

CONTEMPORARY MATHEMATICS

361

Nonstandard Models of Arithmetic and Set Theory

AMS Special Session
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January 15–16, 2003
Baltimore, Maryland

Ali Enayat
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American Mathematical Society
Providence, Rhode Island

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2000 *Mathematics Subject Classification*. Primary 03C62, 03C20, 03H05, 03H15, 03D50, 26E30, 03C55, 03E25, 03E99, 03E35.

Library of Congress Cataloging-in-Publication Data

AMS Special Session Nonstandard Models of Arithmetic and Set Theory (2003 : Baltimore, Md.)
Nonstandard models of arithmetic and set theory : AMS Special Session Nonstandard Models of Arithmetic and Set Theory, January 15–16, 2003, Baltimore, Maryland / Ali Enayat, Roman Kossak, editors.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 361)

ISBN 0-8218-3535-1 (alk. paper)

1. Set theory. I. Enayat, Ali, 1959– II. Kossak, Roman, 1953– III. Title. IV. Contemporary mathematics (American Mathematical Society) ; v. 361.

QA248.A634 2003

511.3'22—dc22

2004051986

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10 9 8 7 6 5 4 3 2 1 09 08 07 06 05 04

Contents

Preface	ix
Non-Standard Models in a Broader Perspective H. GAIFMAN	1
Coding in $I\Delta_0$ P. D'AQUINO and J. KNIGHT	23
Automorphisms, Mahlo Cardinals, and NFU A. ENAYAT	37
AC Fails in the Natural Analogues of V and L that Model the Stratified Fragment of ZF T. FORSTER	61
Working with Nonstandard Models H. FRIEDMAN	71
Internally Iterated Ultrapowers K. HRBACEK	87
On Some Questions of Hrbacek and Di Nasso R. JIN	121
Turing Upper Bounds of Jump Ideals and Scott Sets A. MCALLISTER	129
Diversity in Substructures J. SCHMERL	145
Automorphisms of Countable Recursively Saturated Models of Set Theory A. TOGHA	163

Preface

We are pleased to present the proceedings of the AMS special session on non-standard models of arithmetic and set theory, held in Baltimore during January 15 and 16, 2003. The volume opens with a contribution from Haim Gaifman, whose work has strongly shaped the development of the model theory of both arithmetic and set theory. Gaifman's essay probes the concept of *non-standardness* in mathematics and provides a fascinating mix of historical and philosophical insights into the nature of nonstandard mathematical structures. In particular, Gaifman compares and contrasts the discovery of nonstandard models with other key mathematical innovations, such as the introduction of various number systems, the modern concept of function, and non-Euclidean geometries. Before introducing the other papers in this volume, we wish to identify, very briefly, some major developments in the study of nonstandard models of arithmetic and set theory that are relevant to the contents of this volume.

Although the discovery of nonstandard models dates back to Skolem's pioneering work during the 1920's and 1930's, their systematic investigation did not get under way until the 1950's, when several researchers, including Feferman, Rabin, Scott, and Tennenbaum initiated the analysis of the *complexity* of nonstandard models. In particular, Scott [Sc] characterized the families of subsets of natural numbers that can be coded within countable models of arithmetic (known nowadays as *standard systems*) as precisely those that, in the modern language of subsystems of second arithmetic [Si], give rise to a model of $RCA_0 + WKL$. Tennenbaum, on the other hand, blended Scott's coding technique with Kleene's effectively inseparable r.e. sets in order to demonstrate that no nonstandard model of PA can be recursively presented [BJ]. Another major trend in the study of models of arithmetic was initiated by MacDowell and Specker [MS] who refined Skolem's method to show that *every* model of PA has an elementary end extension. A notable landmark in this line of research is Gaifman's [G], which opened up the possibility for the systematic study and classification of isomorphism types of nonstandard models of PA , complete types of their elements (equivalently: ultrafilters on the algebra of definable subsets), the lattice of their elementary submodels, and morphisms amongst them. In particular, Gaifman introduced the powerful machinery of *minimal types*, which can be used to construct a variety of models of arithmetic, e.g., models of PA whose automorphism group is *isomorphic* to that of a prescribed linear order¹. In a different direction, Friedman [Fr] obtained a number of deep results on the structure of *countable* nonstandard models of various systems of arithmetic

¹The classical Ehrenfeucht-Mostowski method of indiscernibles only allows one to construct, given a linear order L , a model whose automorphism group *embeds* the automorphism group of L . It is worth noting that in a recent paper [Sch] Schmerl has extended Gaifman's theorem by

and set theory. One of the features of Friedman's paper is the striking blend of *external* back-and-forth and saturation arguments with *internal* considerations of definability, exemplified by his discovery that every countable nonstandard model of PA or ZF is isomorphic to a proper initial segment of itself. Another seminal paper in the study of models of arithmetic is Paris and Harrington's [PH], which utilized the elaborate model theoretic techniques of the study of cuts developed by Paris, Kirby, and others in order to establish the first non-Gödelian incompleteness theorems within Peano arithmetic. This work led to an avalanche of combinatorial independence results in various systems, ranging from arithmetic to set theory (e.g., Borel diagonalization theorems of Friedman). To summarize: we now know that the study of nonstandard models of arithmetic has deep, unexpected connections with computability theory, the theory of ultrafilters, finite combinatorics, lattice theory, and set theory.

