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Atomic dilations

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ABSTRACT. For the first time we show the existence of dilations of completely positive semigroups to $*$-endomorphism semigroups which are not minimal but still 'atomic' in the sense that they have no 'smaller' dilations. We also show that such a phenomenon can't occur for unital semigroups or for regular dilations.

1. Introduction

In this short note we describe through examples the delicate nature of the notion of minimality for dilations of semigroups of contractive completely positive maps to semigroups of $*$-endomorphisms. Various characterizations of minimality of dilation have been obtained earlier by Arveson [**Ar**] and SeLegue [**Se**] for unital semigroups. The non-unital case seems to be much more complicated.

In this article our Hilbert spaces will be complex, separable with an inner product $\langle \cdot, \cdot \rangle$, which is antilinear in the first variable. For any two vectors x, y in a Hilbert space by $|x\rangle\langle y|$, we will denote the operator which sends any vector z to $\langle y, z\rangle x$. Let $\mathbb{T}$ be the semigroup $\mathbb{Z}_+$ of non-negative integers or the semigroup $\mathbb{R}_+$ of non-negative real numbers. Let $\mathcal{H}$ be a complex separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded operators on $\mathcal{H}$. A quantum dynamical semigroup on $\mathcal{B}(\mathcal{H})$ is a semigroup $\tau = \{\tau_t : t \in \mathbb{T}\}$ of normal contractive completely positive maps of $\mathcal{B}(\mathcal{H})$. (If $\mathbb{T} = \mathbb{R}_+$, one usually demands continuity in weak or strong operator topology of the map $t \mapsto \tau_t(X)$ for each X in $\mathcal{B}(\mathcal{H})$, however except for the existence of dilation in a separable Hilbert space in the dilation theorem we do not need any sort of continuity in this article). Now let $\mathcal{K}$ be another Hilbert space. An E-semigroup on $\mathcal{B}(\mathcal{K})$ is a semigroup $\theta = \{\theta_t : t \in \mathbb{T}\}$ of $*$-endomorphisms of $\mathcal{B}(\mathcal{K})$. If further $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$, and τ can be got from E-semigroup θ by compression:

$$\tau_t(X) = P\theta_t(X)P, \ X \in \mathcal{B}(\mathcal{H}), t \in \mathbb{T}, \tag{1.1}$$

where $P = P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ on to $\mathcal{H}$ (Here and elsewhere in this note for $\mathcal{H} \subset \mathcal{K}$, we identify $\mathcal{B}(\mathcal{H})$ with $P\mathcal{B}(\mathcal{K})P$ in the natural way, and for this reason $\theta_t(X)$ makes sense for X in $\mathcal{B}(\mathcal{H})$) then we call $(\mathcal{K}, \theta)$ as a dilation of $(\mathcal{H}, \tau)$ or more simply as θ is a dilation of τ.

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Motivated by papers of Arveson and Powers dealing on various aspects of E-semigroups, one such dilation was constructed in Bhat [**Bh1, Bh2**]. This relied heavily on Bhat-Parthasarathy [**BP1, BP2**], and there the purpose was to develop a non-commutative Markov process theory. The idea that quantum dynamical semigroups are non-commutative analogues of transition probability semigroups and there should be associated Markov processes is not new. There have been many papers along these lines by many different authors; Accardi, Belavkin, Davis, Emch, Kümmerer, Sauvageot to name a few. We just refer to the survey article [**Ku**]. More references can be found in the papers mentioned here and generally in quantum probability literature. A comparison of our approach with that of Kümmerer can be found in [**Go**]. Some recent versions considering more general C^* or von Neumann algebras can be seen in [**BS**], [**MS**]. We recall the main theorem from [**Bh1, Bh2**] without proof.

THEOREM 1.1 (Dilation Theorem). *Let τ be a quantum dynamical semigroup on $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, with a dilation θ on $\mathcal{B}(\mathcal{K})$ of τ such that,*

$$F(s)\theta_t(X)F(s) = \theta_s(\tau_{t-s}(X)), X \in \mathcal{B}(\mathcal{H}), \tag{1.2}$$

where $F(s)$ is the projection on to $\overline{\text{span}}\,\{\theta_{r_1}(X_1)\cdots\theta_{r_n}(X_n)u : s \geq r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, r_i \in \mathbb{T}, X_i \in \mathcal{B}(\mathcal{H}), n \geq 0, u \in \mathcal{H}\}$, for $0 \leq s \leq t$ in $\mathbb{T}$; and

$$\overline{\text{span}}\,\{\theta_{r_1}(X_1)\cdots\theta_{r_n}(X_n)u : r_1 \geq \cdots \geq r_n, r_i \in \mathbb{T}, X_i \in \mathcal{B}(\mathcal{H}), n \geq 0, u \in \mathcal{H}\} = \mathcal{K}. \tag{1.3}$$

Moreover if $(\mathcal{K}', \theta')$ is another such dilation then there exists a unitary $U\colon \mathcal{K} \to \mathcal{K}'$, such that $Uu = u$ for all $u \in \mathcal{H}$ and $\theta'_t(Z) = U\theta_t(U^*ZU)U^*$ for all $Z \in \mathcal{B}(\mathcal{K}')$ and $t \in \mathbb{T}$.

