

CONTEMPORARY MATHEMATICS

528

Symmetry for Elliptic PDEs

(30 years after a conjecture of De Giorgi,
and related problems)

May 25–29, 2009

INdAM School

Rome, Italy

Alberto Farina
Enrico Valdinoci
Editors



American Mathematical Society

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Preface

“Coniectura igitur est positiva assertio,
in alteritate veritatem, uti est, participans.”
(Nicolaus Cusanus)

This book collects some of the contributions to the INdAM School on symmetry for elliptic PDEs, which was held in Rome in May 2009.

The school was a way to mark “30 years after a conjecture of De Giorgi, and related problems” and several experts gathered together to clarify the state of the art and the open questions on the subject.

Motivated by the classical rigidity properties of the minimal surfaces, De Giorgi proposed the study of the one-dimensional symmetry of the monotone solutions of a semilinear, elliptic partial differential equation, and he posed the following striking question:

Let $u \in C^2(\mathbb{R}^n, [-1, 1])$ satisfy

$$\Delta u + u - u^3 = 0 \text{ and } \partial_n u > 0$$

in the whole \mathbb{R}^n .

Is it true that all the level sets of u are hyperplanes, at least if $n \leq 8$?

Much impressive advance has been recently performed in this field, though many problems still remain open. Several generalizations to more complicated operators have been attracted the attention of pure and applied mathematicians, both for their important theoretical problems and for their relation, among others, with the gradient theory of phase transitions and the dynamical systems. For instance, it is now very fashionable, as well as very important for the applications, to consider equations driven by complicated differential or pseudo-differential terms, such as subelliptic Laplacians, p -Laplacians, fractional Laplacians, and fully nonlinear operators, or to set the problem in very general kind of manifolds. The symmetry problem posed by De Giorgi is also very important to determine the shape of blow-up solutions, or to understand the rigidity problems of overdetermined systems, and many developments have been done in this field for these cases too.

The INdAM School was the occasion to present a fruitful introduction and a multifaceted approach to these and other hard but beautiful research problems, and we believe that the same captivating spirit has passed to this book.

We would like to thank all the speakers of the school and the contributors of these volume for the great effort they have put into this project and for the clarity of their explanations.

We are also very indebted to the participants, many of whom were very young and promising researchers, for their enthusiastic learning and their important feedback. We hope that the school, as well as this book, may be an occasion for them to develop their magnificent mathematical potentials. The field is rich of difficult and fascinating problems and we need these youngsters to drive the research towards the new frontiers.

Last but not least, we thank INdAM for the financial and organizational support.

Alberto & Enrico

Speakers of the INdAM School

I. Birindelli,
X. Cabré,
M. del Pino,
L. Dupaigne,
F. Ferrari,
E. Lanconelli,
L. Modica,
O. Savin,
B. Sciunzi,
Y. Sire
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