The specific study of nonstandard models of systems of set theory also dates back to the 1950's, with the work of Rosser and Wang [RW]. The subject has developed in several interrelated directions, including what may be described as the *Keisler School* and the *Robinson School*. The Keisler School is guided by the Tarskian model theoretic paradigm and studies the behaviour of models of set theory from the point of view of classical model theory, with an eye towards applications to the study of extensions of first order logic. For example, the countable compactness and abstract completeness (recursive enumerability of the set of valid sentences) of the logic $L(Q)$, where Q is the quantifier "there exists uncountably many", can be proved using the Keisler-Morley analogue [KM] of the MacDowell-Specker theorem for ZF^2 . The Robinson School, on the other hand, is distinguished by its focus on the applications of nonstandard models to the various facets of the practice of *nonstandard analysis*. A distinguished feature of this approach is the use of sophisticated ultrapower constructions to build exotic models of set theory with various saturation properties in order to elucidate the relationship between nonstandard objects and Cantorian set theory³.

We now offer brief introductions to the papers presented in our volume.

- D'Aquino and Knight use Gödel-Rosser type sentences devised by Arana to give an interesting construction of an end chain of models of $I\Delta_0 + \Pi_2$ -consequences of PA such that Σ_1 -theories are strictly increasing and none is in the Scott set. Thus although for each n , the standard system of a nonstandard model M of PA contains its complete Σ_n -theory, this is no longer true for *fragments* of PA . Results in this direction were proved earlier by Adamowicz, Paris, and Wilkie.
- Forster shows that the axiom of choice fails in certain natural models that satisfy the *stratified* fragment of ZF (roughly speaking, a formula of set theory is stratified if it is obtained from a formula of type theory by erasing the type-superscripts). This is achieved by a new method of producing

showing that the result is true for arbitrary linearly ordered structures, rather than just linear orders.

²See [E] for a survey of analogues of the MacDowell-Specker theorem for set theory, and [H] for applications of the model theory of set theory to general model theory

³See Di Nasso's survey article [D] for a thorough discussion of the significant advances in this area

models of set theory using permutations, which is reminiscent of, but distinct from, the methods of Fraenkel-Mostowski, and Rieger-Bernays.

- Friedman provides three distinct applications of the use of nonstandard models in foundational work. Specifically, he uses a masterful blend of techniques to calibrate the consistency strength of certain statements in General Algebra, Descriptive Set Theory, and Boolean Relation Theory in terms of large cardinal hypotheses, employing proofs that make significant use of nonstandard models.
- Hrbacek develops a delicate adaptation of the Gaifman-Kunen method of iterated ultrapowers from the classical theory of measurable cardinals in order to construct a nonstandard model of set theory to show that Péraire's recently introduced system of nonstandard set theory is a conservative extension of ZFC .
- Jin's work also deals with foundations of nonstandard analysis, and answers certain open questions posed by Di Nasso and Hrbacek regarding combinatorial principles in nonstandard universes.
- McAllister presents a timely survey of the known results on the Turing upper bounds of arithmetic sets and functions by a number of researchers (including Ash, Jockush, Knight, Lerman, Lachlan, Marker, Simpson, Soare, and Solovay). He then extends some of these results to the case of arbitrary countable jump ideals and Scott sets, and closes his paper with a rich set of open problems.
- Schmerl provides a technique for constructing cofinal extensions with prescribed lattices of elementary submodels. He then gives new results concerning conditions under which certain finite lattices can be realized as substructure lattices of *pairwise nonisomorphic* elementary submodels. The results are further generalized to "equivalenced" substructure lattices with the isomorphism of substructures as the equivalence relation. Note that the construction of a model with a prescribed substructure lattice is quite nontrivial: it is still open whether there is a finite lattice that cannot be represented as a substructure lattice.
- The papers of Togha and Enayat both deal with automorphisms of models of ZFC set theory. Since no well-founded model of extensionality has a nontrivial automorphism, the models involved in these papers are all nonstandard. Togha's paper shows that a classical result of Smorynski, characterizing initial segments of countable recursively saturated models of PA that are the longest initial segment pointwise fixed by some automorphism, has an exact analogue for models of ZFC . Enayat's work, on the other hand, examines an intimate connection between Mahlo cardinals and automorphisms of models of set theory. He then uses these results to refine the work of Solovay on the calibration of the consistency strength of the extension $NFUA$ of the Quine-Jensen system NFU of set theory.

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Other articles in the book present results related to nonstandard models in arithmetic and set theory, including a survey of known results on the Turing upper bounds of arithmetic sets and functions. The volume is suitable for graduate students and research mathematicians interested in logic, especially model theory.

ISBN 0-8218-3535-1



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