

Galois invariants of K_1 -groups of Iwasawa algebras.

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ABSTRACT. We study Galois descent of K_1 of group algebras with coefficients in certain subrings of the ring of integers of \mathbb{C}_p , the completion of an algebraic closure of \mathbb{Q}_p .

Introduction

Let S be an arbitrary ring with unit, then $K_1(S)$ is defined by the exact sequence

$$1 \longrightarrow E(S) \longrightarrow GL(S) \longrightarrow K_1(S) \longrightarrow 1,$$

where $GL(S) = \varinjlim GL_n(S)$ denotes the general linear group of S and $E(S)$ denotes the subgroup of elementary matrices over S .

In [FK, Rem. 3.4.6] Fukaya and Kato formulate the following expectation, which plays an important role in the definition of so called ε -isomorphisms concerning the local Iwasawa theory of ε -constants: For an *adic ring* Λ in the sense of 1.4.1 in (loc. cit.) the sequence

$$1 \longrightarrow K_1(\Lambda) \longrightarrow K_1(\tilde{\Lambda}) \xrightarrow{1-\varphi} K_1(\tilde{\Lambda}) \longrightarrow 1$$

should be exact, where $\tilde{\Lambda} := \widehat{\mathbb{Z}_p^{ur}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$ denotes the completed tensor product and φ denotes the Frobenius morphism acting on the ring of integers $\widehat{\mathbb{Z}_p^{ur}}$ of the p -adic completion $\widehat{\mathbb{Q}_p^{ur}}$ of the maximal unramified extension of \mathbb{Q}_p . For any finite group G and $\Lambda = \mathbb{Z}_p[G]$ this amounts to the statement:

$$i_* : K_1(\mathbb{Z}_p[G]) \cong K_1(\widehat{\mathbb{Z}_p^{ur}}[G])^{\varphi=id},$$

where i_* is induced by the inclusion $i : \mathbb{Z}_p \rightarrow \widehat{\mathbb{Z}_p^{ur}}$, and the Frobenius map φ acts coefficientwise on $\widehat{\mathbb{Z}_p^{ur}}[G]$ and hence on the K_1 -group.

The original motivation of this note was to show Fukaya and Kato's expectation in this specific case. A more general question would rather be whether the following statement:

$$(0.1) \quad i_* : K_1(S^\Delta[G]) \cong K_1(S[G])^\Delta$$

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holds whenever S is a ring and Δ is a group acting on S by ring automorphisms. But surprisingly, it turns out, that neither of the above statements does hold in general (see subsection 3.2). In this paper we restrict our attention to the case, where Δ is the Galois group of some algebraic field extension related to the extension S over S^Δ of a specific class of rings S contained in the completion \mathbb{C}_p of $\overline{\mathbb{Q}_p}$. We obtain partial results toward (a corrected version of) (0.1), see Theorem 2.35. In particular, we show, that in general the following sequence is exact

$$1 \longrightarrow SK_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{Z}_p[G]) \xrightarrow{i_*} K_1(\widehat{\mathbb{Z}_p^{ur}}[G])^{\varphi=id} \longrightarrow 1$$

and induces an isomorphism of the rational K -groups

$$K_1(\mathbb{Z}_p[G])_{\mathbb{Q}} \cong K_1(\widehat{\mathbb{Z}_p^{ur}}[G])_{\mathbb{Q}}^{\varphi=id}.$$

If S is a finite algebraic extension of \mathbb{Z}_p and $SK_1(S[G]) = 1$, the isomorphism (0.1) reduces to Galois descent of the determinantal image:

$$i_* : \text{Det}(S^\Delta[G]) \cong \text{Det}(S[G])^\Delta$$

as has been proved by M. Taylor in the case, where S is unramified. But the case of infinite extensions of \mathbb{Z}_p and infinite groups Δ seems not to be covered in the literature, not even by the fairly general recent treatment [CPT], where only finite group actions are considered, as was pointed out to us by M. Taylor. Actually one has to check that the techniques of integral group logarithms extend to this situation, either by extending Taylor’s original definition as pursued in (loc. cit.) or by using Snaith’s version in [Sn] - both in the case of p -groups and then use standard induction techniques to reduce the general case of finite groups to it, as e.g. in [F]. Both approaches work, and for the convenience of the reader we show, that the methods of [CPT] extends easily to our setting, recalling the main steps of their proof, but noting that for ramified extensions Snaith’s construction might be better adapted.

The reason for the defect in (0.1) relies on the surprising vanishing

$$SK_1(\widehat{\mathbb{Z}_p^{ur}}[G]) = 1$$

for all finite groups G . In particular, SK_1 - in contrast to the Det-part - does not have good Galois descent in general, see Corollary 2.34 for a more precise statement.

The paper is organized as follows: In the first section we recall for the convenience of the reader Galois descent results for group rings with coefficients in local or global fields using Fröhlich’s Hom-description. In the second section, the heart of the paper, we first concentrate on descent results for the Det-part. In particular, we obtain a rather general result not only for finite groups, but also for compact p -adic Lie groups and their Iwasawa algebras, which turns out to be quite useful in number theory, see [BV]. Then we deal with the SK_1 -part recalling and generalizing results from [O 1]. Altogether both parts lead to the desired descent description for K_1 . Finally, in the last section we derive similar descent results over the corresponding residue class fields.

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1. The case of “local” and “number” fields

The goal of this section is to prove the following theorem which is certainly known to experts but for lack of a reference we treat it here, because it forms the prototype for the descent results in the integral cases later.

We fix an embedding $\overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}_p$. Let L be a finite Galois extension of \mathbb{Q}_p and M be either an arbitrary (possibly infinite) Galois extension M^0 of \mathbb{Q}_p or the p -adic completion of a Galois extension M^0 of \mathbb{Q}_p , such that $\mathbb{Q}_p \subseteq L \subseteq M \subseteq \mathbb{C}_p$. Furthermore, we set $\Delta := \text{Gal}(M^0/L)$.

REMARK 1.1. In the case of completions we shall several times use the theorem of Ax-Sen-Tate in the following, which says that

$$M^\Delta = (M^0)^\Delta = L \quad \text{and} \quad \mathcal{O}_M^\Delta = (\mathcal{O}_{M^0})^\Delta = \mathcal{O}_L.$$

THEOREM 1.2. *In the situation as above let us assume that M^0 is of finite absolute ramification index over \mathbb{Q}_p and let Γ be a finite group. Then*

$$(1.1) \quad i_* : K_1(L[\Gamma]) \cong K_1(M[\Gamma])^\Delta,$$

where Δ acts on the K_1 -group coefficientwise.

For the proof of Theorem 1.2 we need the following

PROPOSITION 1.3. *Let N be either an arbitrary (possibly infinite) algebraic extension of \mathbb{Q}_p or the completion of an algebraic extension of finite absolute ramification index over \mathbb{Q}_p . Let Γ be a finite group. Then*

- (i) *The map $i_* : K_1(N[\Gamma]) \rightarrow K_1(\overline{N}[\Gamma])$ is injective,*
- (ii) *If N is a finite Galois extension of \mathbb{Q}_p and $G_N = \text{Gal}(\overline{N}/N)$ is the absolute Galois group, then*

$$i_* : K_1(N[\Gamma]) \cong K_1(\overline{N}[\Gamma])^{G_N}.$$

PROOF. The first statement is well known for the local fields, i.e. finite extensions of \mathbb{Q}_p , and more generally for the perfect discrete valued fields (see [Q, Prop. 2.8]; [MN, Thm. 1 and the Rem. after Thm. 2]). The infinite algebraic extensions can always be written as a direct limit of their finite subextensions. Since the direct limit is exact on the category of abelian groups and K_1 commutes with the direct limit (see [Ro, Exer. 2.1.9]), (i) is true for infinite algebraic extensions. This completes the proof of (i).

Let $R_\Gamma = R_\Gamma(\overline{N})$ denote the Grothendieck group of finitely generated $\overline{N}[\Gamma]$ -modules. Alternatively R_Γ will be viewed as the group of virtual \overline{N} -valued characters of Γ . Since \overline{N} is algebraically closed, R_Γ is a free abelian group on the irreducible characters.