DEFINITION 1.2. Given a quantum dynamical semigroup τ, the unique (up to unitary equivalence) dilation constructed in Theorem 1.1 is called as *the minimal dilation* of τ.

In the context of non-commutative Markov process theory the increasing family of projections $F(t)$ is the 'filtration', and the family of $*$-homomorphisms j_t defined by $j_t(X) = \theta_t(X), X \in \mathcal{B}(\mathcal{H})$ is the 'Markov process'. So the condition (1.2) is the 'Markov property'. The condition (1.3) is known as the 'minimality condition'. The semigroup θ is known as the time-shift of the Markov process, as it satisfies $\theta_t(j_s(X)) = j_{s+t}(X)$.

In the following we assume that we are provided with a dilation θ of τ. The problem we wish to consider is that of identifying dilations which can be considered as 'smallest' in some sense. In this context we have the following notion from Arveson [**Ar**], SeLegue [**Se**]. For the sake of clarity we have chosen the word 'atomic' to replace their 'minimal'. Otherwise there will be a conflict with Definition 1.2. They deal with only unital quantum dynamical semigroups and one of their results is that then the two notions coincide. We will have a different proof of this result in Section 2.

DEFINITION 1.3. A dilation $(\mathcal{K}, \theta)$ of $(\mathcal{H}, \tau)$ is called *atomic* if there does not exist a proper subspace $\mathcal{L}$ of $\mathcal{K}$ such that $\mathcal{L}$ contains $\mathcal{H}$ and $(\mathcal{L}, \psi)$ is a dilation of $(\mathcal{H}, \tau)$, where ψ is the compression of θ to $\mathcal{B}(\mathcal{L})$, defined by,

$$\psi_t(Y) = P_{\mathcal{L}}\theta_t(Y)P_{\mathcal{L}},\ Y \in \mathcal{B}(\mathcal{L}), t \in \mathbb{T}, \tag{1.4}$$

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and $P_{\mathcal{L}}$ is the projection on to $\mathcal{L}$.

Note that in this Definition we demand that ψ_t defined by (1.4) is a $*$ - endomorphism. It can be checked easily that this happens if and only if $\theta_t(Y)$ leaves $\mathcal{L}$ invariant for every Y in $\mathcal{B}(\mathcal{L})$. We also want $\psi = \{\psi_t\}_{t\in\mathbb{T}}$ to form a semigroup. For determining this the following version of Sarason's theorem is useful.

THEOREM 1.4. *Let θ be a semigroup of $*$-endomorphisms of $\mathcal{B}(\mathcal{K})$ and suppose ψ is a compression of θ by a projection $P_{\mathcal{L}}$ as in* (1.4) *for a subspace $\mathcal{L}$ of $\mathcal{K}$. Then ψ is a semigroup of $*$-endomorphisms if and only if for every $Y \in \mathcal{B}(\mathcal{L})$ and $t \in \mathbb{T}$, $\theta_t(Y)$ leaves $\mathcal{L}$ invariant and $P = P_1 - P_2$, for some projections P_1, P_2 with $P_1 \geq P_2$ such that $\theta_t(P_1) \leq P_1$ and $\theta_t(P_2) \leq P_2$ for all t.*

PROOF. See Corollary 2.3 of [**Bh3**]. □

Finally, we introduce one more notion of 'smallness' for dilations.

DEFINITION 1.5. A dilation $(\mathcal{K},\theta)$ of a quantum dynamical semigroup $(\mathcal{H},\tau)$ is called a *primary* dilation if $\overline{\text{span}}\,\{\text{range}\,\theta_t(P_{\mathcal{H}}) : t \in \mathbb{T}\} = \mathcal{K}$.

In general $\theta_t(P_{\mathcal{H}})$ need not be a commuting family. However, it is to be noted that if τ is unital then for any dilation θ we have $\theta_t(P_{\mathcal{H}}) \geq P_{\mathcal{H}}$ for all t. So here θ is primary if and only if $\theta_t(P_{\mathcal{H}})$ increases to identity in strong operator topology as t increases to infinity.

