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441

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on Hopf Algebras at the Crossroads
of Algebra, Category Theory, and Topology
October 23–24, 2004
Evanston, Illinois

Louis H. Kauffman
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# Preface

Hopf algebras and bialgebras arise in many different mathematical contexts as fundamental structures. The theory of Hopf algebras is rather rich at this point and has applications to many areas. This volume concerns Hopf algebras and their generalizations.

Hopf algebras first arose in algebraic topology in the study of cohomology rings of Lie groups, later generalized for H-spaces. The Hopf algebras of algebraic topology are graded objects. Hopf algebras which are not graded include a very wide range of examples. The first to be considered were group algebras, universal enveloping algebras of Lie algebras, and affine Hopf algebras. The latter, with their morphisms, constitute a category which is in anti-equivalence with the category of affine algebraic groups. More generally, there is an anti-equivalence between the category of finitely generated commutative Hopf algebras and the category of affine group schemes (group objects in the category of schemes). These non-graded examples of Hopf algebras are either commutative or cocommutative.

Hopf algebras have a rich representation theory. One reason is that a Hopf algebra $H$ affords a natural module structure on the tensor product of $H$-modules. Finite-dimensional Hopf algebras have a one-dimensional ideal of *integrals*, which plays a key role in their representation theory. The complexity of the structure of a Hopf algebra accounts for many possible actions and coactions.

In the 1980s, connections were discovered between some aspects of Hopf algebras, topology, and theoretical physics which developed into what might be called "quantum mathematics". Included are the subjects of quantum groups, quantum topology, noncommutative geometry, and certain categories with additional structure. The Hopf algebras found in these subjects are, for the most part, neither commutative nor cocommutative.

*Quantum groups* constitute a very rich family of Hopf algebras. They arise in a number of ways. For example: Through *quantum R-matrices*, that is, solutions to the quantum Yang-Baxter equation; via *quantization*, that is, *deformation* of enveloping algebras of some Lie algebras and Kac-Moody algebras; deformations of the Hopf algebra of regular functions on an algebraic group (a group variety) are seen as descriptions of quantizations of that algebraic group; as Hopf algebras of endomorphisms of some quadratic Koszul algebras called *quantum linear spaces*; and by means of some $C^*$-algebras in non-commutative geometry, being originally called *compact matrix pseudogroups*.

Quantum groups have very important applications in the topology of manifolds where they account for algebraic coefficients for the so-called *quantum invariants*, which are studied by *quantum topology*. In fact, quantum groups provided an interdisciplinary approach to topology of manifolds that started from applications of methods from theoretical physics to topological objects and, gradually, abstracted the methods and initial results. Some of the paths towards quantum groups mentioned above originated in statistical mechanics and quantum inverse scattering. As quantum groups generalize some notions of symmetries, they have very important applications to some approaches in the realm of *quantum gravity*, particularly to loop quantum gravity, and to *quantum field theory*, especially in conformal field theory and topological quantum field theory. It is worthwhile mentioning that there is an on-going attempt to formalize renormalization that involves Hopf algebras. Coalgebras have also played a meaningful role in theoretical computer science.

As it is typical in quantum mathematics, the study of quantum groups and, more generally, Hopf algebras incorporated the study of whole categories of such objects. With this perspective, the categories of modules of an algebra have some additional structure precisely when that algebra has some additional operations or special elements. For example, those categories inherit the tensor-product structure of the underlying vector spaces when that algebra is a bialgebra, becoming what is called a *monoidal* or *tensor* category; they acquire a notion of *duality* for the finite-dimensional modules when that algebra is a Hopf algebra, becoming an *autonomous* category when the algebra is finite-dimensional; a *braiding* when that algebra is a *quasitriangular bialgebra*, which is endowed with an R-matrix; and so forth, passing through *tortile* (*ribbon*), *semisimple* and *modular* categories. On the one hand, these categories with additional structure include the categories of representations of some quantum groups and, on the other hand, they serve to produce link and 3-manifold invariants and can even provide categorical descriptions of some topological objects. Thus, they explain the use of quantum groups in quantum topology and, in part, evolved from that use. Categories of representations of quantum groups also revived the subject of Tannaka-Krein duality theory.

Graphical calculi appear in many areas of mathematics and science, allowing relatively short proofs and computations. Some freely generated categories with additional structure are diagrammatic, providing graphical calculi for those categories and their applications. In the early 1990s, graphical calculi specific for visualizing Hopf modules and Hopf algebras in symmetric and, more generally, braided monoidal categories emerged. They led to some universal Hopf-algebra objects of diagrammatic nature, and to theorems on Hopf-algebra objects that are defined in rather general settings. Such techniques inspired further topological invariants and even research programmes that have tried (with partial success) to algebraize 3-manifolds as certain morphisms related to suitable Hopf-algebra objects.

A number of generalizations of Hopf algebras (besides those already mentioned) and related objects are known. Their motivations are of varied nature, including algebraic, categorical, geometrical, topological and physical-mathematical. Well-known examples of these generalizations (some of which are connected to some of the others) are: The *quasi-Hopf algebras*, whose coassociativity is relaxed, and which were created to relate the monodromy of the Knizhnik-Zamolodchikov equations to quantum groups via braid groups; the *weak Hopf algebras*, whose unit and counit satisfy weaker axioms, and which occur in the study of operator algebras; various notions of *Hopf algebroids*, *quantum grupoids* and *Hopf monads*; *dynamical quantum grupoids*, which are associated with the quantum dynamical Yang-Baxter equation; *Hopfish algebras*, which appear in Poisson geometry and are defined in terms of morphims that are bimodules "between" algebras; and *Hopf operads*, which relate to algebraic topology. Some of these algebraic objects have nice interpretations and/or results at the level of category theory.

This volume was motivated by the *Special Session on Hopf Algebras at the Crossroads of Algebra, Category Theory, and Topology* of the 2004 Fall Central Section Meeting of the American Mathematical Society (AMS Meeting # 1001), which took place in Evanston, IL on October 23–24, 2004. This special session was organized by the editors of this volume, and had a focus on the current interdisciplinary nature of research on Hopf algebras and their many applications. Researchers working in Canada, France, Japan, Mexico and, of course, the USA participated in that special session.

As the editors identified the speakers interested in a proceedings volume, and invited a few selected authors to contribute to it as well, the volume's emphasis on Hopf algebras and their generalizations increased, and reflected the variety of approaches to their study. This volume includes contributions by researchers working in Australia, Germany, Japan, Romania, Sri Lanka and the USA. Each article in this volume was refereed. The number of referees was two per paper for all but two of the articles, for which there were one and three referees, respectively.

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Louis H. Kauffman and David E. Radford (University of Illinois at Chicago),
Fernando J. O. Souza (Universidade Federal de Pernambuco),
April, 2007.

Hopf algebras have proved to be very interesting structures with deep connections to various areas of mathematics, particularly through quantum groups. Indeed, the study of Hopf algebras, their representations, their generalizations, and the categories related to all these objects has an interdisciplinary nature. It finds methods, relationships, motivations and applications throughout algebra, category theory, topology, geometry, quantum field theory, quantum gravity, and also combinatorics, logic, and theoretical computer science.

This volume portrays the vitality of contemporary research in Hopf algebras. Altogether, the articles in the volume explore essential aspects of Hopf algebras and some of their best-known generalizations by means of a variety of approaches and perspectives. They make use of quite different techniques that are already consolidated in the area of quantum algebra. This volume demonstrates the diversity and richness of its subject. Most of its papers introduce the reader to their respective contexts and structures through very expository preliminary sections.

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