Using the Wedderburn’s decomposition of $\overline{N}[\Gamma]$ we get an isomorphism of G_N -modules

$$(1.2) \quad K_1(\overline{N}[\Gamma]) \cong \prod_{\chi} \overline{N}^\times \cong \text{Hom}(R_\Gamma, \overline{N}^\times),$$

where χ are irreducible \overline{N} -valued characters and the action of G_N on the Hom-group is given by the actions on R_Γ and on \overline{N}^\times in the standard way

$$f^g(\chi) = (f(\chi^{g^{-1}}))^g, \forall f \in \text{Hom}(R_\Gamma, \overline{N}^\times), g \in G_N, \chi \in R_\Gamma.$$

From [T, part 1, §2] we obtain the corresponding Hom-description for $K_1(N[\Gamma])$

$$(1.3) \quad K_1(N[\Gamma]) \cong \text{Hom}_{G_N}(R_\Gamma, \overline{N}^\times).$$

Then the second statement is obvious, as i_* is a Galois homomorphism. □

Now let L and M be as in Theorem 1.2. We introduce a commutative diagram

$$(1.4) \quad \begin{array}{ccccc} K_1(L[\Gamma]) & \longrightarrow & K_1(\overline{L}[\Gamma]) & \xrightarrow{\cong} & \text{Hom}(R_\Gamma(\overline{L}), \overline{L}^\times) \\ \downarrow i_* & & \downarrow & & \downarrow \\ K_1(M[\Gamma]) & \longrightarrow & K_1(\overline{M}[\Gamma]) & \xrightarrow{\cong} & \text{Hom}(R_\Gamma(\overline{M}), \overline{M}^\times), \end{array}$$

The rows are injective by the Proposition 1.3. The right hand side column is injective as $\overline{L}^\times \subseteq \overline{M}^\times$ and each $\psi \in R_\Gamma(\overline{M})$ being a character of a finite group is the composition of one of the characters $\chi \in R_\Gamma(\overline{L})$ with the inclusion map $i : \overline{L} \rightarrow \overline{M}$, so that

$$R_\Gamma(\overline{L}) \cong R_\Gamma(\overline{M}),$$

which we take as an identification henceforth. It follows, that the left hand side column is also injective.

Taking invariants under the action of Δ , which is a left exact functor, we obtain the inclusion of Theorem 1.2

$$(1.5) \quad i_* : K_1(L[\Gamma]) \subseteq (K_1(M[\Gamma]))^\Delta.$$

To prove the surjectivity of i_* we take invariants under the action of Δ of the following commutative diagram

$$\begin{array}{ccc} K_1(L[\Gamma]) & \xrightarrow{\cong} & \text{Hom}_{G_L}(R_\Gamma, \overline{L}^\times) \\ \downarrow i_* & & \downarrow \\ K_1(M[\Gamma]) & \xrightarrow{\cong} & \text{Hom}_{G_{M^0}}(R_\Gamma, \overline{M}^\times), \end{array}$$

so that the right hand side injection becomes an isomorphism (cf. Remark 1.1), hence also the the left hand side map. This finishes the proof of Theorem 1.2.

REMARK 1.4. Unfortunately we cannot prove Theorem 1.2 in full generality, i.e. for M being the completion of an arbitrary Galois extension of \mathbb{Q}_p . For instance, it is not known to us, whether the theorem holds for M being the completion of the infinite purely ramified extension $\mathbb{Q}_p(\mu_{p^\infty})$ of \mathbb{Q}_p .

REMARK 1.5. The proof above also works for “number” fields, i.e. algebraic (possible infinite) extensions of \mathbb{Q} . We just have to replace \mathbb{Q}_p by \mathbb{Q} in Theorem 1.2. Then, letting L be a finite Galois extension of \mathbb{Q} and M be an arbitrary (possible infinite) Galois extension of \mathbb{Q} , we follow the proof of Theorem 1.2 using the same arguments and references to get

$$i_* : K_1(L[\Gamma]) \cong K_1(M[\Gamma])^\Delta.$$

The only difference is, that the elements of Hom-groups in the proof are to be totally positive on all symplectic representations.

2. The case of rings of integers of “local” fields

Let G be a finite group. If S is an integral domain of characteristic zero with field of fractions L , then \overline{L} will denote a chosen algebraic closure of L . We have a map induced by base extension $\overline{L} \otimes_S -$

$$\text{Det} : K_1(S[G]) \rightarrow K_1(\overline{L}[G]) = \prod_x \overline{L}^\times \cong \text{Hom}(R_G, \overline{L}^\times),$$

where the direct product extends over the irreducible \bar{L} -valued characters of G . We write $SK_1(S[G])$ for $\ker(\text{Det})$. Since the Det -map factorizes over $K_1(L[G])$ and the map from $K_1(L[G])$ to $K_1(\bar{L}[G])$ induced by $\bar{L} \otimes_L -$ is injective (see Proposition 1.3 (i)), we have an exact sequence

$$(2.1) \quad 1 \longrightarrow SK_1(S[G]) \longrightarrow K_1(S[G]) \longrightarrow \text{Det}(K_1(S[G])) \longrightarrow 1.$$

Therefore we shall consider the two parts of K_1 , namely the Det -part and the SK_1 -part, separately.

REMARK 2.1. From the exact sequence (2.1) and the fact, that K_1 commutes with direct limits (see [Ro, Exer. 2.1.9]), we deduce, that Det and SK_1 also commute with direct limits.

2.1. The Det -part. We keep the notation of the introduction. In this subsection let $S = \mathcal{O}_L$, where L is either an arbitrary (possibly infinite) Galois extension L^0 of finite absolute ramification index over \mathbb{Q}_p or the p -adic completion of such L^0 . Then S is a Noetherian local ring, i.e. S has the unique maximal ideal, and $S[G]$ is semilocal, i.e. the quotient $S[G]/\text{rad}(S[G])$ of the ring by its Jacobson radical is left Artinian (see [L, Prop. 20.6]). We have the following

PROPOSITION 2.2. *Let Λ be a semilocal ring (for example $S[G]$). The maps*

$$\Lambda^\times = GL_1(\Lambda) \hookrightarrow GL(\Lambda) \twoheadrightarrow K_1(\Lambda)$$

induce an equality

$$\text{Det}(\Lambda^\times) = \text{Det}(GL(\Lambda)) = \text{Det}(K_1(\Lambda)).$$

PROOF. See [CR 2, Thm. 40.31]. □

CONJECTURE 2.3. *Let $S = \mathcal{O}_L$ and G be as above. Let Δ be an open subgroup of $\text{Gal}(L^0/\mathbb{Q}_p)$ acting coefficientwise on $S[G]$ and hence on Det -groups. Then*

$$i_* : \text{Det}(S^\Delta[G]^\times) \cong \text{Det}(S[G]^\times)^\Delta.$$

The proof of Conjecture 2.3 proceeds in two steps. At present we can prove step 2 and thus Conjecture 2.3 only under further assumptions on S (see Theorem 2.11). We first do the proof for finite extensions and completions of infinite extensions, since S is p -adically complete in these cases, and then we generalize the statement to infinite algebraic extensions using direct limits (see Remark 2.10).

REMARK 2.4. The map i_* in the conjecture is always a monomorphism, as the following diagram commutes and respects the action of Δ

$$\begin{array}{ccc} \text{Det}(S^\Delta[G]^\times) & \hookrightarrow & K_1(L^\Delta[G]) \\ \downarrow i_* & & \downarrow i_* \\ \text{Det}(S[G]^\times) & \hookrightarrow & K_1(L[G]), \end{array}$$

and the right hand side map is injective (see section 1).

2.1.1. *Step 1. Reduction of the general case to the p -group case.* Let S and G be as in the conjecture (for infinite Galois extensions see Remark 2.10, so we assume, that S is p -adically complete). Let Δ be an open subgroup of $Gal(L^0/\mathbb{Q}_p)$, so that $R := S^\Delta$ is the ring of integers of a finite extension of \mathbb{Q}_p . Then S is a local, Noetherian, normal ring satisfying

- (i) S is an integral domain, which is torsion free as an abelian group,
- (ii) the natural map $S \rightarrow \varprojlim S/p^n S$ is an isomorphism, so that S is p -adically complete,
- (iii) S supports a lift of Frobenius, that is to say an endomorphism $F : S \rightarrow S$ with the property that for all $s \in S$

$$F(s) \equiv s^p \pmod{\mathfrak{M}},$$

where \mathfrak{M} is the maximal ideal of S . Note that with S also R satisfies (i)-(iii).