2. Regular dilations

In this Section we see as to what is the order relation between various notions of 'smallness' introduced on dilations in the Introduction. It is convenient to have some notation. Let $(\mathcal{K},\theta)$ be a dilation of a quantum dynamical semigroup $(\mathcal{H},\tau)$. We make the short hand notation $\theta(\underline{r},\underline{X})$ for the operator $\theta_{r_1}(X_1)\cdots\theta_{r_n}(X_n)$ for $\underline{r} = (r_1,\ldots,r_n), \underline{X} = (X_1,\ldots,X_n)$ with $r_i \in \mathbb{T}, X_i \in \mathcal{B}(\mathcal{H})$. Now take,

$$\begin{aligned}\mathcal{H}_a &= \overline{\text{span}}\,\{\theta(\underline{r},\underline{X})u : (\underline{r},\underline{X},u) \in D_a\}\\ \mathcal{H}_b &= \overline{\text{span}}\,\{\theta(\underline{r},\underline{X})u : (\underline{r},\underline{X},u) \in D_b\}\\ \mathcal{H}_c &= \overline{\text{span}}\,\{\theta_t(X)x : t \in \mathbb{T}, X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{K}\}\end{aligned}$$

where $D_a = \{(\underline{r},\underline{X},u) : r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, r_i \in \mathbb{T}, X_i \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}, n \geq 0\}$, $D_b = \{(\underline{r},\underline{X},u) : r_1, r_2, \ldots, r_n \in \mathbb{T}, X_i \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}, n \geq 0\}$. Here $\theta(\underline{r},\underline{X})u$ in $\mathcal{H}_a$ or $\mathcal{H}_b$ is to be interpreted as vector u, when we have empty tuple $(\underline{r},\underline{X})$, that is, when $n = 0$, so that $\mathcal{H}$ is a subspace of $\mathcal{H}_a, \mathcal{H}_b$. Then it is clear that we have $\mathcal{H} \subset \mathcal{H}_a \subset \mathcal{H}_b \subset \mathcal{H}_c \subset \mathcal{K}$.

THEOREM 2.1. *Let $(\mathcal{K},\theta)$ be a dilation of a quantum dynamical semigroup $(\mathcal{H},\tau)$.*

(i) *If θ satisfies the minimality condition* (1.3) *then it is an atomic dilation.*
(ii) *If θ is an atomic dilation then it is a primary dilation.*

PROOF. Suppose $\mathcal{L}$ is a subspace of $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{L} \subset \mathcal{K}$ and the compression $(\mathcal{L},\psi)$ of $(\mathcal{K},\theta)$ to $\mathcal{L}$ (as in the definition of atomic dilation) is a dilation. Then by Theorem 1.4, $\theta_t(Z)$ leaves $\mathcal{L}$ invariant for every Z in $\mathcal{B}(\mathcal{L})$. In particular, $\theta_t(X)$ leaves $\mathcal{L}$ invariant for every X in $\mathcal{B}(\mathcal{H})$. Therefore $\mathcal{H}_b$ is contained in $\mathcal{L}$. So if θ satisfies the minimality condition then $\mathcal{H}_a = \mathcal{H}_b = \mathcal{L} = \mathcal{K}$. This proves the first part. To see the second part consider a rank one operator

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$|\theta_{t_1}(X_1)x_1\rangle\langle\theta_{t_2}(X_2)x_2|$, for any $t_1, t_2 \in \mathbb{T}, X_1, X_2 \in \mathcal{B}(\mathcal{H}), x_1, x_2 \in \mathcal{K}$. Then for $t \in \mathbb{T}$, range$(\theta_t(|\theta_{t_1}(X_1)x_1\rangle\langle\theta_{t_2}(X_2)x_2|)) \subset$ range $(\theta_{t+t_1}(X_1)) \subset \mathcal{H}_c$. It follows that range $\theta_t(Z) \subset \mathcal{H}_c$, for every $Z \in \mathcal{B}(\mathcal{H}_c)$. In particular, $\theta_t(P_{\mathcal{H}_c}) \leq P_{\mathcal{H}_c}$. Taking $P_1 = P_{\mathcal{H}_c}, P_2 = 0$, and applying Theorem 1.4, we see that θ compressed by $\mathcal{H}_c$ is a dilation of τ. Then by atomicity we must have $\mathcal{H}_c = \mathcal{K}$. So θ is a primary dilation. □

Observe that θ_t compressed by $\mathcal{H}_b$ is always a $*$-endomorphism for every t. However, in general this may not be a semigroup and so we may not have a dilation. We see one such example in the next Section. Arguments along the lines of the proof of (ii) above lead us to the following remark.

REMARK 2.2. The smallest subspace $\mathcal{L}$ of $\mathcal{K}$, containing $\mathcal{H}$, such that $\theta_t(P_{\mathcal{L}}) \leq P_{\mathcal{L}}$, is $\mathcal{H}_c$.

To see when we can go in the converse direction in Thoerem 2.1(i), we have the following definition.

DEFINITION 2.3. A dilation $(\mathcal{K}, \theta)$ of $(\mathcal{H}, \tau)$ is called *regular* if $\theta_t(1 - P_{\mathcal{H}}) \leq 1 - P_{\mathcal{H}}$, for all $t \in \mathbb{T}$.

