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Jet Bundles, symmetries, Darboux transforms

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Abstract. We review some aspects of the geometric approach to differential equations due to mathematicians such as Sophus Lie, Gaston Darboux and Elie Cartan. We consider some basic aspects of the formulation of differential equations using jet bundles and, as a non-trivial example, we state Gromov's h-principle and apply it to systems of differential equations: following some recent work on holonomic approximations due to Eliashberg and Mishachev, we present a geometric theorem on local existence of approximate solutions to PDEs. We then review the theory of symmetries of differential equations with particular emphasis on internal and nonlocal symmetries. We advance a very natural approach to nonlocal symmetries using exterior differential systems, and we argue, motivated by calculations carried out in the context of the Camassa-Holm equation, that nonlocal symmetries can be considered as generalizations of the internal symmetries introduced by E. Cartan. Finally we explain, using the associated Camassa-Holm equation, how to derive Darboux transformations using nonlocal symmetries and pseudo-potentials.

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1. Introduction

Partial differential equations (PDEs) first appeared as essential tools for the analytic study of physical models; posteriorly, as explained for instance in [**7**], they became fundamental for mathematics itself. Of the very many deep instances of interaction between PDEs and other areas of mathematics we mention, as just one example, U. Pinkall and I. Sterling's proof of the existence of an infinite number

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of constant mean curvature tori [**41**], carried out by carefully analyzing the sinh–Gordon equation

$$u_{xx} + u_{tt} + 2\sinh 2u = 0$$

and its generalized symmetries. On the other hand, other fields of mathematics have played important roles in the understanding of differential equations as mathematical objects of their own. Functional analysis and topology come immediately to mind, see [**7, 21**], but we can also study differential equations profitably by geometric means.

The geometric approach to differential equations goes back to the fundamental contributions of Sophus Lie on the symmetry theory for general systems of differential equations [**32, 36**] and owes much of its earlier development to the research on differential geometry itself, as the impressive treatise on the geometry of surfaces by Gaston Darboux [**15**] testifies. During the first half of the XXth Century, part of this research evolved into the modern theory of exterior differential systems in the hands of E. Cartan, see for example his classic book [**11**] and the modern treatise [**8**], while another part, the one more directly related to Lie's work, became a well developed and far reaching formal geometric theory of differential equations with manifold applications [**5, 10, 32, 36**].

This paper presents some basic aspects of the rich interplay among topics appearing in the research of Lie, Darboux and Cartan. Specifically, we consider Lie symmetries and their (local/nonlocal) generalizations [**5, 32, 36**], we present briefly Cartan's internal symmetries (see [**5**]), we explain why nonlocal symmetries can be considered as a general kind of internal symmetries, and we construct Darboux transformations. The precise contents of this work are as follows:

Section 2 is on the differential geometric framework used in the study of differential equations. Most of the material in this section is well-known to specialists, but it may be of interest to the general reader, and it is needed for our discussion on symmetries. In Section 3 we present an all-too-brief account of Gromov's h-principle, as a direct application of the structures introduced in Section 2. We include a short discussion on holonomic approximation, following Eliashberg and Mishachev [**20**], and we state one of its applications: the construction of approximated solutions to PDEs. In Section 4 we consider local and nonlocal symmetries of differential equations. The local theory is of course classical [**5, 10, 32, 36**], but there is still room for research in the nonlocal domain. We present a recent approach to nonlocal symmetries [**45, 44, 25, 26**] which is in turn based on the theory of coverings and diffieties of J. Krasil'shchik, A. Vinogradov and their coworkers [**49, 30, 31, 32**], and we advance a possible generalization of the theory by means of exterior differential systems. As a computational application, we recall a recent classification of nonlocal symmetries of the Camassa-Holm and associated Camassa-Holm equations after [**25, 26**]. Finally, in Section 5, we show by means of an example how we can construct Darboux transformations using nonlocal symmetries.