The reduction step is then an easy generalization of paragraphs 5 and 6 of [CPT]. Explicitly, we have to relax the condition (iii) on the ring in Hypothesis on page 2 in (loc. cit.). Since this procedure consists on the formal check of the arguments and needs no new ideas, we skip it.

REMARK 2.5. In the hope to prove Conjecture 2.3 we need the following conditions to be satisfied for every finite p -group G (S, R, Δ being as above):

- (1) There exists a homomorphism ν defined using the lift of Frobenius on S

$$\nu : \text{Det}(1 + I(S[G])) \rightarrow L[\mathcal{C}_G],$$

such that $\mathcal{L} = \nu \circ \text{Det}$ (for the definition of \mathcal{L} see pp. 12-13 in [CPT]). Here $I(S[G])$ denotes the augmentation ideal of the group ring $S[G]$ and \mathcal{C}_G denotes the set of conjugacy classes of G .

- (2) Let ν' denote the restriction of the homomorphism ν to $\text{Det}(1 + \mathcal{A}(S[G]))$, where $\mathcal{A}(S[G])$ is the kernel of the natural map from $S[G]$ to $S[G^{ab}]$, then ν' is an isomorphism

$$\text{Det}(1 + \mathcal{A}(S[G])) \xrightarrow{\cong} p\phi(\mathcal{A}(S[G])),$$

where $\phi : L[G] \rightarrow L[\mathcal{C}_G]$ denotes the L -linear map obtained by mapping each element of G to its conjugacy class.

- (3) We have the exact sequence

$$0 \longrightarrow \phi(\mathcal{A}(S[G])) \xrightarrow{(\nu')^{-1} \circ (p \cdot)} \text{Det}(S[G]^\times) \longrightarrow S[G^{ab}]^\times \longrightarrow 1.$$

- (4) We have the isomorphism

$$i_* : \text{Det}(S^\Delta[G]^\times) \cong \text{Det}(S[G]^\times)^\Delta,$$

where Δ acts coefficientwise on Det -groups.

These conditions are essential ingredients of step 2 and unfortunately are known to us at the present day only in the unramified case (see Remark 2.8) below.

2.1.2. *Step 2. The p -group case.* In contrast to the first step the second one can be proved only under further assumptions on S . In particular, we have to aggravate the condition (iii) of step 1. So let G be a finite p -group. Let S be a unitary ring satisfying the following conditions:

- (i) S is an integral domain, which is torsion free as an abelian group,
- (ii) the natural map $S \rightarrow \varprojlim S/p^n S$ is an isomorphism, so that S is p -adically

complete,

(iii) S supports a lift of Frobenius, that is to say an endomorphism $F : S \rightarrow S$ with the property that for all $s \in S$

$$F(s) \equiv s^p \pmod{pS}.$$

For this step we generalize the ideas of [CPT] to the case of an infinite group Δ .

REMARK 2.6. Proposition 2.2 holds also for S satisfying the conditions (i)-(iii) above and G being a finite p -group (see [CPT, Thm. 1.2]).

Now we are ready to formulate the main theorem of step 2.

THEOREM 2.7. *Let G be a finite p -group. Let S be a unitary ring satisfying the conditions (i)-(iii) and Δ be a group acting on S by the ring automorphisms, such that $R = S^\Delta$ also satisfies the conditions (i)-(iii). We do not suppose, that the lift of Frobenius F_R is compatible with the lift of Frobenius F_S , so that $F_S \upharpoonright_R$ need not equal F_R . Then we have the isomorphism*

$$i_* : \text{Det}(R[G]^\times) \cong \text{Det}(S[G]^\times)^\Delta,$$

where Δ acts on Det -groups coefficientwise.

PROOF. Since Δ acts on S by the ring automorphisms, we have the equality $(S^\times)^\Delta = R^\times$ and for any finitely generated free S -module $M = \bigoplus_i S e_i$, on which Δ acts coefficientwise, M^Δ is the finitely generated free R -module $M^\Delta = \bigoplus_i R e_i$.

S and R both satisfy the conditions (i)-(iii), which are precisely the Hypothesis in [CPT], thus the proof of the theorem is identical with the proof of Theorem 4.1 in [CPT]. Note that this proof does not depend on the condition whether Δ is a finite group or not, it only uses the equalities above. \square

REMARK 2.8. In particular, Theorem 2.7 holds for $S = \mathcal{O}_L$, where L is either a finite unramified extension L^0 of \mathbb{Q}_p or the completion of an infinite unramified extension L^0 of \mathbb{Q}_p - in other words for the ring of Witt vectors $W(\kappa)$ of any algebraic extension κ of \mathbb{F}_p , and for Δ being an open subgroup of $\text{Gal}(L^0/\mathbb{Q}_p)$.

2.1.3. *Conjecture 2.3: proved cases and one generalization.* Because of our restrictions in the p -group case we have proved Conjecture 2.3 only for $S = W(\kappa)$ the ring of Witt vectors of an algebraic extension κ of \mathbb{F}_p . There are some possible generalizations of this result as will be explained now.

REMARK 2.9. Using the reduction step described in [T, p. 92] we can prove Conjecture 2.3 for $S = \mathcal{O}_L$, where L is the completion of a tamely ramified extension L^0 of finite absolute ramification index over \mathbb{Q}_p , i.e., finite tamely ramified extension of the ring of Witt vectors of an algebraic extension κ of \mathbb{F}_p , and Δ being an open subgroup of $\text{Gal}(L^0/\mathbb{Q}_p)$ containing the inertia group.

REMARK 2.10. The infinite Galois extensions can always be written as a direct limit (or simply union) of their finite subextensions, and so their rings of integers, too. Thus, if we have proved Conjecture 2.3 for some class of finite Galois extensions of \mathbb{Q}_p , we can obtain it also for the corresponding ind-objects (infinite extensions), i.e. direct limits of objects in the original class, as the Det -map commutes with direct limits (see Remark 2.1).

Explicitly, let S be the ring of integers of an infinite extension L of \mathbb{Q}_p . Let $L = \bigcup_i L_i$ and $S = \bigcup_i S_i$, where L_i are finite extensions and S_i their rings of integers, and we have the statement of Conjecture 2.3 for all S_i . Further, let Δ be an open subgroup of $Gal(L/\mathbb{Q}_p)$, then

$$\begin{aligned} \text{Det}(S[G]^\times)^\Delta &= \text{Det}\left(\bigcup_i S_i[G]^\times\right)^\Delta = \left(\bigcup_i \text{Det}(S_i[G]^\times)\right)^\Delta = \bigcup_i \left(\text{Det}(S_i[G]^\times)^\Delta\right) \\ &= \bigcup_i \left(\text{Det}(S_i^\Delta[G]^\times)\right) = \text{Det}\left(\bigcup_i S_i^\Delta[G]^\times\right) = \text{Det}(S^\Delta[G]^\times), \end{aligned}$$

where Δ acts on S_i through the corresponding quotient group and the union commutes with such defined Δ -action. The maps between Det-groups induced by inclusions of rings are inclusions by Remark 2.4.

For example, Conjecture 2.3 holds for $S = \mathcal{O}_L$, where L is the maximal unramified extension of \mathbb{Q}_p and Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ or using the previous remark L is the maximal tamely ramified extension of \mathbb{Q}_p and Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ containing the inertia group.

Remarks 2.8, 2.9 and 2.10 imply

THEOREM 2.11. *Let G be a finite group. Let $S = \mathcal{O}_L$, where L is either an arbitrary (possibly infinite) tamely ramified extension L^0 of \mathbb{Q}_p (type 1) or the completion of a tamely ramified extension L^0 of finite absolute ramification index over \mathbb{Q}_p (type 2), and let Δ be an open subgroup of $Gal(L^0/\mathbb{Q}_p)$ containing the inertia group. Then*

$$i_* : \text{Det}(S^\Delta[G]^\times) \cong \text{Det}(S[G]^\times)^\Delta.$$

We conclude this subsection with the following result generalizing Theorem 2.11 to the case of compact p-adic Lie groups and their Iwasawa algebras. Let \mathcal{G} be a compact p-adic Lie group and let $S = \mathcal{O}_L$ be as in the theorem but unramified, as for the infinite and tamely ramified extensions we can use Remarks 2.9 and 2.10. We denote by $R = S^\Delta$ the ring of integers of a finite unramified extension K over \mathbb{Q}_p , where Δ is the Galois group $Gal(L^0/K)$, and write

$$\Lambda_S(\mathcal{G}) := S[[\mathcal{G}]] \quad (\text{resp. } \Lambda_R(\mathcal{G}) := R[[\mathcal{G}]])$$

for the Iwasawa algebra of \mathcal{G} with coefficients in S (resp. in R). Note that it is a Noetherian pseudocompact ring (resp. a Noetherian compact ring). For the notion of pseudocompact rings and algebras see [B]. Generalizing the ideas of Proposition 5.2.16 in [NSW] we deduce, that $\Lambda_R(\mathcal{G})$ and $\Lambda_S(\mathcal{G})$ are semilocal rings.