PROPOSITION 2.4. *If $(\mathcal{K}, \theta)$ is the unique minimal dilation of $(\mathcal{H}, \tau)$, then it is regular.*

PROOF. Take $P = P_{\mathcal{H}}$. We have $\mathcal{H}_a = \mathcal{K}$. So vectors of the form $\theta(\underline{r}, \underline{X})u$ with $r_1 \geq r_2 \geq \cdots r_n \geq 0$, is total in $\mathcal{K}$. It follows that vectors of the form $\theta(\underline{r}, \underline{X})u - P\theta(\underline{r}, \underline{X})u$ with r_i's decreasing are total in $\mathcal{H}^{\perp}$. Now for $y = \theta(\underline{r}, \underline{X})u - P\theta(\underline{r}, \underline{X})u$, $z \in \mathcal{H}^{\perp}$, we have

$$P\theta_t(|y\rangle\langle z|) = P\theta_t(\theta(\underline{r}, \underline{X})P - P\theta(\underline{r}, \underline{X})P)\theta_t(|u\rangle\langle z|).$$

By Markov property (See [**Bh1, Bh2**]), $P\theta(\underline{r}, \underline{X})P = \epsilon(\underline{r}, \underline{X})$, where

$$\epsilon(\underline{r}, \underline{X}) = \tau_{r_n}(\cdots \tau_{r_2 - r_3}(\tau_{r_1 - r_2}(X_1)X_2) \cdots X_n).$$

So $P\theta_t(P\theta(\underline{r}, \underline{X})P) = P\theta_t(\epsilon(\underline{r}, \underline{X}))$. Similarly, $P\theta_t(\theta(\underline{r}, \underline{X})P) = PF(t)\theta(\underline{r} + t, \underline{X})$ $F(t)\theta_t(P) = P\theta_t(\epsilon(\underline{r}, \underline{X}))\theta_t(P) = P\theta_t(\epsilon(\underline{r}, \underline{X}))$. Therefore $P\theta_t(|y\rangle\langle z|) = 0$. By continuity it follows that $P\theta_t(Z) = 0$, for all $Z \in \mathcal{B}(\mathcal{H}^{\perp})$. By taking adjoints we also have $\theta_t(Z)P = 0$. In particular, $\theta_t(1 - P) \leq (1 - P)$, for all t. □

THEOREM 2.5. *Suppose $(\mathcal{K}, \theta)$ is a regular dilation of $(\mathcal{H}, \tau)$. Then it is atomic if and only if it is the unique minimal dilation.*

PROOF. Suppose $(\mathcal{K}, \theta)$ is regular and atomic. Taking $P = P_{\mathcal{H}}, Q = 1 - P$, we have $P\theta_t(Q)P = \theta_t(Q)P = P\theta_t(Q) = 0$, for all t. Now for $c \leq a \leq b$ in $\mathbb{T}$, and X, Y, Z in $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned}\theta_a(X)\theta_b(Y)\theta_c(Z) &= \theta_c(\theta_{a-c}(X\theta_{b-a}(Y))Z) = \theta_c(\theta_{a-c}(X\theta_{b-a}(Y)(P+Q))PZ) \\ &= \theta_c(\theta_{a-c}(X\theta_{b-a}(Y)P)Z) = \theta_c(\theta_{a-c}(X\tau_{b-a}(Y))Z) \\ &= \theta_a(X\tau_{b-a}(Y))\theta_c(Z).\end{aligned}$$

Similarly for $a \leq c \leq b$, $\theta_a(X)\theta_b(Y)\theta_c(Z) = \theta_a(X)\theta_c(\tau_{b-c}(Y)Z)$. It follows by induction that inner products between vectors of the form $\theta(\underline{r}, \underline{X})u$ in $\mathcal{H}_b$ can be computed in terms of τ, and it matches with the values one gets (See [**Bh1, Bh2**]) for the minimal dilation. In other words θ compressed by $\mathcal{H}_b$ gives us the minimal dilation. In particular $\mathcal{H}_a = \mathcal{H}_b$. By atomicity this space is equal to $\mathcal{K}$. The converse has already been proved in Theorem 2.1. □

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Suppose τ is unital. Then for any dilation θ, $P \leq \theta_t(P)$, for all t, since by endomorphism property we have $\theta_t(1) \leq 1$, θ is automatically regular. So as a corollary to Theorem 2.5 we obtain that notions of atomicity and minimal dilation coincide for unital completely positive semigroups. This has been noted earlier by Arveson [**Ar**] and SeLegue [**Se**].

We also would like to remark that we do not know whether the minimality condition (1.3) alone is sufficient to have uniqueness. Actually the dilation property determines norms of vectors of the form $\theta(\underline{r}, \underline{X})u$, with r_i's decreasing, in terms of the semigroup τ. From this it was hastily concluded in [**Bh2**], that we have the unique minimal dilation if the minimality condition is satisfied. But from the example in the next Section it is clear that inner products between these vectors are not determined by the dilation property! Because of all these complications perhaps it is better to restrict one self to regular dilations. The proof of Theorem 2.5 actually shows that any regular dilation contains an atomic dilation which is unitarily equivalent to the unique minimal dilation of Theorem 1.1.

