CONTEMPORARY MATHEMATICS

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Beyond Topology

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Preface

Often in mathematics, the context in which a problem is first studied is not the most adapted to the problem at hand. For instance, difficult questions in the context of real numbers become easy to tackle in the realm of complex numbers, problems of planar geometry requiring intricate arguments become trivial when considered as projections of a three dimensional situation, etc. Topological questions are no exception, even though this fact is not widely recognized yet.

After various attempts at axiomatizing the notions of *nearness* and *convergence* in the early 20th century, the concept of topology introduced by Felix Hausdorff in 1914 was relatively quickly accepted as *the* answer to the problem of finding solid foundations for analysis and geometry. There are, of course, reasons why topology has been widely accepted as the standard structure to describe nearness, convergence and continuity. Not the least of them is the fact that topologies can be introduced in so many equivalent ways: system of open sets, of closed sets, of neighborhoods at each point, closure operator or interior operator, in terms of covers, of convergent filters, to name a few.

However, working with topological spaces has its shortcomings, many of which will be presented in this volume, together with various approaches to remedy them. Let me just mention two examples that the reader will repeatedly encounter in this book. Firstly, while a quotient set can be canonically endowed with a quotient topology, this operation does not yield very satisfactory results. To be more specific, consider an equivalence relation $\sim$ on a topological space X and denote by $q\colon X \to X/{\sim}$ the map associating to each element of X its equivalence class: The quotient topology on $X/{\sim}$ is the finest topology on $X/{\sim}$ that makes q continuous. This construction is not hereditary in the following sense: if B is a subset of the quotient $X/{\sim}$ the induced topology by $X/{\sim}$ does not necessarily coincide with the quotient topology induced by $q|_{q^{-1}(B)}\colon q^{-1}(B) \to B$. A second fundamental problem is the lack of a canonical function space topology that would yield as nice a duality as the usual algebraic duality. If X and Y and Z are sets and Z^X denotes the set of all functions from X to Z, the sets of functions are well-behaved in the sense that

$$Z^{X\times Y} = \left(Z^X\right)^Y,$$

where the equality represents the bijection $f \mapsto {}^tf$ where ${}^tf(y)(x) = f(x,y)$. But if X, Y and Z are topological spaces and $C(X,Z)$ denotes the set of continuous functions from X to Z, there is in general no topology on $C(X,Z)$ that ensures that $f\colon X \times Y \to Z$ is continuous if and only if the companion map ${}^tf\colon Y \to C(X,Z)$ of f is continuous.

This situation can be viewed in two ways: either you consider that the class of topological spaces is too large and leaves room for too much pathology, in which case you will try to remedy the above problems by finding a subclass of topological spaces behaving better, or you realize that the class of topological spaces is too small to perform certain operations in a natural way, just like the field of real numbers is too small to factor any polynomial into linear factors. The former approach led among others to the theory of k-spaces, which became reasonably popular, notably in homotopy theory, even though this solution suffers from obvious problems, like the necessity of using a product that is different from the usual topological product.

The present book is about the latter, less widely known, approach. It presents to a reader with only a basic knowledge of point-set topology various generalizations of topologies, each addressing one or several particular shortcomings of topologies. Written with the graduate student in mind, this volume should also be an eye-opener for the working mathematician, the day-to-day user of topology: there is sometimes much to gain in looking *beyond topology*.

Any reference to category theory has been carefully avoided so far, because the present volume is not a book on category theory. However, the book focuses on several related structures on a set and the natural way to describe structured sets and their relationships is in the language of category theory. Roughly speaking and restricting ourselves to the present context, a category (of structured sets) is composed of objects that are sets with a structure (think of groups, rings, vector spaces, topological spaces, topological vector spaces, etc.) and morphisms between these objects, usually maps preserving the structure in some sense (for the previous examples: group homomorphisms, ring homomorphisms, linear maps, continuous maps, continuous linear maps, etc.). The categories to be considered in this book all contain the category Top of topological spaces and continuous maps. The language of category will be necessary to describe how Top sits in the category at hand, how these categories relate to each other, and some qualities that Top lacks but are enjoyed by these categories. Therefore, an introductory chapter on categorical topology presents the necessary categorical background. This chapter can be seen as an appendix to refer to when running into an unknown categorical notions while reading another chapter. Taking this into account, each chapter is self contained and can be read independently of the others, despite ocasional overlaps. Each one of them should be seen as an introduction to a field, and a guide for the interested reader who wants to go further.

Finally, I wish to express my deep appreciation to all of the authors who contributed to this book, and to my co-editor Elliott Pearl whose tremendous work and technical skills made it possible to finish this book.

Frédéric Mynard
August 2008

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Titles in This Series

The purpose of this collection is to guide the non-specialist through the basic theory of various generalizations of topology, starting with clear motivations for their introduction. Structures considered include closure spaces, convergence spaces, proximity spaces, quasi-uniform spaces, merotopic spaces, nearness and filter spaces, semi-uniform convergence spaces, and approach spaces. Each chapter is self-contained and accessible to the graduate student, and focuses on motivations to introduce the generalization of topologies considered, presenting examples where desirable properties are not present in the realm of topologies and the problem is remedied in the more general context. Then, enough material will be covered to prepare the reader for more advanced papers on the topic. While category theory is not the focus of the book, it is a convenient language to study these structures and, while kept as a tool rather than an object of study, will be used throughout the book. For this reason, the book contains an introductory chapter on categorical topology.

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