In the following we will use Froehlich’s Hom-description as it has been adapted to Iwasawa theory by Ritter and Weiss in [RW]. We have the following commutative diagram

$$\begin{array}{ccc} K_1(\Lambda_R(\mathcal{G})) & \xrightarrow{\text{Det}} & \text{Hom}_{G_K}(R\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times) \\ \downarrow & & \downarrow \\ K_1(\Lambda_S(\mathcal{G})) & \xrightarrow{\text{Det}} & \text{Hom}_{G_{L^0}}(R\mathcal{G}, \mathcal{O}_{\mathbb{C}_p}^\times), \end{array}$$

where $G_{L^0} = Gal(\overline{L}/L^0)$, $G_K = Gal(\overline{K}/K)$ and R_G as before is the free abelian group on the isomorphism classes of irreducible $\overline{\mathbb{Q}_p}$ -valued Artin representations of \mathcal{G} .

Now we are ready to formulate

THEOREM 2.12. *With the notation as above we have*

$$i_* : \text{Det}(K_1(\Lambda_R(\mathcal{G}))) \cong \text{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta,$$

where Δ acts on the K_1 -groups coefficientwise.

PROOF. From the diagram

$$\begin{CD} \Lambda_R(\mathcal{G})^\times @>\text{Det}>> \text{Hom}_{G_K}(R_G, \mathcal{O}_{\mathbb{C}_p}^\times) \\ @VVV @VVV \\ \Lambda_S(\mathcal{G})^\times @>\text{Det}>> \text{Hom}_{G_{L^0}}(R_G, \mathcal{O}_{\mathbb{C}_p}^\times), \end{CD}$$

which is commutative by the construction, and from Proposition 2.2 we get the first obvious inclusion

$$\text{Det}(\Lambda_R(\mathcal{G})^\times) = \text{Det}(K_1(\Lambda_R(\mathcal{G}))) \subseteq \text{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta = \text{Det}(\Lambda_S(\mathcal{G})^\times)^\Delta.$$

For the opposite inclusion we use Theorem 2.11, then we only have to show, how the general case can be reduced to the case of finite groups. To this end write $\mathcal{G} = \varprojlim_n G_n$ as inverse limit of finite groups. By Theorem 2.11 we have compatible continuous maps

$$R[G_n]^\times \xrightarrow{\text{Det}} \text{Det}(K_1(\Lambda_S(G_n)))^\Delta \hookrightarrow \text{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)^\Delta,$$

where the topology on $\text{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)$ is induced from the valuation topology on \mathbb{C}_p . Taking the inverse limit yields, by the compactness of $\Lambda_R(\mathcal{G})^\times = \varprojlim_n (R/\pi^n[G_n])^\times$ and by letting $R_G = \varinjlim_n R_{G_n}$, a factorization of the homomorphism Det into

$$\Lambda_R(\mathcal{G})^\times \xrightarrow{\text{Det}} \left(\varprojlim_n \text{Det}(K_1(\Lambda_S(G_n))) \right)^\Delta \hookrightarrow \text{Hom}_{G_K}(R_G, \mathcal{O}_{\mathbb{C}_p}^\times).$$

The claim follows, because denoting by

$$\text{res}_n : \text{Hom}_{G_{L^0}}(R_G, \mathcal{O}_{\mathbb{C}_p}^\times) \rightarrow \text{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times)$$

the restriction we obtain from the universal mapping property for

$$\varprojlim_n \text{Hom}_{G_{L^0}}(R_{G_n}, \mathcal{O}_{\mathbb{C}_p}^\times) \cong \text{Hom}_{G_{L^0}}(R_G, \mathcal{O}_{\mathbb{C}_p}^\times)$$

inclusions

$$\text{Det}(K_1(\Lambda_S(\mathcal{G}))) \subseteq \varprojlim_n \text{Im}(\text{res}_n \circ \text{Det}) \subseteq \varprojlim_n \text{Det}(K_1(\Lambda_S(G_n))),$$

whence

$$\text{Det}(K_1(\Lambda_S(\mathcal{G})))^\Delta \subseteq \text{Det}(\Lambda_R(\mathcal{G})^\times) = \text{Det}(K_1(\Lambda_R(\mathcal{G}))).$$

□

For applications of the theorem above in number theory see [BV].

2.2. The SK_1 -part. From [O 1] we have the following

THEOREM 2.13. *Let R be the ring of integers in any finite extension K of \mathbb{Q}_p . Then for any p -group G , there is an isomorphism*

$$\Theta_{RG} : SK_1(R[G]) \xrightarrow{\cong} H_2(G)/H_2^{ab}(G),$$

where $H_2^{ab}(G) = \text{Im}[\sum \{H_2(H) : H \subseteq G, H \text{ abelian}\} \xrightarrow{\sum \text{Ind}} H_2(G)]$

- (i) $i_* : SK_1(R[G]) \rightarrow SK_1(S[G])$ (induced by inclusion) is an isomorphism, if L/K is totally ramified; and
- (ii) $\text{trf} : SK_1(S[G]) \rightarrow SK_1(R[G])$ (the transfer) is an isomorphism, if L/K is unramified.

PROOF. See [O 1, Thm. 8.7]. □

We see, that $SK_1(S[G])$ as an abstract finite group is independent of S . Note also, that i_* and trf are Galois homomorphisms, hence $SK_1(S[G])$ has trivial Galois action. In order to treat infinite algebraic extensions of \mathbb{Q}_p and the analogous descent statement of the introduction for SK_1 -groups we have to describe the maps i_* induced by inclusions, as they appear in the direct limits (see Remark 2.10 and Remark 2.1).

Now we assume that K is unramified over \mathbb{Q}_p , then from Proposition 21 (i) in [O 2], which is also valid for i_* by the same argument, we have commutative squares

$$\begin{CD} SK_1(R[G]) @>\Theta_{RG}>> H_2(G)/H_2^{ab}(G) \\ @V i_* VV @VV ? V \\ SK_1(S[G]) @>\Theta_{SG}>> H_2(G)/H_2^{ab}(G) \end{CD}$$

and

$$\begin{CD} SK_1(R[G]) @>\Theta_{RG}>> H_2(G)/H_2^{ab}(G) \\ @A \text{trf} AA @AA id A \\ SK_1(S[G]) @>\Theta_{SG}>> H_2(G)/H_2^{ab}(G) \end{CD}$$

where Θ_{RG} , Θ_{SG} and trf are isomorphisms. To describe i_* and $?$ we need the following

LEMMA 2.14. *With the previous notation the map*

$$\text{trf} \circ i_* : K_1(R[G]) \longrightarrow K_1(S[G]) \longrightarrow K_1(R[G])$$

is multiplication by $n = [L : K]$.

PROOF. By [O 1, Prop. 1.18] the composite $\text{trf} \circ i_*$ is induced by tensoring over $R[G]$ with $S[G] \cong S \otimes_R R[G]$ regarded as an $(R[G], R[G])$ -bimodule, where the bimodule structure on $S \otimes_R R[G]$ is given through the second factor in the natural way. Since S is a free R -modules of rank n , $S \otimes_R R[G] \cong R[G]^n$ as $(R[G], R[G])$ -bimodules, and so $\text{trf} \circ i_*$ is multiplication by n on $K_1(R[G])$ (written additively). □

The map $trf \circ i_*$ on $K_1(R[G])$ corresponds via Θ_{RG} to the map

$$H_2(G)/H_2^{ab}(G) \xrightarrow{n} H_2(G)/H_2^{ab}(G),$$

and since trf corresponds to the identity map, i_* corresponds to the multiplication by n . From [O 1, Thm. 3.14] we know, that $SK_1(R[G])$ (hence $H_2(G)/H_2^{ab}(G)$) is a finite p -group, so that we have proved the

THEOREM 2.15. *With the notation as above i_* is an isomorphism in the following two cases*

- (i) *if L/K is totally ramified,*
- (ii) *if L/K is unramified and $p \nmid n$. In this case i_* corresponds via Θ_{RG} to the multiplication by n on the finite p -group (written additively).*

If L/K is unramified and $p|n$, then i_ still corresponds via Θ_{RG} to the multiplication by n on the finite p -group, which is neither surjective nor injective, as finite p -groups always have p -torsion elements.*

COROLLARY 2.16. *The groups $SK_1(R[G])$ and $SK_1(S[G])$ are always isomorphic (as abstract groups with the (trivial) action of $Gal(L/K)$), but the statement of the introduction for SK_1 -groups, i.e.*

$$(2.2) \quad i_* : SK_1(R[G]) \cong SK_1(S[G])^\Delta \quad (\Delta = Gal(L/K)),$$

holds only in the cases (i) and (ii) of Theorem 2.15.