3. Examples

Here we look at some regular as well as some non-regular dilations. We consider only discrete time examples, that is, we are taking $\mathbb{T} = \mathbb{Z}_+$, and we are considering dilations of powers of a single completely positive map. All our examples come from multivariable operator theory.

Given a normal contractive completely positive map $\tau \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, through a simple application of Stinesprings representation theorem (and representation theory of $\mathcal{B}(\mathcal{H})$) we can deduce that τ can be expressed in the form (Kraus decomposition):

$$\tau(X) = \sum_{i \in \Lambda} T_i X T_i^*, \ X \in \mathcal{B}(\mathcal{H}),$$

where $\Lambda = \{1, 2, \dots, n\}$ for some n or $\Lambda = \{1, 2, \dots, \infty\}$, and $\sum T_i T_i^* \leq 1$. (If Λ is infinite, the series $\sum T_i X T_i^*$ converges in strong operator topology). Now by Bunce [**Bu**], Frazho [**Fr**] and Popescu [**Po**], there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ with a family of isometries $\{V_i\}_{i \in \Lambda}$, such that

(i) $V_i^* V_j = \delta_{ij}, \ i, j \in \Lambda$;
(ii) $V_i^* u = T_i^* u, i \in \Lambda, u \in \mathcal{H}$.

Here the property (i) says that V_i's are isometries with orthogonal ranges and the content of (ii) is that V_i^*'s leave $\mathcal{H}$ invariant and tuple $\{V_i\}_{i \in \Lambda}$ forms a dilation of tuple $\{T_i\}_{i \in \Lambda}$ in the sense

$$T_{i_1} \dots T_{i_k} = P_{\mathcal{H}} V_{i_1} \dots V_{i_k}|_{\mathcal{H}}$$

for all $i_1, \dots i_k$ in Λ and $k \geq 1$. Such a dilation we call as an isometric dilation of $\{T_i\}_{i \in \Lambda}$. It is also known that up to unitary equivalence there is unique minimal isometric dilation, where here minimality means that the space

$$\mathcal{H}_d := \overline{\operatorname{span}} \{V_{i_1} \dots V_{i_k} u : i_1, \dots i_k \in \Lambda, k \geq 0, u \in \mathcal{H}\}$$

is whole of $\mathcal{K}$. We may call the minimal isometric dilation as the standard (non-commuting) dilation of $\{T_i\}_{i \in \Lambda}$. An isometric dilation $\{V_i\}$ immediately gives us a $*$-endomorphism θ of $\mathcal{B}(\mathcal{K})$ by setting

$$\theta(Z) = \sum_i V_i Z V_i^* \ Z \in \mathcal{B}(\mathcal{K}).$$

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It is easily verified that θ is a $*$-endomophism. Moreover, for $X \in \mathcal{B}(\mathcal{H}), u, v \in \mathcal{H}$

$$\langle u, \theta(X)v\rangle = \sum_i \langle V_i^* u, XV_i^* v\rangle = \sum_i \langle T_i^* u, XT_i^* v\rangle = \sum_i \langle u, T_i XT_i^* v\rangle.$$

In other words $\tau(X) = P\theta(X)P$ for $X \in \mathcal{B}(\mathcal{H})$. In a similar fashion we see that $\tau^n(X) = P\theta^n(X)P$, for all $n \geq 0$. So $(\mathcal{K}, \theta)$ is a dilation of $(\mathcal{H}, \tau)$. At first sight one may think that θ must be the minimal dilation of τ if $\{V_i\}$ is minimal. However this is not true in general, as we may not have chosen the tuple $\{T_i\}$ in an optimal way. Then how does the minimality of $\{V_i\}$ reflected in θ? This has been answered by SeLegue [**Se**] in the unital case. We also refer to Gohm [**Go**] for this result. Here we extend it to include the non-unital case as well.

THEOREM 3.1. *Suppose θ is a dilation of τ obtained through an isometric dilation $\{V_i\}$ as above. Then θ is a primary dilation if and only if $\{V_i\}$ is minimal.*

PROOF. We will show that $\mathcal{H}_c = \mathcal{H}_d$. This clearly proves the Theorem. For $X \in \mathcal{B}(\mathcal{H})$, from the expression $\theta^n(X) = \sum_{i_1, i_2, \ldots, i_n \in \Lambda} V_{i_1} V_{i_2} \ldots V_{i_n} X V_{i_n}^* \ldots V_{i_2}^* V_{i_1}^*$, we clearly have (we may first consider rank one operators and then take limits) $\mathcal{H}_c \subset \mathcal{H}_d$. Conversely, for any unit vector $u \in \mathcal{H}$, and $i_1, i_2, \ldots, i_n \in \Lambda$, $V_{i_1} V_{i_2} \ldots V_{i_n} u = \theta^n(|u\rangle\langle u|) V_{i_1} V_{i_2} \ldots V_{i_n} u$. Hence $\mathcal{H}_d \subset \mathcal{H}_c$. $\square$

We remark that as in the unital case, we can ensure minimality by taking minimal isometric dilation of linearly indedpendent tuple $\{T_i\}$, but we do not elaborate on this here.

