CONTEMPORARY MATHEMATICS

436

Interactions between Homotopy Theory and Algebra

Summer School on Interactions between
Homotopy Theory and Algebra
University of Chicago
July 26–August 6, 2004
Chicago, Illinois

Luchezar L. Avramov
J. Daniel Christensen
William G. Dwyer
Michael A. Mandell
Brooke E. Shipley
Editors

American Mathematical Society

http://dx.doi.org/10.1090/conm/436

Interactions between Homotopy Theory and Algebra

CONTEMPORARY MATHEMATICS

436

Interactions between Homotopy Theory and Algebra

Summer School on Interactions between
Homotopy Theory and Algebra
University of Chicago
July 26–August 6, 2004
Chicago, Illinois

Luchezar L. Avramov
J. Daniel Christensen
William G. Dwyer
Michael A. Mandell
Brooke E. Shipley
Editors

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 55–XX, 13Dxx, 18Exx, 18Gxx; Secondary 13H10, 20J06, 14H52.

Library of Congress Cataloging-in-Publication Data

Summer School on Interactions between Homotopy Theory and Algebra (2004 : University of Chicago)

Interactions between homotopy theory and algebra : Summer School on Interactions between Homotopy Theory and Algebra, July 26–August 6, 2004, University of Chicago, Chicago, Illinois / Luchezar L. Avramov ... [et al.], editors.

p. cm. — (Contemporary mathematics ; ISSN 0271-4132 ; v. 436)

Includes bibliographical references.

ISBN 978-0-8218-3814-3 (alk. paper)

1. Homotopy theory—Congresses. 2. Commutative algebra—Congresses. I. Avramov, L. L. (Luchezar L.), 1948– II. Title.

QA612.7.S86 2004

514′.24—dc22 2007060682

♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 12 11 10 09 08 07

Contents

Preface

This book is based on the material covered in the *Summer School on Interactions between Homotopy Theory and Algebra* held at the University of Chicago, from July 26 to August 6, 2004.

The first week was dedicated to three lecture series that introduced general constructions and techniques that have been used in both fields:

- Paul Goerss (Northwestern University)
  Simplicial Methods and Model Categories
- Craig Huneke (University of Kansas)
  Local Cohomology
- Henning Krause (University of Paderborn, Germany)
  Differential Graded Methods and Derived Categories

Papers based on these lecture series form the first part of the book.

The second week included lecture series on selected topics in these fields:

- Alejandro Adem (University of British Columbia, Canada)
  Group cohomology
- Ragnar-Olaf Buchweitz (University of Toronto, Canada)
  What Arrows can Commutative Algebra pull from Quivers?
- John Greenlees (University of Sheffield, England)
  First steps in brave new commutative algebra
- Kathryn Hess (EPFL, Switzerland)
  Rational homotopy theory
- Mark Hovey (Wesleyan University, Connecticut)
  Cotorsion Theories
- Srikanth Iyengar (University of Nebraska, Lincoln)
  André-Quillen homology of commutative algebras
- Jack Morava (Johns Hopkins University)
  Quillen's setup for cobordism

The second part of this book includes accounts based on all of the above lectures except those of Buchweitz and Morava.

It is impossible in a two-week-long meeting to touch upon many aspects of the complex and intense flow of ideas between algebra and homotopy theory. Two papers in the second part of the book are based on series of lectures delivered at other recent events:

- Igor Burban (University of Mainz, Germany)
  Coherent sheaves on an elliptic curve
- John Greenlees (University of Sheffield, England)
  Spectra for commutative algebraists

Burban lectured at the summer school *Derived categories in representation theory* held at Tsinghua University in Beijing in August 2005. Greenlees gave his talks in February, 2003, during the 2002-2003 MSRI program *Commutative algebra.* Finally, we have added a short account that ties together topics found in the articles by Huneke, Greenlees, and Adem:

- Bill Dwyer (University of Notre Dame)
  Local cohomology in commutative algebra,
  homotopy theory and group cohomology

The goal of this book is to create a resource where one can learn about some deep connections between homotopy theory and various algebraic fields, such as commutative algebra, representation theory, and algebraic geometry, and find background for these directions of current research. The lectures at the summer school were presented to an audience consisting of beginning researchers with varied mathematical backgrounds, and lecturers were asked to deliver their talks with novices in mind. This emphasis is preserved, and often enhanced, in the written accounts collected here.

* * *

The three expositions that make up the first part of this book provide introductions to topics that are crucial in the connections between homotopy theory and various algebraic fields. They are intended for non-experts in the specific topic.