COROLLARY 2.17. *Let M be an infinite algebraic extension of \mathbb{Q}_p and let M^0 be the maximal unramified extension of \mathbb{Q}_p contained in M . We write M as the direct limit (union) of its finite subextensions and use Remark 2.1 and the theorem above to get the following result:*

If p^∞ divides $[M^0 : \mathbb{Q}_p]$ (as supernatural numbers), then $SK_1(\mathcal{O}_M[G]) = 1$ for every finite p -group G .

REMARK 2.18. From Corollary 2.17 we can obtain a generalization of Corollary 2.16 for infinite extensions: Let L be an infinite algebraic extension of \mathbb{Q}_p and let $K = L^\Delta$, where Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$. Then the statement (2.2) holds only in the cases (i) and (ii) of Theorem 2.15, here $p \nmid n$ as supernatural numbers. In the case, where p^∞ divides $[L^0 : \mathbb{Q}_p]$ and $SK_1(R[G]) \neq 1$, $SK_1(R[G])$ and $SK_1(S[G])$ are not isomorphic even as abstract groups. See [O 1, Exam. 8.11] for an example of a non-trivial $SK_1(\mathbb{Z}_p[G])$.

Now we generalize our results to the case of an arbitrary finite group G . Note, that Theorem 3.14 in [O 1] and Lemma 2.14 still hold in this case. From [O 3] we have the

THEOREM 2.19. *Let R be the ring of integers in any finite extension K of \mathbb{Q}_p and let G be a finite group. Let $L \supseteq K$ be a finite extension, and let $S \subseteq L$ be the ring of integers, then*

- (ii) *$i_* : SK_1(R[G]) \rightarrow SK_1(S[G])$ is an isomorphism, if L/K is totally ramified;*
- (iii) *$trf : SK_1(S[G]) \rightarrow SK_1(R[G])$ is onto, if L/K is unramified.*

PROOF. See [O 3, Thm. 1]. □

We need some more notation. For any finite group G and any fixed prime p G_r will denote the set of p -regular elements in G , i.e., elements of order prime to p .

$H_n(G, R(G_r))$ denotes the homology group induced by the conjugation action of G on the free R -module $R(G_r)$ on the set G_r .

When R is the ring of integers in a finite unramified extension of \mathbb{Q}_p , then Φ denotes the automorphism of $H_n(G, R(G_r))$ induced by the map $\Phi(\sum_i r_i g_i) = \sum_i \varphi(r_i) g_i^p$ on coefficients. We set $H_n(G, R(G_r))_\Phi = H_n(G, R(G_r))/(1 - \Phi)$. In analogy with the p -group case, we define

$$H_2^{ab}(G, R(G_r))_\Phi = \text{Im} \left[\sum \{H_2(H, R(H_r)) : H \subseteq G, H \text{ abelian}\} \xrightarrow{\sum \text{Ind}} \xrightarrow{\sum \text{Ind}} H_2(G, R(G_r))_\Phi \right].$$

We use the notation above to formulate the

THEOREM 2.20. *Let R be the ring of integers in a finite unramified extension of \mathbb{Q}_p . Then for any finite group G there is an isomorphism*

$$\Theta_G : SK_1(R[G]) \xrightarrow{\cong} (R/(1 - \varphi)R) \otimes_{\mathbb{Z}_p} H_2(G, \mathbb{Z}_p(G_r))_\Phi / H_2^{ab}(G, \mathbb{Z}_p(G_r))_\Phi.$$

This new tensor product decomposition of $SK_1(R[G])$ in terms of $R/(1 - \varphi)R$ and group cohomology comes from the results announced in [CPT 1] and we are very grateful to T. Chinburg, G. Pappas and M. J. Taylor for sharing this insight with us, which not least also influenced our results below.

PROOF. See [O 1, Thm. 12.10] and use the facts

$$H_2(G, R(G_r))_\Phi \cong (R/(1 - \varphi)R) \otimes_{\mathbb{Z}_p} H_2(G, \mathbb{Z}_p(G_r))_\Phi$$

and

$$H_2^{ab}(G, R(G_r))_\Phi \cong (R/(1 - \varphi)R) \otimes_{\mathbb{Z}_p} H_2^{ab}(G, \mathbb{Z}_p(G_r))_\Phi.$$

□

Arguing as in the p -group case we deduce the following

THEOREM 2.21. *With the notation as above i_* is*
 (i) *an isomorphism, if L/K is totally ramified;*
 (ii) *a monomorphism, if L/K is unramified and $p \nmid n$.*

REMARK 2.22. In the case (ii) we cannot say whether i_* is surjective or not, since, in general, trf is only an epimorphism and not an isomorphism.

COROLLARY 2.23. *The statement (2.2) holds in the cases (i)-(ii) of Theorem 2.21.*

PROOF. The statement is obvious in the case (i), since i_* is a Galois isomorphism. For (ii) we use the isomorphism of Theorem 2.20 for S and R noting that

$$R/(1 - \varphi)R \cong \mathbb{Z}_p \cong S/(1 - \varphi)S$$

in this situation, hence $SK_1(S[G])$ is isomorphic to $SK_1(R[G])$ (as an abstract finite group). Since i_* is a Galois monomorphism in the case (ii), we get the statement. □

REMARK 2.24. Corollary 2.23 generalizes immediately to the case of infinite algebraic extensions L .

To study more general rings (for example completions of infinite extensions of \mathbb{Q}_p) we need generalizations of Oliver’s results on SK_1 to such rings as are announced to appear in [CPT 1] in a very general setting. Meanwhile we outline an ad hoc description sufficient for our purposes. We just note that the same arguments as used below also should work for any ring R as in the beginning of subsection 2.1.2 and satisfying the surjectivity of $1 - F$.

Let p be an odd prime number. For the rest of this section we assume that R is the ring of Witt vectors of a p -closed algebraic extension κ of \mathbb{F}_p , i.e. κ does not allow any extension of degree p . The main example we have in mind being $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$. We note, that such a ring satisfies Hypothesis in [CPT] and is a discrete valuation ring. We write \mathfrak{m} for its maximal ideal pR and start with a crucial (certainly well-known) observation:

LEMMA 2.25. *We have an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow R \xrightarrow{1-\varphi} R \longrightarrow 0,$$

where φ denotes the Frobenius endomorphism of R .

