CONTEMPORARY MATHEMATICS

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# Homotopy Theory of Function Spaces and Related Topics

Oberwolfach Workshop
April 5–11, 2009
Mathematisches Forschungsinstitut
Oberwolfach, Germany

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## Contents

## Problem List

# Preface

This collection of articles is the proceedings volume for a conference entitled *Homotopy Theory of Function Spaces and Related Topics*, which was held 5th–11th April 2009 at the Mathematisches Forschungsinstitut Oberwolfach (MFO), Germany. The conference attracted an international group of 23 participants that included leading practitioners in the field. We would like to thank the MFO for including our conference in its workshop program for the 2009 year and for providing a congenial and productive atmosphere for the conference. As organizers, we were especially appreciative of the excellent level of facilities—technical, professional, and domestic—offered by the MFO.

Function spaces have been objects of central interest to homotopy theorists, and their study has seen steady activity for over sixty years. Current research in this area is remarkably diverse with connections to other areas ranging from geometry to analysis to robotics. The conference brought together researchers with expertise in a wide breadth of such topics. The lectures at the conference provided a snapshot of the current state-of-the-art of the subject, whilst the problem sessions suggested many promising directions for future work. We are grateful to all those who participated in the conference for a very stimulating week.

This volume contains 14 original research articles on function spaces and related topics. Each of the research articles was carefully refereed. We would like to thank the referees for their gracious acceptance and timely execution of this task. The volume also includes two general interest articles: a survey by Smith and a problem list, curated by Félix, which is an expanded and edited version of problems discussed in sessions held at the conference.

Several main themes of research in the area of homotopy theory of function spaces are represented by the selection of articles here. Also represented are a number of important connections to other areas. The survey article of Smith gives a fairly complete picture of this landscape; we restrict ourselves to a brief summary here, and refer to that article for details and more extensive discussion.

Of course, the basic problem is to understand the homotopy type of a function space $\mathrm{map}(X, Y)$. Generally, this is a disconnected space and so one focuses on a path component, denoted here by $\mathrm{map}(X, Y; f)$, for some choice of map $f\colon X \to Y$. Different components generally display different homotopy types. In order to progress, hypotheses on the spaces $X$ and $Y$ are necessary: a popular choice is to restrict to the case in which $X$ is a finite CW complex and $Y$ is a nilpotent CW complex (e.g., a simply connected CW complex).

Classification of the homotopy types of components is clearly a deep and difficult problem. The subject has progressed through results that either focus on

some particular aspect of the homotopy type of a component, or apply to particular choices of spaces $X$ and $Y$. Again, the survey article contains a wealth of information about all this. Here, we briefly highlight some of those themes and directions represented by articles in this proceedings.

Rational homotopy theory has been used intensively in recent years to study function spaces. The articles of Buijs and Gatsinzi-Kwashira focus on the rational homotopy theory of a general component of a function space. Those of Kuribayashi and Félix-Tanré use rational homotopy theory to study topics that have many points of contact with function spaces, namely classifying spaces of fibrations and configuration spaces—actually an extension of such, respectively.

As described in the survey article, many results about $\mathrm{map}(X, Y)$ have been proved in the cases in which either $X$ or $Y$ is a classifying space. Here, the article of Kono-Tsukuda studies the case of $\mathrm{map}(X, BG)$. A connection to geometry and physics arises here due to a theorem of Gottlieb, which establishes an equivalence between the gauge group of a principal $G$-bundle over $X$ and the loop space $\Omega\mathrm{map}(X, BG; h)$, where $h\colon X \to BG$ is the classifying map of the bundle. The article of Smith-Schochet is also on this subject; they focus on extending localization properties of the spaces concerned.

The free loop space $\mathrm{map}(S^1, Y)$ is an object of perennial interest from numerous points of view. The surge of activity around string topology has made it even more ubiquitous. The articles of Chataur, Crabb, Gatsinzi, Giansiracusa-Salvatore, Golasiński-Gonçalves-Wong and Wockel all bear in some way on this very active area.