Goerss introduces Quillen model categories as a tool for understanding resolutions. Goerss then explains how Quillen used this tool in his foundational work on the homology of commutative algebras, now called André-Quillen homology. More recent work on the use of André-Quillen homology to characterize regular homomorphisms and locally complete intersection homomorphisms appears in Iyengar's article. Another of Quillen's original applications of model category technology was to rational homotopy theory; an introduction to this subject appears in Hess' article. Model categories have also been an important ingredient in Morel and Voevodsky's motivic homotopy theory which applies homotopy theory to algebraic geometry [**12, 13, 14**].

Huneke's *Lectures on local cohomology* is an introduction to a topic in commutative algebra that is both well-established and is still a focus for contemporary research. The vanishing and the structure of local cohomology modules characterize two main classes of commutative rings—Gorenstein rings and Cohen-Macaulay rings. (The other two fundamental classes, regular rings and complete intersection rings, are characterized by the vanishing of André-Quillen homology, discussed in the papers of Goerss and of Iyengar.) Huneke's starts by constructing the formal machinery of local cohomology, then goes on to "discover" the classes of Gorenstein rings and Cohen–Macaulay rings. Such an approach may be habitual to algebraic topologists and unusual for algebraists; a fact that by itself should provide each group with a topic for meditation. On a mathematical level, the notes are aimed at providing a complete proof of an important theorem on connectedness of intersections of projective varieties, due to Fulton and Hansen. For systematic expositions of local cohomology, the reader can consult both Grothendieck's foundational text and the modern treatment [**8**]. For a spectacular recent application, (s)he should read Huneke and Lyubeznik's short paper [**7**].

The article *Derived categories, resolutions, and Brown representability* by Krause investigates the differential graded resolutions and derived categories that are precursors to the simplicial resolutions and homotopy categories developed for doing homotopy theory in non-abelian categories as discussed in Goerss' article. The early sections of this paper discuss triangulated categories and Brown representability, which provide a framework and important tools for the kind of homological algebra that arises in representation theory, algebraic geometry, and stable homotopy theory. The later sections give a characterization of algebraic triangulated categories in terms of the derived categories of differential graded algebras or dg-categories. These topics are closely associated to Keller's generalization of Rickard's work on Morita theory [**9**]. For an expository discussion of the extension of Morita equivalences to the ring spectra from stable homotopy theory see [**16**]. For other current related work on dg-categories, see the survey [**10**].

* * *

The second part of this book includes some survey articles and some more specialized accounts based on current research. The order of the articles attempts to reflect some connections between the topics. The fact that it is somewhat arbitrary reflects the multiple connections.

Greenlees' article *Spectra for commutative algebraists* models the development of the category of spectra in stable homotopy theory on the derived category of a ring. The study of spectra, with a suitable analogue of a tensor product, is sometimes referred to as brave new algebra or spectral algebra. Analogues of many constructions from algebra have been studied in this new context: topological André-Quillen homology [**3, 15**], topological Hochschild homology [**11, 4, 17**] and topological modular forms [**5, 6**]. Applications of these ideas include many of the papers written in stable homotopy theory in the past five years. See also Greenlees' second article, *First steps in brave new commutative algebra.*

In her article *Rational homotopy theory: A brief introduction*, Hess surveys an area that has seen a sustained and very successful exchange of ideas between homotopy theory and commutative algebra. Differential graded algebras were first used in algebraic topology to compute cohomology rings. When Tate introduced them into commutative algebra in 1957 he specifically mentioned that origin. In the mid 1970s, after Quillen and Sullivan constructed commutative DG algebra models for the rational singular cochains, the latter reintroduced Tate's technique back into topology. The similarity of the technical machinery used to study rational homotopy invariants of topological spaces and homological invariants of local rings, in arbitrary characteristic, has led to significant advances on both sides of the ledger. Examples include proofs of an elliptic/hyperbolic dichotomy for Betti numbers of loop spaces of finite CW complexes and of residue fields of local rings, and solutions of long standing problems of Ganea on Lusternik-Shnirelmann category and of Quillen on André-Quillen homology. Hess provides a fast-paced, very readable introduction to these developments, starting with an overview of the construction of Sullivan models.

