [**AB87a, AB87b, AB88, AB89**].
- Abhyankar (1990) wrote a book on algebraic geometry specifically for engineers [**Abh90**].
- Cox, Little, and O'Shea (1996,1998) wrote two books on algebraic geometry that have had a substantial influence on an important segment of the geometric modeling community [**CLO96, CLO98**].
- Cox (1998-2001) has also lent his expertise to help in the exploration and analysis of syzygies for the efficient implicitization of rational surfaces [**Cox01, CZG00**].

Recently, geometric modeling has begun to return the favor to algebraic geometry. Computational and numerical issues involving classical resultants have been investigated for the first time by Manocha and Canny [**MC92a, MC92b**] and by Winkler and Goldman [**WG03**]. New representations for sparse resultants have been developed by Chionh, Goldman, and Zhang [**Chi01, CZG00, ZG00**] and also by Zube [**Zub00**]. Cox, Sederberg, and Chen introduced the notion of a μ-basis for the classification of rational curves [**CSC98**]. Conjectures on syzygies initiated by geometric modelers [**CZG00, SC95**] have recently been proved by algebraic geometers [**CZG00, D'A01**].

The first wave of interactions between algebraic geometry and geometric modeling – spanning roughly the years from 1983-1995 – focused mainly on classical algebraic geometry: standard resultants, complex projective spaces, conic sections and quadric surfaces, intersections, and singularities. Building on these past successes, we decided in light of some recent new developments in both fields to initiate a second wave of synergy between these two research communities focusing this time on modern algebraic geometry, including sparse resultants, toric varieties, and real algebraic geometry.

In the summer of 2002, NSF sponsored a four day workshop on algebraic geometry and geometric modeling in Vilnius, Lithuania. The primary purpose of this workshop was to bring together some of the top people in these two research communities in order to examine a broad range of themes of interest to both fields. Vilnius was chosen as the site of this workshop because a strong group of mathematicians and computer scientists working at Vilnius University had taken the lead in investigating applications of toric varieties to geometric modeling [**KK99, Kra97, Kra01, Kra02, Zub00**]. By holding the workshop in Vilnius, we hoped to strengthen not only the links between the algebraic geometry and geometric modeling communities, but also the ties between researchers in the USA and researchers in Eastern Europe.

The organizers of the workshop had several goals. We wanted to reinvigorate geometric modeling by bringing to bare more recent developments from algebraic geometry. We also hoped to interest algebraic geometers in problems that arise in geometric modeling. We felt too that there are some important insights that geometric modeling could contribute to algebraic geometry. But before we could accomplish any of these goals, people from the two disciplines needed to talk to each other, to understand each others languages, problems, tools, and requirements. To foster these objectives, we included survey talks and tutorials as well as papers on basic research.

This volume is an outgrowth of the Vilnius Workshop on Algebraic Geometry and Geometric Modeling; most of the papers collected here are based on talks presented at this workshop. The purpose of this book is to provide a service to both communities by bringing the topics addressed at this workshop to a wider audience.

Topics in Algebraic Geometry and Geometric Modeling is divided into five sections:

i. Modeling Curves and Surfaces
ii. Multisided Patches
iii. Implicitization and Parametrization
iv. Toric Varieties
v. Mixed Volume and Resultants

In keeping with our goal of forging new links between algebraic geometry and geometric modeling, each section contains papers from both disciplines, though some sections are necessarily weighted more heavily to one or the other of these fields.

Modeling curves and surfaces is one of the primary ambitions of geometric modeling, so we begin with three papers on this general theme. Analysis algorithms for Bezier curves and surfaces, intersection algorithms for conic sections and quadric surfaces, and a new surface modeling paradigm with roots in differential topology are presented here. Much of this material will be familiar to researchers in geometric modeling, but our intention is that this section may serve as a brief introduction to the field of geometric modeling for researchers in algebraic geometry. We hope too that people in geometric modeling will gain some new insights from some of the fresh perspectives presented here.

Multisided patches are an emerging theme is geometric modeling [**KK99, Kra02, War92**]. Until recently most of the attention in this field has focused on modeling three sided and four sided patches, but general n-sided patches are often required to fill n-sided holes. Modeling multisided patches is one area where we expect that toric varieties will make an important contribution to geometric modeling. The papers in this section not only survey the current state of the art, but also break some new ground in this domain.

Implicitization and parametrization are fundamental both in geometric modeling and in algebraic geometry. The first two papers in this section deal with a new implicitization method. Initially introduced in geometric modeling for implicitizing surfaces with base points, the method of moving surfaces (or syzygies) can be analyzed rigorously only by invoking powerful techniques from algebraic geometry and commutative algebra. The first two papers presented here discuss what is currently provable about this new method as well as the latest developments in this new technique. The third paper in this section treats approximate implicitization. Approximate implicitization owes as much to approximation theory and numerical analysis as it does to algebraic geometry and geometric modeling, so this paper presents an important attempt to merge insights from several fields. This section closes with a paper that discusses the current state of the art in surface parametrization.

Toric varieties are a major new paradigm in modern algebraic geometry. We observed in Section 1 and Section 2 that toric varieties may have practical applications in geometric modeling to the design of multisided patches. Here we devote an entire section to toric varieties. A tutorial is provided first for the uninitiated, and then three research papers are presented. We hope that this section will serve as an introduction to toric varieties for people in geometric modeling who find it difficult to penetrate the sometimes esoteric literature of algebraic geometry. We also expect that some of the new results presented here will contribute to further developments both in algebraic geometry and in geometric modeling.

Mixed volume and resultants are our final topic. Elimination theory is one of the primary computational tools of classical algebraic geometry, and sparse resultants are their modern incarnation. Some of the latest results on sparse resultants are presented here. The connection between mixed volume and blossoming is discussed in Section 1. In this section the intuition behind the link between mixed volume and the number of roots of polynomial equations is explained; rigorous proofs are also provided. This section closes with a new elimination algorithm for Laurent polynomials together with a new formula for mixed volume. As a special treat, we have included in this section a translation of Minding's 1841 paper *On the determination of the degree of an equation obtained by elimination*, which foreshadows the modern application of mixed volume. This paper has intellectual as well as historical interest. In a commentary after the paper, the translators explain how Minding's formula relates to the mixed area of lattice polygons. They also explain how this approach relates to the way Newton used Newton polytopes.

The ultimate purpose of this book is to promote closer cooperation between two distinct cultures and disciplines – between algebraic geometers and geometric modelers, between theoreticians and practitioners, between mathematicians and computer scientists – in order to further the goals and aspirations of both fields. By focusing the attention of these two research communities on problems that are of both theoretical and practical importance, we hope to establish stronger links and longer range ties between these two disparate groups, and to foster as well more focused interdisciplinary research that will have a substantial, long lasting, impact on both fields.

In this spirit of cooperation and with the understanding that much of the difficult work still remains to be done, we urge you to read on, study hard, and enjoy these mathematical meditations!

Ron Goldman
Rimvydas Krasauskas

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The NSF sponsored the four-day "Vilnius Workshop on Algebraic Geometry and Geometric Modeling", which brought together some of the top experts in the two research communities to examine a wide range of topics of interest to both fields. This volume is an outgrowth of that workshop. Included are surveys, tutorials, and research papers. In addition, the editors have included a translation of Minding's 1841 paper, "On the determination of the degree of an equation obtained by elimination", which foreshadows the modern application of mixed volumes in algebraic geometry.

The volume is suitable for mathematicians, computer scientists, and engineers interested in applications of algebraic geometry to geometric modeling.

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