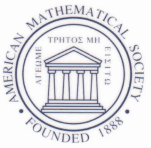
CONTEMPORARY MATHEMATICS

357

Variational Methods: Open Problems, Recent Progress, and Numerical Algorithms

Variational Methods: Open Problems,
Recent Progress, and Numerical Algorithms
June 5–8, 2002
Northern Arizona University
Flagstaff, Arizona

John M. Neuberger
Editor



American Mathematical Society

http://dx.doi.org/10.1090/conm/357

CONTEMPORARY MATHEMATICS

357

Variational Methods: Open Problems, Recent Progress, and Numerical Algorithms

Variational Methods: Open Problems,
Recent Progress, and Numerical Algorithms
June 5–8, 2002
Northern Arizona University
Flagstaff, Arizona

John M. Neuberger
Editor

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 35Axx, 35Jxx, 35J20, 35J60, 65K10, 65Nxx, 65N25, 65N30, 65N38.

Library of Congress Cataloging-in-Publication Data

Variational methods: open problems, recent progress, and numerical algorithms, June 5–8, 2002, Northern Arizona University, Flagstaff, Arizona / John Neuberger, editor.

p. cm.– (Contemporary mathematics, ISSN 0271-4132; 357)

Includes bibliographical references.

ISBN 0-8218-3339-1 (alk. paper)

1. Differential equations, Elliptic–Congresses. 2. Variational inequalities (Mathematics)–Congresses. I. Neuberger, John, 1962– II. Contemporary mathematics (American Mathematical Society); v. 357.

QA377.V37 2004

515′.3533–dc22 2004049919

Contents

Introduction to the Proceedings

1. Preface

This article introduces the proceedings of the conference *Variational Methods: Open Problems, Recent Progress, and Numerical Algorithms*, which was held at Northern Arizona University in Flagstaff, Arizona, June 5^{th} through June 8^{th}, 2002. Alfonso Castro, Goong Chen, and Wei-Ming Ni each gave three one-hour keynote presentations. There were 17 contributed 30-minute talks, a numerical workshop, and lots of time for discussion.

This volume contains 15 mathematical papers; 4 articles authored and/or co-authored by the 3 keynote speakers, and 11 articles authored and/or co-authored by 13 of the 17 contributed speakers. Most of the material concerns variational methods as applied to nonlinear elliptic PDE, although several articles concern nonlinear PDE that are nonvariational and/or nonelliptic. There was a focus on open problems; hence papers with conjectures were encouraged. Survey papers were also solicited, as were works featuring applications. This conference successfully brought together those doing cutting-edge nonlinear functional analysis with researchers more computationally oriented; hence many of the papers in this volume feature numerical investigations. It was our intent to survey what is known in the general subject area, list the important open questions, and suggest analytical and numerical techniques that might be beneficial to those seeking to make progress towards solving those open problems.

Many of the contributers have included an extensive list of relevant references, thoughtfully cited throughout the narrative. We hope that the reader will find this volume an excellent starting point for considering new problems and relevant variational techniques in nonlinear elliptic and related PDE.

This conference would not have been possible without support from NSF DMS Grants 0074326 and 0124121. The organizer would also like to thank Northern Arizona University's Department of Mathematics and Statistics, Dean of the College of Arts and Sciences, and Provost for their generous contributions.

2. Reviews of the Articles by the Editor

The first article by keynote speaker Alfonso Castro [**1**] is an overview of the solvability of semilinear elliptic equations where the linear part has discrete spectrum. This survey paper is intended to be accessible to the non expert and provides an extensive bibliography referencing much of the relevant literature. The paper provides a brief introduction to many of the definitions, theorems, and concepts in critical point theory required by those interested in researching in this area.

For example, resonance, growth, boundary conditions, compactness, critical exponents, the Landesman-Lazer condition, and various classes of nonlinearities are discussed. The influence of the spectrum on nonlinear problems is demonstrated through examples such as Hammerstein integrals and simple hyperbolic problems. Nodal structure theorems, applications, several open problems, and possible extensions are also included. One of the key motivating factors behind organizing the conference was the desire to make progress towards "solving" the often conjectured "infinitely many solutions to the superlinear elliptic Dirichlet problem". This paper gets right to the heart of that issue, clearly explaining the role of "crossing eigenvalues" and providing some insight into N. Dancer's famous counterexample of the so-called Lazer-Mckenna conjecture.

The next three articles [**4, 5, 6**] are co-authored by keynote speaker Goong Chen, Jianxin Zhou, and others, including keynote speaker Wei-Ming Ni. Combined, the works cite an extensive bibliography. The authors provide a unique blend of analytical and numerical results, proving existence, multiplicity, and nodal structure theorems as well as technical convergence, convergence rate, and error estimate theorems. Their development of numerical algorithms together with this hard analysis is a combination rarely found in the literature.