André-Quillen homology of commutative algebras is the title of a survey by Iyengar, where the goal is to demonstrate how the theory is applied to specific problems of commutative algebra. As in Goerss' article, the theory is constructed from scratch, but with changes in emphasis. For one, efforts are made to keep the size of simplicial resolutions under control, with an eye towards extending to simplicial algebras the noetherian character of finitely generated algebras over noetherian rings. Another aspect is an examination of exact sequences in low dimensions, which recover and extend fundamental properties of modules of Kähler differentials. The payoff comes as a characterization, in terms of the vanishing of André-Quillen homology functors, of homomorphisms of finite type that are regular or locally complete intersections; this is applied to prove a generalized form of the Hochschild-Kostant-Rosenberg theorem on Hochschild homology. The topics are chosen so as to highlight the interplay of homotopy theory and commutative ring theory. The hypothesis of finite type can be removed, but new ideas are required; the interested reader should consult [**1**] and [**2**].

Local cohomology in commutative algebra, homotopy theory, and group cohomology by Dwyer ties together the notions of local cohomology appearing in Huneke's article and Greenlees' article *First steps in brave new commutative algebra.* Dwyer also compares the notions of Gorenstein rings appearing in these two articles. Finally, Dwyer explains some of the connections between local cohomology, Gorenstein rings, and group cohomology.

Greenlees' *First steps in brave new commutative algebra* considers constructions from commutative algebra extended to the context of stable homotopy theory. Specifically, he considers localization and completion first with respect to ideals in the coefficient ring of a ring spectrum and then entirely in the derived category. The goal of the first part of this paper is an exposition of the connections between local cohomology and equivariant cohomology theory. In the second part, Greenlees explains how completion and localization arise from Morita adjunctions and develops the notion of Gorenstein ring spectra.

The title of Hovey's paper *Cotorsion pairs and model categories* links two areas of research that, until recently, had little (if anything) in common. The unusual angle here is that model categories are used to study categories of modules directly, rather than through the intermediary of their derived categories. For algebraists with a classical tilt towards abelian categories this may be the most inviting introduction to the language, methods, and use of model categories. The mechanics of the interaction are emphasized by describing instances where parallel notions and constructions appeared independently in completely different setups and for completely different reasons, and by explaining statements of results and structures of arguments that are naturally suggested once a link is established.

Several of the papers presented above gravitate around interactions between algebra and topology that take place through various triangulated categories, such as categories of spectra, derived categories of modules, or derived categories of differential graded modules. Similar categorical constructions have linked recent developments in algebraic geometry and mirror symmetry to representation theory of algebras. Buchweitz's lectures at the workshop followed some of those links, but have not been written up for these proceedings. The article *Coherent sheaves on an elliptic curve* by Brüning and Burban provides a glimpse into that rapidly developing area of research. The paper contains a modern account of the classical description of indecomposable sheaves on an elliptic curve, due to Atiyah. Along the way, the authors present several fundamental algebro-geometric constructions and discuss them from non-traditional points of view.

Group cohomology is a prime example of a subject in which homotopy theory and algebra come together; in fact, one of the reasons for inventing it in the first place was a desire to give an algebraic interpretation of an intrinsically topological object (the cohomology ring of the classifying space of a group). Since then, the connections have developed in many directions. For instance,for a finite group G, Quillen related the algebraic geometry of the ring $R = H^*(BG)$ to the subgroup structure of the group G, and then later on Alperin, Carlson, Benson and others found ways to study G-modules by assigning to them subvarieties of the prime ideal spectrum of R. Benson and Carlson have studied various other commutative algebraic properties of this ring R, and this has developed, in the hands of Greenlees, and others, into an investigation of the cochain algebra of the classifying space with the tools of stable homotopy theory. Local cohomology and its homotopy theoretic versions play a large role in this (see Greenlees' second article).

Adem's *Lectures on the Cohomology of Finite Groups* takes a very elementary approach. He defines the cohomology ring of a finite group; goes in some detail into methods for computing these rings; gives an introduction to questions about depth, dimension, and duality properties; and ends up by sketching how some of these techniques can be applied to study a geometric question: which finite groups can act freely on various products of spheres?

* * *

Acknowledgments. We would like to thank the National Science Foundation for financial support for the summer school and these proceedings as part of Shipley's Grant No. 0134938. The initial idea for this summer school occurred while Luchezar Avramov and Brooke Shipley were both in the department of mathematics at Purdue University, which has a long tradition of supporting interactions between homotopy theory and commutative algebra. Finally, we would like to thank Judy Garza at the University of Chicago for wonderful administrative support during the summer school.

Bibliography

[1] M. André, *Localization de la lissité formelle*, Manuscripta Math. **13** (1974), 297–307.

[2] L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*, Ann. of Math. (2) **150** (1999), 455–487.

[3] M. Basterra, *André-Quillen cohomology of commutative S-algebras*, J. Pure Appl. Algebra **144** (1999), no. 2, 111–143.