THEOREM 3.2. *Let $(\mathcal{K}, \theta)$ be a dilation of a contractive completely positive map $(\mathcal{H}, \tau)$. Then θ is a regular dilation if and only if θ comes from an isometric dilation $\{V_i\}$ for some tuple $\{T_i\}$ such that $\theta(Z) = \sum_i V_i Z V_i^*$, for $Z \in \mathcal{B}(\mathcal{K})$, and $\tau(X) = \sum_i T_i X T_i^*$ for $X \in \mathcal{B}(\mathcal{H})$.*

PROOF. Suppose θ comes from an isometric dilation as above. We have each V_i^* leaving $\mathcal{H}$ invariant. If P denotes the projection onto $\mathcal{H}$, then $V_i^* P = PV_i^* P$ or, $PV_i(1-P) = 0$ for all i. Hence $P\theta(1-P)P = P(\sum_i V_i(1-P)V_i^*)P = 0$. Therefore, $\theta(1-P) \leq (1-P)$. Conversely, suppose $(\mathcal{K}, \theta)$ is a regular dilation. As θ is a $*$-endomorphism, $\theta(Z) = \sum_i V_i Z V_i^*$, for some isometries $\{V_i\}$ with orthogonal ranges. From $\theta(1-P) \leq (1-P)$, for any fixed i we obtain, $V_i(1-P)V_i^* \leq \sum_j V_j(1-P)V_j^* \leq (1-P)$. So $PV_i(1-P)V_i^* P = 0$, which yields $(1-P)V_i^* P = 0$. So each V_i^* leaves $\mathcal{H}$ invariant. Define $T_i \in \mathcal{B}(\mathcal{H})$ by setting $T_i^* u = V_i^* u,\ u \in \mathcal{H}$. Then $\{V_i\}$ is a dilation of $\{T_i\}$. Furthermore by compressing $\theta(X) = \sum_i V_i X V_i^*$ to $\mathcal{H}$ we see that $\tau(X) = \sum_i T_i X T_i^*$ for X in $\mathcal{B}(\mathcal{H})$. $\square$

Finally we have an example of an atomic dilation which is not minimal. This example arose in the context of showing that Theorem 1.4 does not hold in general for quantum dynamical semigroups [**Bh3**].

Take $\mathcal{M} = \mathbb{C}^2$, with standard ortho-normal basis $\{e_1, e_2\}$. Define $\beta \colon \mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{M})$ by

$$\beta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{N} \begin{bmatrix} 2(a+d)-(b+c) & a \\ a & a \end{bmatrix} \tag{3.1}$$

where N is a suitable positive scalar to make β contractive. We see that $\beta(X) = \sum_{i=1}^3 S_i X S_i^*$ where

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$$S_1 = \frac{1}{\sqrt{2N}}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, S_2 = \frac{1}{\sqrt{6N}}\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, S_3 = \frac{1}{\sqrt{3N}}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So β is a completely positive map on $\mathcal{B}(\mathcal{M})$. Take $\mathcal{H} = \mathbb{C}e_1$, and let α be the compression of β by $\mathcal{H}$:

$$\alpha(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = \frac{2}{N}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $(\mathcal{K}, \theta)$ be the minimal dilation of β. Now we consider $(\mathcal{K}, \theta)$ as a dilation of $(\mathcal{H}, \alpha)$. For $n \geq 2$,

$$\beta^n(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \frac{x}{4}\left(\frac{2}{N}\right)^n \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.2}$$

where $x = 2(a+d) - (b+c)$. In particular β^n compressed by $\mathcal{H}$ gives α^n for all n. It follows that θ is a dilation of α. To analyze this dilation define spaces $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ as in the beginning of Section 2, for this dilation, keeping in mind that the space $\mathcal{H}$ is $\mathbb{C}e_1$, and not $\mathbb{C}^2$. Let E_{ij} denote the matrix unit $|e_i\rangle\langle e_j|$ on $\mathcal{M}$ (also thought of as operators on $\mathcal{K}$). Then $\mathcal{H}_a = \{\theta^{r_1}(E_{11})\theta^{r_2}(E_{11})\cdots\theta^{r_n}(E_{11})e_1 : r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, n \geq 0\}$. And similar formulae hold for $\mathcal{H}_b, \mathcal{H}_c$.