Although spaces of equivalences were not a direct focus of the conference, they are related to both the classifying space of a fibration and the Gottlieb groups of a space. Several articles have already been noted that bear on the former topic. The articles of Gatsinzi and Strom-Oprea are concerned with the latter topic.

One further direction of application for the study of $\mathrm{map}(X, Y)$ is represented by the article of Kozlowski-Yamaguchi. A number of seminal results establish equivalences of one sort or another—weak homotopy equivalences, equivalences after stabilization in a certain sense—between the space of maps that preserve some pertinent structure and the corresponding space of continuous maps. An emblematic result of this type is one of Segal that relates the space of based holomorphic maps $\mathrm{Hol}_*(S^2, \mathbb{C}P^n)$ with the based function space $\mathrm{map}_*(S^2, \mathbb{C}P^n)$ in this way. Such results then allow knowledge of the ordinary function space to be applied to yield information about the space of more structured maps, an object which, *a priori*, one might expect to be more difficult to analyze or requiring techniques different from those of the Homotopy Theory of Function Spaces.

We refer the reader to the survey article and to the problem list for further information about themes of research in the area and directions of application to, or connections with, other areas.

The AMS publications department has been very encouraging throughout the preparation of this volume. We would like especially to thank Christine Thivierge for her guidance at each stage of the process.

Y. Félix, G. Lupton, S. B. Smith

## Conference Participants

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Christoph Wockel
Georg-August-Universität, Germany

## Conference Presentations

In keeping with standard practice at MFO, the number of talks was kept relatively low, so as to allow generous amounts of time for discussion amongst participants.

The following four talks were expository (solicited as such by the organizers):

- David Chataur: *Division Functors and Mapping Spaces*
- Katsuhiko Kuribayashi: *Models for Function Spaces and Applications*
- John Oprea: *Gottlieb Groups, Evaluation Maps and Geometry*
- Shuichi Tsukuda: *Survey on Gauge Groups*

The remaining 13 were research talks on the presenters' work.

- Méadhbh Boyle: *An Algebraic Model for the Homology of Pointed Mapping Spaces out of a Closed Surface*
- Urtzi Buijs: *The Homotopy Lie Algebra of Function Spaces and Spaces of Sections (joint with A. Murillo)*
- Jean-Baptiste Gatsinzi: *Rational Homotopy Groups of Function Spaces*
- Marek Golaziński: *Fox and Gottlieb Groups and Whitehead Products (joint with D. Gonçalves, J. Mukai and P. Wong)*
- Daniel Gottlieb: *Self Coincidence Numbers and the Fundamental Group*
- Andrey Lazarev: *Characteristic Classes of Operadic Algebras*
- Paolo Salvatore: *Cyclic Formality of the Operad of Framed Little Discs, with Implications for Spaces of Knots*
- Jonathan Scott: *On the Geodesic Conjecture (joint with K. Hess)*
- Jeffrey Strom: *Miller Spaces*
- Svjetlana Terzić: *The Integral Pontrjagin Homology of the Based Loop Space on a Flag Manifold*
- Shuichi Tsukuda: *On the Configuration Space of a Certain n-arms Machine in Euclidean Space*
- Antonio Viruel: *Equivalences of a Product and Mal'cev Quasi-rings*
- Christoph Wockel: *Non-Integral Central Extension of Loop Groups via Gerbes*

Two problem sessions were also held, not included on this list.

# Titles in This Series

527 **Ricardo Castaño-Bernard, Yan Soibelman, and Ilia Zharkov, Editors,** Mirror symmetry and tropical geometry, 2010

526 **Helge Holden and Kenneth H. Karlsen, Editors,** Nonlinear partial differential equations and hyperbolic wave phenomena, 2010

525 **Manuel D. Contreras and Santiago Díaz-Madrigal, Editors,** Five lectures in complex analysis, 2010

524 **Mark L. Lewis, Gabriel Navarro, Donald S. Passman, and Thomas R. Wolf, Editors,** Character theory of finite groups, 2010

523 **Aiden A. Bruen and David L. Wehlau, Editors,** Error-correcting codes, finite geometries and cryptography, 2010

522 **Oscar García-Prada, Peter E. Newstead, Luis Álverez-Cónsul, Indranil Biswas, Steven B. Bradlow, and Tomás L. Gómez, Editors,** Vector bundles and complex geometry, 2010