[4] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, **47**, American Mathematical Society, Providence, RI, 1997, xii+249 pp.

[5] M. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 554–565, Birkhäuser, Basel, 1995.

[6] M. Hopkins, *Algebraic topology and modular forms*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 291–317, Higher Ed. Press, Beijing, 2002.

[7] C. Huneke and G. Lyubeznik, Absolute integral closure in positive characteristic, Adv. Math. **210** (2007), 498–504.

[8] S. Iyengar, A.O. Leykin, G. Leuschke, C. Miller, E. Miller, A. Singh, and U. Walther, *Twenty four hours of local cohomology*, Graduate Studies in Math., Amer. Math. Soc., Providence, RI, (to appear).

[9] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[10] B. Keller, *On differential graded categories*, Proceedings of the ICM, (Madrid, 2006), Volume II, EMS Publishing House (Zurich, 2006), 151-190.

[11] I. Madsen, *Algebraic K-theory and traces*, Current developments in mathematics, 1995 (Cambridge, MA), 191–321, Int. Press, Cambridge, MA, 1994.

[12] F. Morel and V. Voevodsky, *A^1-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999) 45–143.

[13] F. Morel, *Théorie homotopique des schémas*, Astrisque **256** (1999), vi+119 pp.

[14] F. Morel, *Homotopy theory of schemes*, Translated from the 1999 French original by James D. Lewis. SMF/AMS Texts and Monographs, 12. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2006 viii+104 pp.

[15] A. Robinson and S. Whitehouse, *Operads and* Γ*-homology of commutative rings*, Math. Proc. Cambridge Philos. Soc. **132** (2002), no. 2, 197–234.

[16] S. Schwede, *Morita theory in abelian, derived and stable model categories*, Structured Ring Spectra, London Mathematical Society Lecture Notes **315**, Cambridge Univ. Press, Cambridge, 2004, 33–86.

[17] B. Shipley, *Symmetric spectra and topological Hochschild homology*, K-theory, **19** (2) (2000), 155–183.

List of Participants

Omid Amini
Ecole Polytechnique

Carl Fredrik Berg
NTNU, Trondheim, Norway

Julie Bergner
University of Notre Dame

Georg Biedermann
Universität Bonn

Andrew Blumberg
University of Chicago

Kristian Bruening
Universität Paderborn

Robert Bruner
Wayne State University

Joseph Campbell
Purdue University

Sunil Chebolu
University of Washington

Boris Chorny
University of Western Ontario

Lars Winther Christensen
University of Copenhagen

Charlie Crosby
Purdue University

Goran Dubajic
University of Manchester

Christopher Dwyer
University of Wisconsin-Madison

Anthony D. Elmendorf
Purdue University Calumet

Christopher French
Grinnell College

Jelena Grbic
University of Aberdeen

Bertrand Guillou
University of Chicago

Ananthnarayan Hariharan
University of Kansas

John Harper
University of Notre Dame

Andreas Heider
Universität Bonn

Bernt Tore Jensen
Leeds University

Inga Johnson
University of Rochester

Niles Johnson
University of Chicago

Valentina Joukhovitski
Northwestern University

Vsevolod (Seva) Joukhovitski
UCLA / IAS

Manoj Kummini
University of Kansas

Matthew LaPadula
Lehigh University

Jue Le
University of Paderborn

Johann Leida
University of Wisconsin-Madison

Kathryn Lesh
Union College

Ronald Mitchell
Purdue University

George Ciprian Modoi
Babes Bolyai University

Fernando Muro
Universidad de Sevilla

Justin Noel
University of Chicago

Simona Paoli
SUNY at Buffalo

Brian Pasko
Kansas State University

Paul Pearson
Northwestern University

Mark Pearson
Hope College

Constanze Roitzheim
Universität Bonn

Steffen Sagave
Universität Bonn

Sean Sather-Wagstaff
University of Illinois at
Urbana-Champaign

Kristen Schemmerhorn
Northwestern University

Karsten Schmidt
Universität Paderborn

Liana Sega
Michigan State University

Markus Severitt
University of Bielefeld

Shoham Shamir
Hebrew University

Julia Singer
Universität Bonn

Jaka Smrekar
University of Ljubljana

Bart Snapp
University of Illinois at
Urbana-Champaign

Dumitru Ioan Stamate
University of Bucharest

Janet Striuli
University of Kansas

Goncalo Tabuada
Institut Mathematiques Jussieu

Akaki Tikaradze
University of Chicago

Enrique Torres Giese
University of Wisconsin, Madison

Mark Ullmann
University of Bielefeld

Bruno Vallette
IRMA (Strasbourg)

