CONTEMPORARY MATHEMATICS

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Mirror Symmetry and Tropical Geometry

NSF-CBMS Conference on
Tropical Geometry and Mirror Symmetry
December 13–17, 2008
Kansas State University
Manhattan, Kansas

Ricardo Castaño-Bernard
Yan Soibelman
Ilia Zharkov
Editors

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http://dx.doi.org/10.1090/conm/527

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2000 *Mathematics Subject Classification.* Primary 14J32, 14T05, 53D37, 53D40, 14N35, 32S35, 58A14, 14M25, 52B70.

Library of Congress Cataloging-in-Publication Data

NSF-CBMS Conference on Tropical Geometry and Mirror Symmetry (2008 : Kansas State University)

Mirror symmetry and tropical geometry : NSF-CBMS Conference on Tropical Geometry and Mirror Symmetry, December 13–17, 2008, Kansas State University, Manhattan, Kansas / Ricardo Castaño-Bernard, Yan Soibelman, Ilia Zharkov, editors.

p. cm. — (Contemporary mathematics ; v. 527)

Includes bibliographical references.

ISBN 978-0-8218-4884-5 (alk. paper)

1. Tropical geometry—Congresses. 2. Calabi-Yau manifolds—Congresses. 3. Algebraic varieties—Congresses. 4. Symmetry (Mathematics)—Congresses. 5. Mirror symmetry—Congresses. I. Castaño-Bernard, Ricardo, 1972– II. Soibelman, Yan S. III. Zharkov, Ilia, 1971– IV. Title.

QA582.N74 2010

516.3′5—dc22 2010017905

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Preface

Mirror symmetry has sparked enormous interest and revolutionized various areas of mathematics—notably, algebraic and symplectic geometry. This has triggered the development of powerful new techniques to understand problems which are important for both string theory and mathematics. Kontsevich's Homological Mirror Symmetry provided a conjectural explanation in terms of triangulated categories associated with mirror dual Calabi-Yau manifolds. It was complemented by Strominger, Yau and Zaslow's more geometric conjecture, suggesting that mirror symmetry could be understood in terms of dual special Lagrangian fibrations. Later, the work of Kontsevich and Soibelman, Gross and Wilson led to view the SYZ conjecture as a limiting statement in which the underlying structures controlling mirror symmetry are integral affine manifolds and piecewise linear objects, which are now called tropical varieties. Tropical geometry has recently started to release its full power to understand the involution of geometric structures posed by mirror symmetry. This *tropicalization* of mirror symmetry is a promising approach that aims at remarkably simplifying problems depending on various parameters in a non-linear way to much simpler combinatorial problems.

The conference covered a variety of topics related to tropical geometry and mirror symmetry. Mark Gross' lectures, devoted to his program with Bernd Siebert, will be published separately by the AMS as a volume of the CBMS Monograph Series. Some of the contributions of other participants are included in the present volume. They illustrate vast connections of mirror symmetry and tropical geometry with other areas of mathematics and mathematical physics. The techniques and methods used by the authors of the volume are at the frontier of this very active area of research. The reader will benefit from Dave Morrison's insightful survey of the evolution and future of the SYZ conjecture. With the flavor of the classical Batyrev and Borisov construction, Doran and Novoseltsev provide an algorithm to compute string-theoretic Hodge numbers for the intersection of two hypersurfaces in 5-dimensional toric varieties. Using tropical degeneration data and logarithmic geometry, Ruddat gives a version of mirror duality for ordinary Hodge numbers, stringy Hodge numbers and the affine Hodge numbers. On the symplectic side, Fukaya, Oh, Ohta and Ono provide a version of Lagrangian Floer homology for anchored Lagrangians and its (higher) product structures; this article may serve as a partial outline for their book *Lagrangian Intersection Floer Theory: Anomaly and obstruction, volume 46, parts I & II, AMS/IP Studies in Advanced Math., 2009.* Kontsevich and Soibelman present a cohesive picture of the study of the categorical, geometric, and formal computational aspects of Donaldson-Thomas theory. Zharkov studies tropical counterpart of the classical theory of theta characteristics.

We expect young researchers will find in this volume a variety of unsolved problems to think about.

Acknowledgements: The CBMS Conference on Tropical Geometry & Mirror Symmetry, Manhattan KS, 2008 was sponsored by NSF grant 0735319. The preparation of this volume was partially supported by NSF 0735319 and NSF-FRG 0854989.