PROOF. By Artin-Schreier theory and the p -closeness of κ we have the obvious exact sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \kappa \xrightarrow{1-\varphi} \kappa \longrightarrow 0,$$

because $(1 - \varphi)(x) = x - x^p$. Inductively, one shows that, for all $n \geq 1$, also

$$0 \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p \longrightarrow R/p^nR \xrightarrow{1-\varphi} R/p^nR \longrightarrow 0$$

is exact. Thus for any given $r = (r_n)_n \in \text{projlim}_n R/p^nR = R$ the sets $S_n := \{s_n \in R/p^nR \mid (1 - \varphi)(s_n) = r_n\}$ are finite and form an inverse system, whence $S := \text{projlim}_n S_n$ is non-empty and any $s \in S$ satisfies $(1 - \varphi)(s) = r$ by construction. \square

Let G be a finite p -group. The split exact sequence

$$1 \longrightarrow I(R[G]) \longrightarrow R[G] \longrightarrow R \longrightarrow 1$$

induces isomorphisms

$$K_1(R[G]) \cong K_1(R[G], I(R[G])) \oplus R^\times$$

and

$$R[\mathcal{C}_G] \cong \phi(I(R[G])) \oplus R,$$

where $\phi : R[G] \rightarrow R[\mathcal{C}_G]$ denotes the canonical map, \mathcal{C}_G denoting the conjugacy classes of G . By $\text{Log}(1 - x)$ we denote the logarithm series. Then the map $\frac{1}{p}\mathcal{L} = \phi(\frac{1}{p}(p - \Psi)(\text{Log}(1 - x)))$ defined on page 13 of [CPT] induces by [CPT, Cor. 3.3, Thm. 3.17] and the lemma above a surjective map

$$\Gamma_{I(R[G])} : K_1(R[G], I(R[G])) \rightarrow \phi(I(R[G])).$$

We use a generalization of Theorem 2.8 in [O 1] (for ideals contained in the Jacobson radical) in order to show, that this map is actually a group homomorphism, which together with the surjective homomorphism

$$\Gamma_R : R^\times \rightarrow R,$$

which sends $x \in 1 + \mathfrak{m}$ to $\frac{1}{p}(p-\varphi)\text{Log}(x)$ and $x \in \kappa^\times$ to zero (note that $\text{Log}(1+pR) = pR$ and that $p-\varphi$ is an isomorphism of R), defines a surjective group homomorphism

$$\Gamma_{R[G]} = \Gamma_{I(R[G])} \oplus \Gamma_R : K_1(R[G]) \rightarrow R[\mathcal{C}_G],$$

which factorizes over

$$\Gamma_{\text{Wh}(R[G])} : \text{Wh}(R[G]) := K_1(R[G]) / (G^{ab} \times \mu_R) \rightarrow R[\mathcal{C}_G].$$

Setting

$$SK'_1(R[G]) := \ker(\Gamma_{\text{Wh}(R[G])})$$

we obtain the following exact sequence

$$(2.3) \quad 1 \longrightarrow SK'_1(R[G]) \longrightarrow \text{Wh}(R[G]) \xrightarrow{\Gamma_{\text{Wh}(R[G])}} R[\mathcal{C}_G] \longrightarrow 1.$$

The relation between $SK'_1(R[G])$ and the original $SK_1(R[G])$ will be cleared below.

Our goal is to prove the following

THEOREM 2.26. *Let G be a p -group. Then $SK'_1(R[G]) = 1$. In particular,*

$$\text{Wh}(R[G]) \cong R[\mathcal{C}_G]$$

is torsion free and a Hausdorff topological group (the second group being a pseudo-compact R -module).

PROOF. The proof proceeds by induction on the order of G . If G is trivial, it is well-known that the $SK'_1(R) = 1$, because the kernel of Γ_R is just μ_R . Now assume G to be non-trivial. Then there exists a central element $z \in G$ of order p . We set $\bar{G} := G / \langle z \rangle$ and write $\alpha : G \rightarrow \bar{G}$ for the canonical projection. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \ker(\text{Wh}(\alpha)) & \longrightarrow & (1-z)R[\mathcal{C}_G] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & SK'_1(R[G]) & \longrightarrow & \text{Wh}(R[G]) & \xrightarrow{\Gamma_{\text{Wh}}} & R[\mathcal{C}_G] \longrightarrow 0 \\ & & \downarrow SK'_1(\alpha) & & \downarrow \text{Wh}(\alpha) & & \downarrow H_0(\alpha) \\ 0 & \longrightarrow & SK'_1(R[\bar{G}]) & \longrightarrow & \text{Wh}(R[\bar{G}]) & \xrightarrow{\Gamma_{\text{Wh}}} & R[\mathcal{C}_{\bar{G}}] \longrightarrow 0, \end{array}$$

in which also the right column is exact by [CPT, Lem. 3.9].

Let I_z denotes the ideal $(1-z)R[G]$. An immediate generalization of [O 1, Prop. 6.4] to our setting tells us that the logarithm induces an exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow K_1(R[G], I_z) \xrightarrow{\log} H_0(G, I_z) \longrightarrow 0$$

(note that τ in (loc. cit.) has to be replaced by the trivial map, because $1-\varphi$ in Lem. 6.3 is surjective on κ ; also [O 1, Thm. 2.8] (for ideals contained in the Jacobson radical) needed for the proof generalizes immediately to our setting, because by [L, (20.4)] $S := M_n(R[G])$ is also semilocal and satisfies $J(S)^N \subseteq pS$ for N sufficiently big).

Thus, letting B denote the kernel of the third arrow in the upper line, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B & \longrightarrow & K_1(R[G], I_z) / \langle z \rangle & \longrightarrow & \ker(\text{Wh}(\alpha)) \longrightarrow 0 \\
 & & \downarrow & & \cong \downarrow \log & & \downarrow \Gamma_{R[G]} \\
 0 & \longrightarrow & H_0(G, (1-z)R[\Omega]) & \longrightarrow & H_0(G, (1-z)R[G]) & \longrightarrow & (1-z)R[\mathcal{C}_G] \longrightarrow 0.
 \end{array}$$

Here, following [O 1] we write

$$\Omega = \{g \in G \mid g \text{ is conjugate to } zg\}.$$

By the Snake-lemma we see that

$$\begin{aligned}
 \ker(SK'_1(\alpha)) &= \ker(\text{Wh}(\alpha)) \cap SK'_1(R[G]) \\
 &= H_0(G, (1-z)R[\Omega]) / \log B \\
 &= R[\Omega] / \psi^{-1}(\log B),
 \end{aligned}$$

where $\psi : R[\Omega] \rightarrow H_0(G, (1-z)R[\Omega])$ is induced by multiplication with $(1-z)$.

We note that our last term in the above equation corresponds to D/C in the proof of [O 1, Thm. 7.1]. Hence, copying literally the same arguments and noting again that $1 - \varphi$ is surjective on R , we see from (c) on p. 176 in (loc. cit.) that $C = \psi^{-1}(\log B) = R[\Omega]$. In other words, $\ker(SK'_1(\alpha))$ is trivial. Since, by our induction hypothesis also $SK'_1(R[\bar{G}])$ vanishes, the theorem is proved. \square

Let L^0 be the unique unramified algebraic extension of \mathbb{Q}_p with residue field κ .

COROLLARY 2.27. *For any open subgroup $\Delta \subseteq \text{Gal}(L^0/\mathbb{Q}_p)$, there is an exact sequence*

$$1 \longrightarrow SK_1(R^\Delta[G]) \longrightarrow K_1(R^\Delta[G]) \longrightarrow K_1(R[G])^\Delta \longrightarrow 1,$$

and isomorphisms

$$H^1(\Delta, \mu_R) \cong H^1(\Delta, K_1(R[G])) \text{ and } (\mu_R)_\Delta \cong K_1(R[G])_\Delta$$

of continuous cochain cohomology groups and coinvariants, respectively.

PROOF. Taking Δ -invariants of the exact sequence (of topological Hausdorff modules, cp. Corollary 2.28 for K_1)

$$1 \longrightarrow G^{ab} \times \mu_R \longrightarrow K_1(R[G]) \xrightarrow{\Gamma_{R[G]}} R[\mathcal{C}_G] \longrightarrow 0$$

and noting the Galois invariance of $\Gamma_{R[G]}$ (if we choose the arithmetic Frobenius in its definition) we obtain the following commutative diagram with exact rows (the first of which is the standard exact sequence as proved in [O 1])

$$\begin{array}{ccccccc}
 1 & \longrightarrow & G^{ab} \times \mu_{R^\Delta} \times SK_1(R^\Delta[G]) & \longrightarrow & K_1(R^\Delta[G]) & \longrightarrow & R^\Delta[\mathcal{C}_G] \longrightarrow G^{ab} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
 1 & \longrightarrow & G^{ab} \times \mu_{R^\Delta} & \longrightarrow & K_1(R[G])^\Delta & \longrightarrow & R[\mathcal{C}_G]^\Delta \longrightarrow H^1(\Delta, \mu_R) \times G^{ab} \longrightarrow \dots
 \end{array}$$

from which the claim follows using [NSW, Prop. 1.7.7]. Note that

$$\begin{aligned}
 H^1(\Delta, R[\mathcal{C}_G]) &\cong \text{proj} \lim_n H^1(\Delta, R/p^n R)^{\#C_G} \\
 &\cong \text{proj} \lim_n (R/p^n R)_\Delta^{\#C_G} = 0
 \end{aligned}$$

by the straight forward generalization of [NSW, Thm. 2.7.5] to pseudocompact modules, again [NSW, prop. 1.7.7] and Lemma 2.25. Alternatively, we may replace the long exact cohomology sequence above by the kernel/cokernel exact sequence arising from the Snake-lemma associated to multiplication by $1 - \tau$ for any topological generator τ of Δ . \square