Oana Veliche
Purdue University

Arne Weiner
University of Bielefeld

Diana White
University of Illinois at
Urbana-Champaign

Wenliang Zhang
University of Minnesota

Titles in This Series

436 **Luchezar L. Avramov, J. Daniel Christensen, William G. Dwyer, Michael A. Mandell, and Brooke E. Shipley, Editors,** Interactions between homotopy theory and algebra, 2007

435 **Krzysztof Jarosz, Editor,** Function spaces, 2007

434 **S. Paycha and B. Uribe, Editors,** Geometric and topological methods for quantum field theory, 2007

432 **Dick Canery, Jane Gilman, Juha Heinoren, and Howard Masur, Editors,** In the tradition of Ahlfors-Bers, IV, 2007

431 **Michael Batanin, Alexei Davydov, Michael Johnson, Stephen Lack, and Amnon Neeman, Editors,** Categories in algebra, geometry and mathematical physics, 2007

430 **Idris Assani, Editor,** Ergodic theory and related fields, 2007

429 **Gui-Qiang Chen, Elton Hsu, and Mark Pinsky, Editors,** Stochastic analysis and partial differential equations, 2007

428 **Estela A. Gavosto, Marianne K. Korten, Charles N. Moore, and Rodolfo H. Torres, Editors,** Harmonic analysis, partial differential equations, and related topics, 2007

427 **Anastasios Mallios and Marina Haralampidou, Editors,** Topological algebras and applications, 2007

426 **Fabio Ancona, Irena Lasiecka, Walter Littman, and Roberto Triggiani, Editors,** Control methods in PDE-dynamical systems, 2007

425 **Su Gao, Steve Jackson, and Yi Zhang, Editors,** Advances in Logic, 2007

424 **V. I. Burenko, T. Iwaniec, and S. K. Vodopyanov, Editors,** Analysis and geometry in their interaction, 2007

423 **Christos A. Athanasiadis, Victor V. Batyrev, Dimitrios I. Dais, Martin Henk, and Francisco Santos, Editors,** Algebraic and geometric combinatorics, 2007

422 **JongHae Keum and Shigeyuki Kondō, Editors,** Algebraic geometry, 2007

421 **Benjamin Fine, Anthony M. Gaglione, and Dennis Spellman, Editors,** Combinatorial group theory, discrete groups, and number theory, 2007

420 **William Chin, James Osterburg, and Declan Quinn, Editors,** Groups, rings and algebras, 2006

419 **Dinh V. Huynh, S. K. Jain, and S. R. López-Permouth, Editors,** Algebra and Its applications, 2006

418 **Lothar Gerritzen, Dorian Goldfeld, Martin Kreuzer, Gerhard Rosenberger, and Vladimir Shpilrain, Editors,** Algebraic methods in cryptography, 2006

417 **Vadim B. Kuznetsov and Siddhartha Sahi, Editors,** Jack, Hall-Littlewood and Macdonald polynomials, 2006

416 **Toshitake Kohno and Masanori Morishita, Editors,** Primes and knots, 2006

415 **Gregory Berkolaiko, Robert Carlson, Stephen A. Fulling, and Peter Kuchment, Editors,** Quantum graphs and their applications, 2006

414 **Deguang Han, Palle E. T. Jorgensen, and David Royal Larson, Editors,** Operator theory, operator algebras, and applications, 2006

413 **Georgia M. Benkart, Jens C. Jantzen, Zongzhu Lin, Daniel K. Nakano, and Brian J. Parshall, Editors,** Representations of algebraic groups, quantum groups and Lie algebras, 2006

412 **Nikolai Chernov, Yulia Karpeshina, Ian W. Knowles, Roger T. Lewis, and Rudi Weikard, Editors,** Recent advances in differential equations and mathematical physics, 2006

411 **J. Marshall Ash and Roger L. Jones, Editors,** Harmonic analysis: Calderón-Zygmund and beyond, 2006

This book is based on talks presented at the Summer School on Interactions between Homotopy Theory and Algebra held at the University of Chicago in the summer of 2004. The goal of this book is to create a resource for background and for current directions of research related to deep connections between homotopy theory and algebra, including algebraic geometry, commutative algebra, and representation theory. The articles in this book are aimed at the audience of beginning researchers with varied mathematical backgrounds and have been written with both the quality of exposition and the accessibility to novices in mind.

ISBN 978-0-8218-3814-3

9 780821 838143

CONM/436