In general θ may not even be a primary dilation. We see this as follows. Let I_2, I denote the identitiy operators in $\mathcal{M}, \mathcal{K}$ respectively. As

$$\beta(I_2) = \frac{1}{N}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix},$$

has eigenvalues $\frac{1}{N}(\frac{5-\sqrt{13}}{2})$, and $\frac{1}{N}(\frac{5+\sqrt{13}}{2})$, β is contractive for $N \geq \frac{5+\sqrt{13}}{2}$. Now for $N > \frac{5+\sqrt{13}}{2}$, the matrix $D := I_2 - \beta(I_2)$ is positive and invertible. Consider the vector $z = (I - \theta(I))D^{-1}e_2$. We claim that $z \in \mathcal{H}_c^{\perp}$. Clearly for $n \geq 1, x \in \mathcal{K}$, $\langle z, \theta^n(E_{11})x\rangle = \langle D^{-1}e_2, (\theta^n(E_{11}) - \theta(I)\theta^n(E_{11}))x\rangle = 0$. Also, by using the regularity property of θ as a dilation of β, we have, $\langle z, e_1\rangle = \langle D^{-1}e_2, (I - \theta(I))e_1\rangle = \langle D^{-1}e_2, (I_2 - \theta(I_2))e_1\rangle = \langle D^{-1}e_2, (I_2 - \beta(I_2))e_1\rangle = \langle D^{-1}e_2, De_1\rangle = 0$. A similar computation shows that $\|z\|^2 = \langle D^{-1}e_2, DD^{-1}e_2\rangle = \|D^{-\frac{1}{2}}e_2\|^2 \neq 0$. So z is a non-trivial vector in $\mathcal{H}_c^{\perp}$. It is not clear as to what happens when $N = \frac{5+\sqrt{13}}{2}$. However, this does not matter for our present purposes as anyway we are going to compress θ to its primary part.

Let $(\mathcal{H}_c, \psi)$ be the compression of $(\mathcal{K}, \theta)$ by $\mathcal{H}_c$. We claim that this is a non-minimal atomic dilation of α for any $N \geq \frac{5+\sqrt{13}}{2}$. This requires some explicit computations. Let (W_1, W_2, W_3) be the minimal isometric dilation of (S_1, S_2, S_3) in $\mathcal{K}$ so that

$$\theta(Z) = \sum_i W_i Z W_i^*, \tag{3.3}$$

for $Z \in \mathcal{B}(\mathcal{K})$. Recalling that $W_i^*u = S_i^*u$, for $u \in \mathcal{M}$, we have

$$\theta(E_{11})e_1 = \sum_i W_i E_{11} W_i^* e_1 = \sum_i W_i e_1 \langle e_1, S_i^* e_1\rangle = \sqrt{\frac{2}{N}} W_1 e_1,$$

or $W_1 e_1 = \sqrt{\frac{N}{2}}\theta(E_{11})e_1$. Similar computations show $W_2 e_1 = \sqrt{\frac{N}{6}}(\theta(E_{11})e_1 + 2\theta(E_{12})e_1)$, and $W_3 e_1 = \sqrt{\frac{N}{3}}(3\theta(E_{11})e_2 - 2\theta(E_{11})e_1 - \theta(E_{12})e_1)$. Also note that

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$\theta(E_{11})\theta^2(E_{11})e_1 = \theta(E_{11}\beta(E_{11}))e_1 = \frac{1}{N}\theta(2E_{11}+E_{12})e_1$. Therefore, $W_1e_1, W_2e_1 \in \mathcal{H}_b$ and $W_3e_1 \in \mathcal{H}_c$. Note that from (3.3), $\{W_1e_1, W_2e_1, W_3e_1\}$ is an ortho-normal basis for the range of $\theta(E_{11})$.

Suppose $\mathcal{L}$ is a subspace of $\mathcal{H}_c$ such that $\mathcal{H}_b \subset \mathcal{L} \subset \mathcal{H}_c$ and $(\mathcal{L}, \eta)$, where η is the compression of ψ (or of θ) by $\mathcal{L}$ is a dilation of α. To show atomicity we need to show $\mathcal{L} = \mathcal{H}_c$. Let $R = P_{\mathcal{L}}$ be the projection on to $\mathcal{L}$. Note that $\theta(X)$ leaves $\mathcal{L}$ invariant for every X in $\mathcal{B}(\mathcal{L})$. In particular $\theta(E_{11})R = R\theta(E_{11})R$. So rank three projection $\theta(E_{11})$ commutes with R. We already have W_1e_1, W_2e_1 in $\mathcal{H}_b \subset \mathcal{L}$. So either $W_3e_1 \in \mathcal{L}$ or $W_3e_1 \in \mathcal{L}^{\perp} \bigcap \mathcal{H}_c$.

Case (i) $W_3e_1 \in \mathcal{L}$. Now we have $\theta(E_{11}) \leq R$. We claim that this forces us to have $\theta(R) \leq R$. Indeed for $z_1, z_2 \in \mathcal{L}, z \in \mathcal{K}$,

$$\theta(|z_1\rangle\langle z_2|)z = \theta(|z_1\rangle\langle e_1|)\theta(E_{11})\theta(|e_1\rangle\langle z_2|)z = \theta(|z_1\rangle\langle e_1|)x,$$

for $x = \theta(E_{11})\theta(|e_1\rangle\langle z_2|)z \in \mathcal{L}$. Therefore $\theta(|z_1\rangle\langle z_2|)z \in \mathcal{L}$ as $\mathcal{L}$ is left invariant by all $\theta(X)$, $X \in \mathcal{B}(\mathcal{L})$. Then by normality of θ, $\theta(R) \leq R$. Clearly this implies $\theta^n(R) \leq R$ for all $n \geq 0$. Then by Remark 2.2, $\mathcal{L} = \mathcal{H}_c$.