COROLLARY 2.28. $SK_1(R[G]) = 1$, i.e., $K_1(R[G]) \cong \text{Det}(R[G]^\times)$. In particular,

$$K_1(R[G]) \cong \text{proj} \lim_n K_1(R/p^n R[G]).$$

The last claim follows from the first one using [CPT, Prop. 1.3]. For the proof of the first claim of the corollary consider the following diagram

$$(2.4) \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & & \downarrow & & & \\ & & & SK_1(R[G]) & & & \\ & & & \downarrow & & & \\ 1 & \longrightarrow & G^{ab} \times \mu_R & \longrightarrow & K_1(R[G]) & \xrightarrow{\Gamma_{R[G]}} & R[\mathcal{C}_G] \longrightarrow 1, \\ & & & & \downarrow \text{Det} & & \downarrow \text{Tr} \\ & & & & \text{Hom}(R_G, \mathbb{C}_p^\times) & \xrightarrow{\Gamma_{\text{Hom}}} & \text{Hom}(R_G, \mathbb{C}_p) \end{array}$$

where Γ_{Hom} is defined as follows: Choose any continuous lift $F : \mathbb{C}_p \rightarrow \mathbb{C}_p$ of the absolute Frobenius automorphism and denote by $\log : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p$ the usual p -adic logarithm. We write ψ^p and ψ_p for the p th Adams operator, which is characterized by

$$\text{tr}(g, \psi^p \rho) = \text{tr}(g^p, \rho) \text{ for all } g \in G,$$

and its adjoint, respectively. Also the rule $f(\rho) \mapsto F(f(\rho^{F^{-1}}))$ induces an operator \tilde{F} on $\text{Hom}(R_G, \mathbb{C}_p^\times)$ and $\text{Hom}(R_G, \mathbb{C}_p)$, which commutes obviously with ψ_p . Now we set

$$\Gamma_{\text{Hom}}(f) = \frac{1}{p}(p - \tilde{F}\psi_p)(\log \circ f).$$

Finally, Tr , the additive analog of Det , is induced by

$$\text{Tr}(\lambda)(\rho) = \text{tr}(\rho(\lambda)),$$

where $\rho : R[G] \rightarrow M_n(\mathbb{C}_p)$ is the linear extension of $\rho : G \rightarrow GL_n(\mathbb{C}_p)$ keeping the same notation. One easily checks that

$$(2.5) \quad \text{Det}(F(\lambda))(\rho) = \tilde{F}(\text{Det}(\lambda))(\rho) = F\text{Det}(\lambda)(\rho^{F^{-1}})$$

and

$$(2.6) \quad \text{Tr}(\Psi(\lambda))(\rho) = \tilde{F}(\text{Tr}(\psi_p \lambda))(\rho) = F\text{Tr}(\lambda)(\psi_p \rho^{F^{-1}})$$

The Corollary will follow immediately from the following

LEMMA 2.29. *The diagram (2.4) commutes and Det restricted to $G^{ab} \times \mu_R$ is injective.*

PROOF. The injectivity being well-known we only check the commutativity similarly as in [Sn, Prop. 4.3.25]. Since $K_1(R[G], I(R[G]))$ and $K_1(R)$ generate $K_1(R[G])$ as has been observed above, it suffices to check this individually on each direct summand. The case of $K_1(R)$ being similar but easier, we assume that a belongs to $K_1(R[G], I(R[G]))$ and calculate using the definitions, (2.6), the continuity of ρ , the fact that \log transforms \det into tr and (2.5):

$$\begin{aligned} (\text{Tr} \circ \Gamma(a))(\rho) &= \text{Tr}\left(\frac{1}{p}(p - \Psi) \log(a)\right)(\rho) \\ &= \text{Tr}(\log(a))(\rho) - \frac{1}{p} F \text{Tr}(\log(a))(\psi_p \rho^{F^{-1}}) \\ &= \text{tr}(\log \rho(a)) - \frac{1}{p} F \text{tr}(\log(\psi_p \rho^{F^{-1}}(a))) \\ &= \log(\det(\rho(a))) - \frac{1}{p} F \log(\det(\psi_p \rho^{F^{-1}}(a))) \\ &= \frac{1}{p} \{p \log \text{Det}(a)(\rho) - F \log \text{Det}(a)(\psi_p \rho^{F^{-1}})\} \\ &= \frac{1}{p} (p - \tilde{F} \psi_p) \log \text{Det}(a)(\rho) \\ &= (\Gamma_{\text{Hom}} \circ \text{Det}(a))(\rho). \end{aligned}$$

□

REMARK 2.30. The case $p = 2$ should be treated separately, since $\text{Log}(R^\times) = \text{Log}(1 + 2R) = 4R + 2(1 - \varphi)R \cong 2R$, so that we have to replace $R[\mathcal{C}_G]$ by $2R[\mathcal{C}_G]$ in the exact sequence (2.3), or, if we keep $R[\mathcal{C}_G]$ in (2.3), then we get a finite cokernel, which we denote by $\mu = \langle -1 \rangle$. Doing required corrections we can prove Corollaries 2.27 and 2.28 also in this case. Note, that the finite cokernel μ also appears in the first row of the commutative diagram in the proof of Corollary 2.27 (see [O 1, Thm. 6.6]).

LEMMA 2.31. *The statement (i) of Theorem 2.13 is true also for R being the ring of Witt vectors of a p -closed algebraic extension κ of \mathbb{F}_p .*

PROOF. The injectivity is obvious $SK_1(R[G])$ being trivial (see Corollary 2.28) and the surjectivity follows from the generalized Proposition 15 in [O 2]. □

Corollary 2.28 and the lemma above imply

COROLLARY 2.32. *$SK_1(S[G]) = 1$ for any totally ramified integral extension S of R , where R is the ring of Witt vectors of a p -closed algebraic extension κ of \mathbb{F}_p .*

Finally, we want to generalize Corollaries 2.17 and 2.32 to the case of an arbitrary finite group G . For this we need

THEOREM 2.33. *Fix a prime p and a Dedekind domain R with field of fractions K , such that $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$. For any finite group G , let g_1, \dots, g_k be K -conjugacy class representatives for elements in G of order $n_i = \text{ord}(g_i)$ prime to p , and set*

$$N_i = N_G^K(g_i) = \{x \in G : xg_ix^{-1} = g_i^a \text{ for some } a \in \text{Gal}(K(\zeta_{n_i})/K)\}$$

and $Z_i = C_G(g_i)$. Then $SK_1(R[G])$ is computable by induction from p -elementary subgroups of G and there is an isomorphism

$$SK_1(R[G]) \cong \bigoplus_{i=1}^k H_0(N_i/Z_i; \varinjlim_{H \in \mathcal{P}(Z_i)} SK_1(R[\zeta_{n_i}][H])),$$

where $\mathcal{P}(Z_i)$ is the set of p -subgroups of Z_i and $R[\zeta_{n_i}]$ denotes the integral closure of R in $K(\zeta_{n_i})$.

PROOF. See [O 1, Thm. 11.8 and Thm. 12.5] □

COROLLARY 2.34. *Let G be an arbitrary finite group. Let S be \mathcal{O}_M , with M as in Corollary 2.17, but of finite absolute ramification index over \mathbb{Q}_p , or S be a finite totally ramified extension of R , where R is the ring of Witt vectors of a p -closed algebraic extension κ of \mathbb{F}_p , then $SK_1(S[G]) = 1$.*

PROOF. Use Corollaries 2.17, 2.32 and Theorem 2.33. Note also, that $R[\zeta_{n_i}]$ is a finite unramified extensions of R . □

End of the SK_1 -part.