The paper [**4**] presents theoretical numerical results for semilinear elliptic boundary value problems (BVP) with *nonlinear oblique* boundary conditions. This research paper introduces the subject and provides new results, including convergence, convergence rate, and error estimate theorems for their numerical scheme. Some computational results are also provided.

The next article [**5**] is also a research article, providing existence and multiplicity results for the elliptic sine-Gordon equation. The physical relevance to Josephson junctions and π-junctions is first reviewed and explained, and then the relevant variational structure is revealed. The effect of the choice of domain is discussed, and then computational results for positive solutions on a dumbbell-shaped domain are provided for visualization purposes.

The final research paper [**6**] by these authors concerns convergence analysis for an optimal scaling algorithm for semilinear elliptic BVP. A nice introduction discusses the strengths and applicability of many related algorithms, including variants of the Mountain Pass Algorithm (MPA), Monotone Iteration Algorithm (MIA), Scaling Iteration Algorithm (SIA), High-Linking Algorithm (HLA), and so-called variational derivative iteration type algorithms such as Sobolev steepest descent and Newton's method (see also [**12**] and [**13**]). After introducing the SIA in some detail, the Optimal Scaling Iteration Algorithm (OSIA) is presented. A convergence result is proven. Numerical computations which reveal the advantages of the scheme are included.

The survey article [**13**] is another paper very much in the spirit of the conference. Via simple examples, it introduces the concept of and need for Sobolev gradient steepest descent methods. The methodology behind enforcing boundary conditions and using projections is explained, and connections between continuous and discrete methods are made. A motivating theme in the authors' presented and cited work is described in their quote (destined to become a classic): "Analytical difficulties and numerical difficulties always come in pairs." A tantalizing list of applications is presented, including Ginzburg-Landau, transonic flow, Monge-Ampere, and minimal surfaces. For each subtopic, problems are stated that range

from "exercises" to "open ended" and perhaps "impossible". Much of the material follows one of the author's book on the subject, definitely recommended reading; some of the applications are new. The applications are far from trivial and are physically significant. It is pointed out that the researcher who makes progress towards solving some of the open problems has the potential to make a real impact on mathematics and science.

The research article [**7**] gives a new semilinear elliptic result on all of $\mathbb{R}^N$. A brief introduction into resonant Schrödinger equations on bounded domains is given and many relevant references are provided; it is noted that resonant problems on the whole space for such equations are almost absent in the literature. The authors' new result is perhaps the first to deal with unbounded nonlinearities in this already novel setting.

The article [**3**] provides a historical survey citing many relevant references, new existence and multiplicity results, and numerical computations for quasilinear BVP. Specifically, the authors are interested in the p-Laplacian and the relationship between so-called eigenvalues of associated quasilinear-linear problems and the existence of solutions to quasilinear-nonlinear problems, much in the same way that semilinear researchers are concerned with linear problems. The provided background into the subject is quite complete and will be extremely useful to both new and experienced researchers in the area. The new results extend what is known concerning parameter regions and matters of existence and multiplicity. Numerical simulations provide complex yet effective visualization and confirmation of their results.

Two of the participants explain in [**8**] the issues understood and conjectured behind the so-called eigenvalue problem mentioned in the summary of the preceeding article. Specifically, they are interested in the variational characterization of eigenvalues of the quasilinear p-Laplacian problem. Enough background and known results are given so that interested new and experienced researchers may begin considering four significant open problems. For example, they ask if their variational characterization describes all possible eigenvalues.

The survey article [**14**] introduces the variational duality whereby nonlinear elliptic PDE can model underlying (and fascinating) geometric phenomena. The general concept is explained in depth, and specific examples are given. In particular, pattern formation is exploited to answer questions concerning triple-junction and minimal length / surface area solutions. In keeping with the theme of the conference, the article is far-reaching and contemplative in nature; seven open problems are loosely stated. The solution of any of these interesting problems would extend the author's successful related results. This article ties in with several other papers in this volume; see for example [**5**]. The historical section contains many references and provides an excellent starting point for those interested in considering open problems related to pattern formation in nonlinear elliptic PDE.

The research paper [**15**] further extends what is known about quasilinear problems. In particular, an existence theorem for periodic solutions of the perturbed p-Laplacian is given. Galerkin techniques are used, suggesting to this editor that a corresponding numerical algorithm might be obtained. After this technical new result is proven, three interesting open problems are presented. The problems are stated in sufficient detail that they might be immediately considered by the interested reader.

The survey paper [**11**] introduces an important concept in real-world applications, namely the application of variational optimization to solve ill-posed problems. Researchers consider such problems notoriously difficult when numerically seeking approximations; many applications in science lead to ill-posed systems that necessitate such efforts. After some historical background and numerical results are given, a detailed explanation of an application from electrical impedance tomography is presented. Several conjectures and open questions are stated, ready for the reader's immediate and serious consideration.

