CONTEMPORARY MATHEMATICS

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Tropical and Idempotent Mathematics

International Workshop TROPICAL-07
Tropical and Idempotent Mathematics
August 25–30, 2007
Independent University of Moscow
and Laboratory J.-V. Poncelet

G. L. Litvinov
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Preface: The Maslov Dequantization and Tropical Mathematics

Tropical and idempotent mathematics is a popular and rapidly developing area of modern mathematics. In a nutshell, it investigates properties of the mathematical structures, which arise when one replaces the usual $+, \times$ arithmetics of numerical fields by a new set of basic operations, with a new idempotent addition $\oplus$. Idempotency means that $a \oplus a = a$. Taking maxima or minima of real numbers yields a typical and important example, which is called max-plus algebra or tropical algebra or $\mathbf{R}_{\max}$, see below.

The main objective of the workshop and this volume is to enhance collaboration between groups of people working on tropical/idempotent mathematics. In this connection, note that the terminology in the papers of the volume may depend on the team.

In a sense, the traditional mathematics over numerical fields can be regarded as a "quantum" theory, while the tropical/idempotent mathematics can be treated as its "classical shadow." The corresponding procedure of "dequantization" is called the *Maslov dequantization*. Tropical mathematics can be treated as a result of the Maslov dequantization applied to the traditional mathematics over numerical fields; in this case the dequantization parameter can be thought of as the Planck constant that takes imaginary values. For example, modern tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to the traditional algebraic geometry (O.Viro, G. Mikhalkin, see, e.g., [**16**, **18**, **19**]).

An important stage of development of the subject was presented in the volume "Idempotent Mathematics and Mathematical Physics"/G.L. Litvinov and V.P. Maslov, eds., Contemp. Math., vol. 377, 2005. To take a new snapshot of the modern tropical and idempotent mathematics, we organized a workshop "Tropical-07" hosted by the Independent University of Moscow and its French-Russian Laboratory "J.-V. Poncelet" in Moscow, Russia, in August 2007. The present volume provides an extended record of this meeting along with a number of invited contributions. Idempotent analysis and tropical geometry are ground stones for the volume. Contributions to idempotent analysis are focused on the Hamilton–Jacobi semigroup, applications of the finite element method, investigations of tropical semirings consisting of plurisubharmonic functions and their germs; a tropical version of the Schauder fixed point theorem is presented. Applications to statistical mechanics and explanation of economical crises are examined in a paper by V.P. Maslov. We also pay attention to useful surveys on linear dependence and matrix ranks over tropical semirings, and on the methods of tropical geometry and tropical convexity. The volume contains a number of original research papers on tropical mathematics and applications.

Let $\mathbf{R}$ be the field of real numbers and $\mathbf{R}_+$ the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). The change of variables $x \mapsto u = h \ln x$, $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to S = \mathbf{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbf{R}$ to S by Φ_h, i.e., let $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$, $\mathbf{0} = -\infty = \Phi_h(0)$, $\mathbf{1} = 0 = \Phi_h(1)$. It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$ and that S forms a semiring with respect to addition $u \oplus v = \max\{u, v\}$ and multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by $\mathbf{R}_{\max}$; it is *idempotent*, i.e. $u \oplus u = u$ for all its elements. The semiring $\mathbf{R}_{\max}$ is actually a semifield. The analogy with quantization is obvious; the parameter h plays the rôle of the Planck constant, so $\mathbf{R}_+$ (or $\mathbf{R}$) can be viewed as a "quantum object" and $\mathbf{R}_{\max}$ as the result of its "dequantization." A similar procedure, for $h < 0$, yields the semiring $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$; in this case $\mathbf{0} = +\infty$, $\mathbf{1} = 0$. This passage to $\mathbf{R}_{\max}$ or $\mathbf{R}_{\min}$ is called the *Maslov dequantization*. The semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$ are isomorphic. It is clear that the corresponding passage from $\mathbf{C}$ or $\mathbf{R}$ to $\mathbf{R}_{\max}$ follows from the Maslov dequantization and the map $x \mapsto |x|$. By abuse of language, *we also call this passage the Maslov dequantization.* Connections with physics and the meaning of imaginary values of the Planck constant are discussed in [**11**, **12**]. The idempotent semiring $\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$, $\odot = \min$ can be obtained as a result of a "second dequantization" of $\mathbf{C}$, $\mathbf{R}$ or $\mathbf{R}_+$. There are many interesting examples of nonisomorphic idempotent semirings, and there is a number of standard methods of deriving new semirings from these (see, e.g., [**1**,**3–8**,**10–13**,**15**]). The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is a passage from fields to idempotent semifields and semirings in mathematical constructions and results.

The "intermediate" algebras $\mathbf{R}_h = \Phi_h(\mathbf{R}_+)$ with operations $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$ for $h \neq 0$ (see above) are called *subtropical algebras*, or *Gibbs–Maslov semirings* (see [**14**]); the paper of V. P. Maslov (this volume) emphasizes their importance.

The Maslov dequantization is related to the well-known logarithmic transformation, which appeared, e.g., in the classical papers of E. Schrödinger (1926) and E. Hopf (1951), and is also known as the *Cole-Hopf transformation.* The subsequent progress of E. Hopf's ideas has culminated in the well-known vanishing viscosity method and the method of viscosity solutions, see, e.g., [**2**, **13**].

The term "tropical semirings" was introduced in computer science to denote discrete versions of the max-plus algebra $\mathbf{R}_{\max}$ or min-plus algebra $\mathbf{R}_{\min}$ and their subalgebras; (discrete) semirings of this type were called tropical semirings by Dominic Perrin in honour of Imre Simon, a Brazilian mathematician and computer scientist, and one of the pioneers in the tropical area, see [**17**].

Recently the situation and terminology have changed. Now the term "tropical mathematics" usually means mathematics over $\mathbf{R}_{\max}$ or $\mathbf{R}_{\min}$, see, e.g., [**11**, **13**, **16**, **18**]. The terms "max-plus" and "min-plus" are often used in the same sense. Tropicalization and tropification (see, e.g., [**9**]) mean exactly dequantization and quantization in our sense. In any case, tropical mathematics is a natural and very important part of idempotent mathematics.[1]

[1] Note that the term "tropical" was initially used in 80-ies by V.P. Maslov in his papers on applications of idempotent analysis to economics, see, e.g., his paper "Is it possible to predict..." in Kommunist, 1989, # 13, p. 89–91 (in Russian).

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G.L. Litvinov and S.N.Sergeev,
Moscow and Birmingham, March 2009.

This volume is a collection of papers from the International Conference on Tropical and Idempotent Mathematics, held in Moscow, Russia in August 2007. This is a relatively new branch of mathematical sciences that has been rapidly developing and gaining popularity over the last decade.

Tropical mathematics can be viewed as a result of the Maslov dequantization applied to "traditional" mathematics over fields. Importantly, applications in econophysics and statistical mechanics lead to an explanation of the nature of financial crises. Another original application provides an analysis of instabilities in electrical power networks.

Idempotent analysis, tropical algebra, and tropical geometry are the building blocks of the subject. Contributions to idempotent analysis are focused on the Hamilton-Jacobi semigroup, the max-plus finite element method, and on the representations of eigenfunctions of idempotent linear operators. Tropical algebras, consisting of plurisubharmonic functions and their germs, are examined. The volume also contains important surveys and research papers on tropical linear algebra and tropical convex geometry.

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