2. Jet bundles and differential equations

First of all, we observe that a differential equation (hereafter understood as either a scalar equation or a system) $\Xi = 0$, in which Ξ depends on independent variables $x_1, \ldots, x_n$, dependent variables $u^1, \ldots, u^m$, and a finite number of partial derivatives of the functions u^α with respect to the variables x_j, can be thought

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of as determining a *locus* in a suitable space in which $x_1, \ldots, x_n$, $u^1, \ldots, u^m$, and the partial derivatives $\partial^k u^\alpha / \partial x_j^k$ are coordinates. By restricting this locus, we can assume that the equation $\Xi = 0$ determines a submanifold of this space called the "equation manifold" of $\Xi = 0$. We now present some basic constructions which formalize these comments. Our main tool is Charles Ehresmann's theory of jets [**18, 19**] and its generalization to infinite dimensions [**32**].

2.1. Jet bundles. We begin with an $(n+m)$–dimensional fiber bundle $\pi : E \to M$. The n-dimensional manifold M is the space of independent variables x_i, $1 \le i \le n$, and the m-dimensional typical fiber is the space of the dependent variables u^α, $1 \le \alpha \le m$. If the reader wishes to assume that the bundle π is simply $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, he/she can safely do so: one of the most interesting characteristics of the theory is that even if it looks like the researcher is "trivializing" everything because he/she is working locally, the fact that we are primarily interested in properties of differential equations and their solutions makes the geometry highly nontrivial. Perhaps the best examples of this are the ideal of contact forms which we will introduce momentarily, and the theory of conservation laws as it appears in [**4, 32, 36**].

Let $s_1(x_i) = (x_i, s_1^\alpha(x_i))$ and $s_2(x_i) = (x_i, s_2^\alpha(x_i))$ be two local sections of the bundle $E \to M$ defined about a point $p = (x_i)$ in M. We say that s_1 and s_2 *agree to order k at $p \in M$* if s_1^α, s_2^α and *all* the partial derivatives of the functions s_1^α and s_2^α, up to order k, agree at p. This notion determines a coordinate-independent equivalence relation on local sections of E. We let $j^k(s)(p)$ represent the equivalence class of the section s at p, and we call this equivalence class *the k–jet of s at p*. We have

Definition 2.1. The *k–order jet bundle of E* is the space

$$J^k E = \bigcup_{p \in M} J^k(p),$$

in which $J^k(p)$ denotes the set of all the k–jets $j^k(s)(p)$ of local sections s at p.

The jet bundle $J^k E$ possesses a natural manifold structure; it fibers over $J^l E$ $(l < k)$ and also over M: local coordinates on $J^k E$ are $(x_i, u_0^\alpha, u_{i_1}^\alpha, u_{i_1 i_2}^\alpha, \ldots, u_{i_1 \ldots i_k}^\alpha)$, in which

(2.1)
$$u_0^\alpha(j^k(s)(p)) = s^\alpha(p), \qquad u_i^\alpha(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x_i}(p), \quad u_{i_1 i_2}^\alpha(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x_{i_1} \partial x_{i_2}}(p)\,,$$

and so forth, where $j^k(s)(p) \in J^k E$ and $(x^i) \mapsto (x^i, s^\alpha(x^i))$ is any local section in the equivalence class $j^k(s)(p)$. In these coordinates, the projection maps $\pi_M^k : J^k E \to M$ and $\pi_l^k : J^k E \to J^l E$, $l < k$, are obvious. For instance, π_M^k is simply $(x_i, u_0^\alpha, u_{i_1}^\alpha, \ldots, u_{i_1 \ldots i_k}^\alpha) \mapsto (x_i)$.

Any local section $s : (x_i) \mapsto (x_i, s^\alpha(x_i))$ of E lifts to a unique local section $j^k(s)$ of $J^k E$ called the *kth prolongation* of s. In coordinates, $j^k(s)$ is the section

$$j^k(s)(x_i) = \left(x_i, s^\alpha(x_i), \frac{\partial s^\alpha}{\partial x_{i_1}}(x_i), \ldots, \frac{\partial^k s^\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}(x_i), \ldots \right).$$

We use prolongations to define a special class of differential forms:

Definition 2.2. A differential form ω on $J^k E$ is a *contact form* if $j^k(s)^* \omega = 0$ for all local sections s of E.