For a finite group G and S being the ring of integers of either an arbitrary tamely ramified extension L of \mathbb{Q}_p (type 1) or the completion of a tamely ramified extension L , whose residue field is p -closed (type 2), both types having finite absolute ramification index over \mathbb{Q}_p . Let $R = S^\Delta$ be the fixed ring of S , where Δ is an open subgroup of $Gal(L/\mathbb{Q}_p)$ containing the inertia group. We write K and C for the kernel and cokernel of the map

$$i_* : SK_1(R[G]) \rightarrow SK_1(S[G])^\Delta,$$

respectively. Note that they are finite p -primary abelian groups. By $K_1(R[G])_{\mathbb{Q}}$ we denote rational K -groups $K_1(R[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Snake-lemma, Theorem 2.11 and the SK_1 -part imply immediately

THEOREM 2.35. *If L is of type 1, then we denote by L^0 the maximal unramified extension of L^Δ contained in L and let p^n be the p -part of $[L^0 : L^\Delta]$ ($0 \leq n \leq \infty$). Then for both types:*

- (1) *The following sequence is exact*

$$1 \longrightarrow K \longrightarrow K_1(R[G]) \xrightarrow{i_*} K_1(S[G])^\Delta \longrightarrow C \longrightarrow 1$$

and induces

$$K_1(R[G])_{\mathbb{Q}} \cong K_1(S[G])_{\mathbb{Q}}^\Delta.$$

- (2) *If S is of type 1 and $n = 0$, then*

$$K = 1 \quad \text{and} \quad C = 1.$$

- (3) *If S is either of type 2 or of type 1 with $n = \infty$, then we have*

$$K \cong SK_1(R[G]) \quad \text{and} \quad C = 1.$$

- (4) *Let G be a p -group and S be of type 1 with $0 < n < \infty$, then*

$$K \cong SK_1(R[G])[p^n] \quad \text{and} \quad C \cong SK_1(R[G])/p^n.$$

3. The case of residue class fields

Let λ be an arbitrary (not necessary finite) Galois extension of \mathbb{F}_p and G be a finite group. Let ϕ denote the Frobenius automorphism on λ , which takes $x \in \lambda$ to x^p , then $Gal(\lambda/\mathbb{F}_p) = \langle \phi \rangle$. Moreover, if $\mathbb{F}_{p^n} \subset \lambda$, then $\mathbb{F}_{p^n} = \lambda^{\langle \phi^n \rangle}$. We fix such an n , set $\kappa = \mathbb{F}_{p^n}$ and $\Delta = \langle \phi^n \rangle$. We are going to prove the following

THEOREM 3.1. *With the notation as above, we have an exact sequence*

$$1 \longrightarrow K \longrightarrow K_1(\kappa[G]) \xrightarrow{i_*} K_1(\lambda[G])^\Delta \longrightarrow C \longrightarrow 1,$$

which induces

$$K_1(\kappa[G])_{\mathbb{Q}} \cong K_1(\lambda[G])_{\mathbb{Q}}^\Delta,$$

where K and C are as in Theorem 2.35 for R and S the unique unramified extensions of \mathbb{Z}_p lifting κ and λ , respectively. As usual ϕ (resp. ϕ^n) acts on the K_1 -groups coefficientwise.

PROOF. From [SV] we have an exact sequence

$$(3.1) \quad 0 \longrightarrow \mathbb{Z}_p[\mathcal{C}_G] \longrightarrow K_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{F}_p[G]) \longrightarrow 1,$$

where $\mathbb{Z}_p[\mathcal{C}_G]$ is a finitely generated free \mathbb{Z}_p -module over the set of conjugacy classes in G .

With the same argument as in [SV], we can obtain (3.1) for finite unramified extensions of \mathbb{Q}_p and their rings of integers, and, since K_1 commutes with the direct limit, also for infinite unramified extensions. Using this fact we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \downarrow & & \\
 & & & & K & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & R[\mathcal{C}_G] & \longrightarrow & K_1(R[G]) & \longrightarrow & K_1(\kappa[G]) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S[\mathcal{C}_G]^\Delta & \longrightarrow & K_1(S[G])^\Delta & \longrightarrow & K_1(\lambda[G])^\Delta \longrightarrow H^1(\Delta, S[\mathcal{C}_G]), \\
 & & & & \downarrow & & \\
 & & & & C & & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
 \end{array}$$

where the bottom row is the part of the long exact sequence in the group cohomology associated to the short exact sequence of Δ -modules, and Δ acts coefficientwise.

The left hand side vertical map is an isomorphism, as $R[\mathcal{C}_G]$ and $S[\mathcal{C}_G]$ are finitely generated free R - and S -modules, respectively. The middle column is exact by Theorem 2.35. Thus, to prove the theorem, it is enough to show, that $H^1(\Delta, S[\mathcal{C}_G]) = 1$.

From [FS, Prop. 2] we know, that $H^1(\text{Gal}(M_1/M_2), \mathcal{O}_{M_1}) = 1$ for every finite unramified extension M_1 of \mathbb{Q}_p . Since Δ can be written as the inverse limit of finite groups corresponding to the finite unramified subextensions of S , and $S[\mathcal{C}_G]$ as a Δ -module is isomorphic to the direct sum of copies of S , we get the statement above by using standard properties of group cohomology. \square

References

- [BV] T. Bouganis and O. Venjakob, *On the non-commutative main conjecture for elliptic curves with complex multiplication*, preprint, 2010. MR2755723 (2011m:11220)
- [B] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, Journal of Algebra **4**, 1966, pp. 442–470. MR0202790 (34:2650)
- [CPT] T. Chinburg, G. Pappas and M. J. Taylor, *K-1 of a p-adic group ring I. The determinantal image*, Journal of Algebra, Manuscript Draft, 2009.
- [CPT 1] T. Chinburg, G. Pappas and M. J. Taylor, *K-1 of a p-adic group ring II. The determinantal kernel SK-1*, to appear.
- [CR 2] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, vol. 2, John Wiley and Sons (New York), 1987. MR892316 (88f:20002)
- [F] A. Froehlich, *Galois module structure of algebraic integers*, Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 1, 1983. MR717033 (85h:11067)
- [FK] T. Fukaya and K. Kato, *A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory*, Proceedings of the St. Petersburg Mathematical Society. Vol. XII, 1–85, AMS Transl. Ser. 2, **219**, 2006. MR2276851 (2007k:11200)
- [FS] Y. Furuta and Y. Sawada, *On the Galois cohomology group of the ring of integers in a global field and its Adele ring*, Nagoya Mathematical Journal, vol. 32, 1968, pp. 247–252. MR0234994 (38:3306)
- [L] T.Y. Lam, *A first course in noncommutative rings*, Springer, Graduate Texts in Mathematics, vol. 131, second edition, 2001. MR1838439 (2002c:16001)
- [MN] Y. Matsushima and T. Nakayama, *Ueber die multiplikative Gruppe einer p-adischen Divisionsalgebra*, Proc. Imp. Acad. Tokyo **19**, 1943, pp. 622–628. MR0014081 (7:238a)
- [NSW] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Springer, A Series of Comprehensive Studies in Mathematics, vol. 323, second edition, 2008. MR2392026 (2008m:11223)
- [O 1] R. Oliver, *Whitehead groups of finite groups*, London Math. Soc., Lecture Note Series **132**, 1988. MR933091 (89h:18014)
- [O 2] R. Oliver, *SK-1 for finite group rings II*, Math. Scand. **47**, 1980, pp. 195–231. MR612696 (82m:18006a)
- [O 3] R. Oliver, *SK-1 for finite group rings III*, Algebraic K-theory (Evanston), 1980. MR612696 (82m:18006a)
- [Q] J. Queyruet, *S-Groupes des Classes d'un Ordre Arithmétique*, Journal of Algebra **76**, 1982, pp. 234–260. MR659222 (84a:12017)
- [RW] J. Ritter and A. Weiss, *Toward equivariant Iwasawa theory*, manuscripta math. **109**, 2002, pp. 131–146. MR1935024 (2003i:11161)
- [Ro] J. Rosenberg, *Algebraic K-theory and Its Applications*, Springer, Graduate Texts in Mathematics **147**, 1994. MR1282290 (95e:19001)
- [SV] P. Schneider, O. Venjakob, *A splitting for K_1 of completed group rings*, preprint, 2010.
- [Sn] V.P. Snaitch, *Explicit Brauer Induction (with applications to algebra and number theory)*, Cambridge studies in advanced mathematics **40**, Cambridge University Press, 1994. MR1310780 (96e:20012)
- [T] M. Taylor, *Classgroups of Group Rings*, London Math. Soc., Lecture Note Series **91**, 1984. MR748670 (86c:11100)

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