Case (ii) $W_3e_1 \in \mathcal{L}^{\perp} \bigcap \mathcal{H}_c$. This time we wish to show that η is not a dilation of α and hence arrive at a contradiction. Firstly we claim that W_1, W_2 leave $\mathcal{L}$ invariant and W_3 sends $\mathcal{L}$ to $\mathcal{L}^{\perp} \bigcap \mathcal{H}_c$. First note that from (3.3), for any Z in $\mathcal{B}(\mathcal{K})$, $\theta(Z)W_i = W_iZ$ for all i. Also recall that $\mathcal{L}$ and $\mathcal{L}^{\perp} \bigcap \mathcal{H}_c$ reduce the representation $X \mapsto \theta(X)$ for X in $\mathcal{B}(\mathcal{L})$. Now take any $x \in \mathcal{L}$. For $i = 1, 2$ we have $W_ix = W_i|x\rangle\langle e_1|e_1 = \theta(|x\rangle\langle e_1|)W_ie_1 \in \mathcal{L}$ as $\theta(|x\rangle\langle e_1|)$ leaves $\mathcal{L}$ invariant. Similarly $W_3x = W_3|x\rangle\langle e_1|e_1 = \theta(|x\rangle\langle e_1|)W_3e_1 \in \mathcal{L}^{\perp} \bigcap \mathcal{H}_c$ as $\theta(|x\rangle\langle e_1|)$ leaves $\mathcal{L}^{\perp} \bigcap \mathcal{H}_c$ invariant. So η, the compression of θ by $\mathcal{L}$, is given by $\eta(X) = W_1XW_1^* + W_2XW_2^*$, for $X \in \mathcal{B}(\mathcal{L})$. Now a simple computation shows $\langle e_1, \eta^2(E_{11})e_1\rangle = \sum_{i,j=1}^{2} |\langle e_1, W_i^*W_j^*e_1\rangle|^2 = \sum_{i,j=1}^{2} |\langle e_1, S_i^*S_j^*e_1\rangle|^2 = \frac{10}{3N^2} \neq \frac{4}{N^2}$. Therefore η is not a dilation of α.

Finally as $\psi(E_{11}) = \theta(E_{11})$ is a rank 3 projection, it is obvious that ψ is not the minimal dilation of α.

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References

[Ar] Arveson, W., Minimal E_0-semigroups, *Operator Algebras and their Applications* (Fillmore, P., and Mingo, J., ed.), Fields Institute communications, AMS, (1997) 1-12.

[Bh1] Bhat, B. V. Rajarama, An index theory for quantum dynamical semigroups, Trans. of Amer. Math. Soc., **348** (1996) 561-583.

[Bh2] ———, Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras, J. Ramanujan Math. Soc., **14** (1999) 109-124.

[Bh3] ———, Minimal isometric dilations of operator cocycles, Integral Equations Operator Theory, **42** (2002) 125-141.

[BP1] Bhat, B.V.Rajarama and Parthasarathy, K.R., Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory, Ann. Inst. Henri Poincaré, Probabilités et Statistiques, **31**, no. 4 (1995) 601-651.

[BP2] ———, Kolmogorov's existence theorem for Markov processes in C^*-algebras, Proc. Ind. Acad. Sci. Math. Sci., **103** (1994), 253-262.

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[BS] Bhat, B. V. Rajarama and Skeide, M., Tensor product systems of Hilbert modules and dilations of completely positive semigroups, Infin. Dimens. Anal. Quantum Probab. Relat. Top., Vol. **3**, Number 4, 519-575(2000)

[Bu] Bunce, J. W., Models for n-tuples of non-commuting operators, J. Funct. Anal. 57 (1984), 21-30.

[Fr] Frazho, A.E., Models for non-commuting operators, J. Funct. Anal. 48(1982), 1-11.

[Go] Gohm, R., Elements of a spatial theory for non-commutative stationary processes with discrete time index, preprint 2002.

[Ku] Kümmerer B., Survey on a theory of non-commutative stationary Markov processes, Quantum Prob. and Appl.-III, Springer Lecture Notes in Math. **1303**, (1987) 154-182.

[MS] Muhly, P.S. and Solel, B., Quantum Markov processes (correspondences and dilations), Internat. J. Math. **13** (2002), 863-906.

[Po] Popescu, G., Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc., 316(1989), 523-536.

[Se] SeLegue, D.B., Minimal Dilations of CP Maps and a C^*-Extension of the Szego Limit Theorem, Ph. D. Thesis, University of California, Berkeley (1997).

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