The organizer and editor's own article [**12**] is a complete survey of his efforts to use and develop numerical algorithms for investigating many of the problems found in the other articles in this volume. Specifically, he presents results from past and current experiments where variants of the MPA and Newton's method were used to explore semilinear elliptic BVP. Much time is spent describing the development, implementation, and application of the so-called GNGA: Gradient Newton Galerkin Algorithm. The experiments require a basis in function space. Often times this is a basis of eigenfunctions, a collection whose generation is a difficult and interesting process in its own right. Current and recent experiments are described, where general domains and symmetry are considered in the quest to explain the existence, multiplicity, and nodal structure of solutions via continuation methods. Numerous well-defined and less well-defined open problems are stated; some are suitable for undergraduate or master's level research, while others represent much longer term efforts to solve problems whose answers have long been sought by experienced researchers. The interplay between existence theory and analysis with numerical investigations was a theme of the conference and is a key component of this paper.

The research article [**2**] discusses critical nonlinearities and symmetry for semilinear elliptic PDE, chiefly on all of $\mathbb{R}^N$. New existence and nodal structure theorems are provided via variational methods. Six open problems are precisely stated, suitable for immediate consideration by the interested reader.

The research article [**10**] considers a class of semilinear elliptic PDE on all of $\mathbb{R}^N$. Specifically, ODE techniques are used to prove the existence of radial solutions when the nonlinearity is nonconvergent. The result is related to a previous result, and an open problem is suggested.

The final article [**9**] in this volume presents a new result considering traveling solitary wave solutions to several equations, namely the Burgers-KdV, combined dissipative double-dispersive, and sine-Gordon equations. Under some conditions, explicit solutions are given in these three instances. A brief statement of a general direction for future research along these lines is given. In particular, it is asked if variational techniques might be used to duplicate or extend the results found in this research article.

The organizer and editor wishes to thank the conference participants for their uniformly excellent talks, and thanks the majority of these participants for contributing survey and research articles. He also thanks the participants and contributors for their patience and understanding as regards his first attempt at managing a project of this magnitude. Any typographical errors or accidental omissions are the sole responsibility of this editor.

John M. Neuberger, Conference Organizer and Proceedings Editor

Bibliography

[1] Castro, Alfonso; *Semilinear Equations with Discrete Spectrum.*

[2] Catrina, Florin; *Critical Nonlinearities and Symmetric Solutions.*

[3] Čepička, Jan; Drábek, Pavel & Girg, Petr; *Quasilinear Boundary Value Problems: Existence and Multiplicity Results.*

[4] Chen, Goong; Deng, Yuanhua; Ni, Wei-Ming; & Zhou, Jianxin; *Semilinear Elliptic Boundary Value Problems with Nonlinear Oblique Boundary Conditions, a Boundary Element Monotone Iteration Approach.*

[5] Chen, Goong; Ding, Zhonghai; Hu, Chia-Ren; Ni, Wei-Ming & Zhou, Jianxin; *A Note on the Elliptic Sine-Gordon Equation.*

[6] Chen, Goong; Englert, Berthold G. & Zhou, Jianxin; *Convergence Analysis of an Optimal Scaling Algorithm for Semilinear Elliptic Boundary Value Problems.*

[7] Costa, David G. & Tehrani, Hossein; *Unbounded Perturbations of Resonant Schrodinger Equations.*

[8] Drábek, Pavel & Robinson, Stephen P.; *Eigenvalue Problems, Resonance Problems, and Open Problems.*

[9] Feng, Zhaosheng; *Traveling Wave Solutions to Nonlinear Evolution Equations.*

[10] Iaia, Joseph A.; *Non-convergent Radial Solutions of a Semilinear Elliptic Equation in* $\mathbb{R}^N$.

[11] Knowles, Ian; *Variational Methods for Ill-Posed Problems.*

[12] Neuberger, John M.; *GNGA: Recent Progress and Open Problems for Semilinear Elliptic PDE.*

[13] Neuberger, J. W. & Renka, R. J.; *Sobolev Gradients: Introduction, Applications, Problems.*

[14] Padilla, Pablo; *Variational, Dynamic and Geometric Aspects of Some Nonlinear Problems.*

[15] Shapiro, Victor L.; *The Perturbed p-Laplacian and Quadratic Growth.*

This volume contains the proceedings of the conference on Variational Methods: Open Problems, Recent Progress, and Numerical Algorithms. It presents current research in variational methods as applied to nonlinear elliptic PDE, although several articles concern nonlinear PDE that are nonvariational and/or nonelliptic. The book contains both survey and research papers discussing important open questions and offering suggestions on analytical and numerical techniques for solving those open problems.

It is suitable for graduate students and research mathematicians interested in elliptic partial differential equations.

ISBN 0-8218-3339-1

9 780821 833391

CONM/357