521 **David Kohel and Robert Rolland, Editors,** Arithmetic, geometry, cryptography and coding theory 2009, 2010

520 **Manuel E. Lladser, Robert S. Maier, Marni Mishna, and Andrew Rechnitzer, Editors,** Algorithmic probability and combinatorics, 2010

519 **Yves Félix, Gregory Lupton, and Samuel B. Smith, Editors,** Homotopy theory of function spaces and related topics, 2010

518 **Gary McGuire, Gary L. Mullen, Daniel Panario, and Igor E. Shparlinski, Editors,** Finite fields: Theory and applications, 2010

517 **Tewodros Amdeberhan, Luis A. Medina, and Victor H. Moll, Editors,** Gems in experimental mathematics, 2010

516 **Marlos A.G. Viana and Henry P. Wynn, Editors,** Algebraic methods in statistics and probability II, 2010

515 **Santiago Carrillo Menéndez and José Luis Fernández Pérez, Editors,** Mathematics in finance, 2010

514 **Arie Leizarowitz, Boris S. Mordukhovich, Itai Shafrir, and Alexander J. Zaslavski, Editors,** Nonlinear analysis and optimization II, 2010

513 **Arie Leizarowitz, Boris S. Mordukhovich, Itai Shafrir, and Alexander J. Zaslavski, Editors,** Nonlinear analysis and optimization I, 2010

512 **Albert Fathi, Yong-Geun Oh, and Claude Viterbo, Editors,** Symplectic topology and measure preserving dynamical systems, 2010

511 **Luise-Charlotte Kappe, Arturo Magidin, and Robert Fitzgerald Morse, Editors,** Computational group theory and the theory of groups, II, 2010

510 **Mario Bonk, Jane Gilman, Howard Masur, Yair Minsky, and Michael Wolf, Editors,** In the Tradition of Ahlfors-Bers, V, 2010

509 **Primitivo B. Acosta-Humánez and Francisco Marcellán, Editors,** Differential algebra, complex analysis and orthogonal polynomials, 2010

508 **Martin Berz and Khodr Shamseddine, Editors,** Advances in $p$-Adic and non-archimedean analysis, 2010

507 **Jorge Arvesú, Francisco Marcellán, and Andrei Martínez-Finkelshtein, Editors,** Recent trends in orthogonal polynomials and approximation theory, 2010

506 **Yun Gao, Naihuan Jing, Michael Lau, and Kailash C. Misra, Editors,** Quantum affine algebras, extended affine Lie algebras, and their applications, 2010

505 **Patricio Cifuentes, José García-Cuerva, Gustavo Garrigós, Eugenio Hernández, José María Martell, Javier Parcet, Alberto Ruiz, Fernándo Soria, José Luis Torrea, and Ana Vargas, Editors,** Harmonic analysis and partial differential equations, 2010

504 **Christian Ausoni, Kathryn Hess, and Jérôme Scherer, Editors,** Alpine perspectives on algebraic topology, 2009

503 **Marcel de Jeu, Sergei Silvestrov, Christian Skau, and Jun Tomiyama, Editors,** Operator structures and dynamical systems, 2009

This volume contains the proceedings of the Workshop on Homotopy Theory of Function Spaces and Related Topics, which was held at the Mathematisches Forschungsinstitut Oberwolfach, in Germany, from April 5–11, 2009.

This volume contains fourteen original research articles covering a broad range of topics that include: localization and rational homotopy theory, evaluation subgroups, free loop spaces, Whitehead products, spaces of algebraic maps, gauge groups, loop groups, operads, and string topology.

In addition to reporting on various topics in the area, this volume is supposed to facilitate the exchange of ideas within Homotopy Theory of Function Spaces, and promote cross-fertilization between Homotopy Theory of Function Spaces and other areas. With these latter aims in mind, this volume includes a survey article which, with its extensive bibliography, should help bring researchers and graduate students up to speed on activity in this field as well as a problems list, which is an expanded and edited version of problems discussed in sessions held at the conference. The problems list is intended to suggest directions for future work.

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