R. Castaño-Bernard, Y. Soibelman, I. Zharkov, Editors

Invited lectures

- *Mohammed Abouzaid (M.I.T.): "Homological Mirror Symmetry for T^4 and applications to Lagrangian embeddings".*
- *Kwok Wai Chan (Harvard): "SYZ transformations and mirror symmetry".* Abstract: In joint work with Conan Leung, we propose a program which is aimed at understanding mirror symmetry by using Fourier-type transformations (SYZ transformations). In this talk, we will discuss the applications of SYZ transformations to mirror symmetry for toric Fano manifolds. In particular, we will see how quantum cohomology is transformed to Jacobian ring and how Lagrangian torus fibers are transformed to matrix factorizations.
- *Charles Doran (U of Alberta): "Algebraic Cycles, Regulator Periods, and Local Mirror Symmetry".*
- *Kenji Fukaya (Kyoto University): "Singularity theory over Novikov ring and Mirror symmetry" (joint with Oh-Ohta-Ono).* Abstract: Generating function of open-closed Gromov-Witten invariant that is potential function with bulk in our sense, is a formal function which is some kinds of formal power series converging in appropriate adic topology, over universal Novikov ring. In the case of toric manifold and its Lagrangian fiber this function provides a nice example of universal family of hypersurface singularities, and becomes a 'rigid analytic analogue' of K. Saito's theory of isolated hypersurface singularity. This is actually a global theory and so is different from classical Saito's theory. Saito's theory is an imortant source of so called Frobenius manifold structure (= Saito's flat structure). Another important source of Frobenius manifold structure is (big) quatum cohomology. We find that they coincides in the case of arbitrary toric manifold.
- *Ilia Itenberg (U of Strasbourg): "Welschinger invariants of toric Del Pezzo surfaces" (joint work with V. Kharlamov and E. Shustin)* Abstract: The Welschinger invariants are designed to bound from below the number of real rational curves passing through a given generic real collection of points on a real rational surface. In some cases these invariants can be calculated using G. Mikhalkin's approach which deals with a corresponding count of tropical curves. Using the tropical approach we establish a logarithmic equivalence of Welschinger and Gromov-Witten invariants in the case of generic collections of real points on a toric Del Pezzo surface equipped with an arbitrary real structure (with non-empty real part).

- *Ludmi Katzarkov (U of Miami): "Conic Bundles Old and New".* Abstract: We will formulate a HMS approach to a classical question of rationality of conic bundles.
- *David Morrison (University of California, SB): "SYZ and the moduli of Calabi–Yau threefolds."*
- *Anvar Mavlyutov (Oklahoma State U): "Deformation of toric varieties and Calabi-Yau hypersurfaces".* Abstract: In the 90's, Klaus Altmann studied deformations of affine toric varieties. He constructed families of deformations of affine toric varieties as complete intersections in another toric variety using Minkowski sums of polyhedra. We found a generalization of this construction for arbitrary toric varities. In a particular important case of complete simplicial toric varieties which are partial crepant resolutions of the projective toric varieties corresponding to reflexive polytopes, this new construction coincides with our previous construction of deformations of such toric varieties obtained by a different method via homogeneous coordinates. These deformations are important as they induce deformations of Calabi-Yau hypersurfaces.
- *Yong-Geun Oh (U of Wisconsin-Madison): "Seidel's exact sequence for closed Calabi-Yau manifolds".* Abstract: In this talk, we will explain how construction of Seidel's long exact sequence of Floer cohomology under the symplectic Dehn twists can be extended to general, especially closed, Calabi-Yau manifolds. The highlight of the talk is our usage of the notion of 'anchored Lagrangian submanifolds' and some study of compactness issue of the moduli space of pseudo-holomorphic sections in the setting of symplectc Lefschetz fibrations.
- *Tony Pantev (U of Pennsylvania): "Mirror symmetry for del Pezzo surfaces".* Abstract: I will discuss the general mirror symmetry question for del Pezzo surfaces in a setup that goes beyond the Hori-Vafa ansatz. I will describe the mirror map explicitly and will describe non-trivial tests for homological mirror symmetry. This is a joint work with Auroux, Katzarkov and Orlov.
- *Bernd Siebert (U of Hamburg): "The tropical vertex".* Abstract: One insight of mirror symmetry is the fact that the enumerative geometry of rational curves is related to the deformation theory of a "mirror variety". Now there is a pro-nilpotent group of automorphisms of the algebraic 2-torus ruling the constructions of maximal degenerations (Kontsevich/Soibelman, Gross/S.). On the mirror side this group should have some enumerative geometry meaning. In the talk I will present joint work with M. Gross (UCSD) and R. Pandharipande (Princeton) showing that this group indeed organizes a class of natural enumerative geometry problems on toric surfaces into an algebraic structure. The correspondence runs via tropical geometry.
- *Yan Soibelman (Kansas State U): "Complex integrable systems, stability structures and invariants of Donaldson-Thomas type".* Abstract: In a recent joint work with Maxim Kontsevich we offered a general approach to Donaldson-Thomas type invariants ("counting of BPS states" in the language of physicists). I am going to discuss an application of our approach to complex integrable systems of Seiberg-Witten type. In particular, I will

explain how the wall-crossing formulas in Seiberg-Witten theory are related to the wall-crossing formulas of our DT-invariants, and how tropical geometry appears in the description of the spectrum of SW-model.

- *Benjamin Young (McGill University): "Counting colored 3D Young diagrams with vertex operators".* Abstract: I will show how to compute some multivariate generating functions for 3D Young diagrams (otherwise known as "plane partitions"). Each box in a 3D Young diagram gets assigned a "color" according to a certain pattern; the variables keep track of how many boxes of each color there are. My generating functions turn out to be orbifold Donaldson-Thomas partition functions for $\mathbb{C}^3/G$, where G is a finite abelian subgroup of $SO(3)$. If time permits, I will discuss recent work on the more general problem of the orbifold topological vertex and the combinatorial crepant resolution conjecture.

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This volume contains contributions from the NSF-CBMS Conference on Tropical Geometry and Mirror Symmetry, which was held from December 13–17, 2008 at Kansas State University in Manhattan, Kansas.

It gives an excellent picture of numerous connections of mirror symmetry with other areas of mathematics (especially with algebraic and symplectic geometry) as well as with other areas of mathematical physics. The techniques and methods used by the authors of the volume are at the frontier of this very active area of research.

ISBN 978-0-8218-4884-5

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CONM/527