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A careful discussion of contact forms appears in [**37**]. The set of contact forms, $\mathcal{I}^k(E)$, is an ideal of the ring of differential forms on J^kE. Locally, $\mathcal{I}^k(E)$ is generated by the *basic contact one–forms*

$$\theta^\alpha_{i_1\dots i_r} = d\,u^\alpha_{i_1\dots i_r} - \sum_{j=1}^n u^\alpha_{i_1\dots i_r j}\,d\,x_j, \qquad 0 \le r \le k-1\ . \tag{2.2}$$

Contact forms allow us to define the *Cartan connection* on J^kE by specifying horizontal subspaces: a vector field X on J^kE is *horizontal* if and only if $i_X\omega = 0$ for all one–forms $\omega \in \mathcal{I}^k(E)$. It follows from the Frobenius theorem that the distribution on J^kE determined by the Cartan connection *is not* integrable. Indeed, it is not hard to check that the exterior derivative of $\theta^\alpha_{i_1\dots i_r}$ is given by

$$d\theta^\alpha_{i_1\dots i_r} = \sum_{j=1}^n dx_j \wedge \theta^\alpha_{i_1\dots i_r j}\ . \tag{2.3}$$

Locally, horizontal vector fields are linear combinations of the *total derivative operators* (here, and whenever convenient hereafter, we use the Einstein summation convention)

$$D_j = \frac{\partial}{\partial x_j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u^\alpha_{i_1 j}\frac{\partial}{\partial u^\alpha_{i_1}} + u^\alpha_{i_1 i_2 j}\frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \cdots\ . \tag{2.4}$$

Now we consider lifts using the Cartan connection. If V is a vector field on the base space M, its *kth prolongation*, pr^kV, is defined as follows, see [**30**]: the action of pr^kV on a smooth function f on J^kE at a point $\sigma = j^k(s)(p) \in J^kE$, $p \in M$, is given by

$$pr^kV\,(\sigma)\cdot f = V(p)\cdot(f\circ j^k(s))\ . \tag{2.5}$$

For instance, the *kth* prolongation of the partial derivative $\partial/\partial x_j$ is precisely the operator D_j defined in (2.4). On the other hand, a vector field V on E can be canonically prolonged to an horizontal vector field $pr^{(k)}V$ on J^kE [**5, 30, 32, 36**]: if

$$V = \sum_{i=1}^n \xi^i(x_j,u^\beta)\,\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^m \varphi^\alpha(x_j,u^\beta)\,\frac{\partial}{\partial u^\alpha},$$

the kth prolongation of V is the vector field $pr^{(k)}V$ given by the formula

$$pr^{(k)}V = \sum_{i=1}^n \xi^i\,\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^m \sum_{0\le \#J\le k} \varphi^\alpha_J\,\frac{\partial}{\partial u^\alpha_J}, \tag{2.6}$$

in which J is a multi-index understood as either an *unordered* k-tuple of integers $J = (j_1,\dots,j_k)$, $1 \le j_1,\dots,j_k \le n$ or $J = 0$, $\#J = k$, $\#0 = 0$, and the functions φ^α_J are obtained inductively by means of

$$\varphi^\alpha_0 = \varphi^\alpha\ ; \qquad \varphi^\alpha_{J\,i} = D_i\,\varphi^\alpha_J - \sum_{k=1}^n D_i(\xi^k)\,u^\alpha_{J\,k}\ . \tag{2.7}$$

We note that if V is a vector field on E, the flow Φ_ϵ of V can be canonically prolonged to a flow $\Phi^{(k)}_\epsilon$ on J^kE, see [**36**, p. 98]. The vector field $pr^{(k)}V$ just defined is precisely the infinitesimal generator of $\Phi^{(k)}_\epsilon$, see [**36**, pp. 101, 110].

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DEFINITION 2.3. A smooth map $\Phi : J^kE \to J^kE'$ is called a *contact transformation* if it preserves contact ideals, that is, if

$$\Phi^*\mathcal{I}^k(E') \subseteq \mathcal{I}^k(E)\ . \tag{2.8}$$

Contact transformations are the most general kind of transformations between jet bundles which "send solutions to solutions". Thinking in terms of coordinates, contact transformations are determined by mappings which mix independent variables, dependent variables and derivatives of the dependent variables. Thus, the standard distinction between independent and dependent variables *is not* preserved by contact transformations. On the other hand, contact transformations are not as general as we would think. We quote a classical theorem by A.V. Bäcklund [**5, 37**]:

THEOREM 2.4. *Let $\Phi : J^kE \to J^kE$ be a contact transformation. If the fibers of E are one-dimensional, then Φ is the $(k-1)$th prolongation of a first order contact transformation, while if the fibers of E are of dimension greater than one, then Φ is the kth prolongation of a transformation $\phi : E \to E$.*

EXAMPLE 2.5. An interesting contact transformation has been discovered by O. Morozov [**35**]: consider the generalized Hunter-Saxton equation

$$u_{xt} = u\,u_{xx} + \kappa u_x^2, \quad k \neq 1, \tag{2.9}$$

and the *linear* Euler-Poisson equation

$$u_{xt} = \frac{1}{\kappa(t+x)}\,u_t + \frac{2(1-\kappa)}{\kappa(t+x)}\,u_x - \frac{2(1-\kappa)}{(\kappa(t+x))^2}\,u\ . \tag{2.10}$$

Morozov proved in [**35**] that the contact transformation

$$\begin{aligned}
\tilde{u} &= (t+x)^{-1/\kappa}\,(\kappa(t+x)u_x + (\kappa-1)u)\\
\tilde{t} &= \kappa^{-1}t\\
\tilde{x} &= -(t+x)^{(\kappa-1)/\kappa}\,(\kappa(t+x)u_x - u)\\
\tilde{u}_{\tilde{t}} &= \kappa^2(t+x)^{-1/\kappa}\,(u_t - u_x)\\
\tilde{u}_{\tilde{x}} &= -(t+x)^{-1}
\end{aligned}$$

maps the linear equation (2.10) into Equation (2.9) written in variables with tildes.

Further information on these important transformations appears for example in [**4, 5, 23, 32, 37**]. For instance, it is shown in [**23**], following earlier work by R. Gardner (the precise reference appears in [**23**]) that the classical division of second order scalar PDEs in two independent variables into equations of hyperbolic, parabolic and elliptic type can be made intrinsically, in a manifestly contact invariant fashion.

2.2. Infinite jet bundles. The *infinite jet bundle* $\pi_M^\infty : J^\infty E \to M$ is the inverse limit of the tower of jet bundles $M \leftarrow E \cdots \leftarrow J^kE \leftarrow J^{k+1}E \leftarrow \cdots$ under the standard projections $\pi_l^k : J^kE \to J^lE$, $k > l$. The limit $J^\infty E$ is a topological space: a basis for a topology on $J^\infty E$ is the collection of all sets of the form $(\pi_k^\infty)^{-1}(W)$, in which W is an open subset of J^kE. We describe the space $J^\infty E$ locally by sequences

$$(x_i, u^\alpha, u^\alpha_{i_1}, ..., u^\alpha_{i_1 i_2 \ldots i_k}, ...), \quad 1 \le i_1 \le i_2 \le \cdots \le i_k \le n, \tag{2.11}$$

obtained from the standard coordinates on the finite–order jet bundles J^kE. We would like to consider $J^\infty E$ as a manifold. However, since (2.11) implies that

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the dimension of $J^\infty E$ must be infinite, we need to re-define all of the standard differential geometry notions in this new context, see [**4, 30, 32**]:

Let $\pi_k^\infty : J^\infty E \to J^k E$ denote the canonical projection from $J^\infty E$ onto $J^k E$. We say that a function $f : J^\infty E \to \mathbb{R}$ is *smooth* if it factors through a finite–order jet bundle, that is, if $f = f_k \circ \pi_k^\infty$ for some smooth function $f_k : J^k E \to \mathbb{R}$. Real-valued smooth functions on $J^\infty E$ are also called *differential functions*, see [**36**]. More generally, if P is some finite-dimensional manifold, a function $f : J^\infty E \to P$ is smooth if there exist $k \geq 0$ and a smooth function $f_k : J^k E \to P$ such that $f = f_k \circ \pi_k^\infty$, and a function $\tilde{f} : P \to J^\infty E$ is smooth if for any finite-dimensional manifold Q and any smooth map $g : J^\infty E \to Q$ the composition $g \circ \tilde{f}$ is smooth. Finally, if $F \to N$ is another fiber bundle, a function $\Phi : J^\infty E \to J^\infty F$ is smooth if for every smooth map $g : J^\infty F \to Q$, the composition $g \circ \Phi : J^\infty E \to Q$ is smooth.

Vector fields X on $J^\infty E$ are derivations on the ring of smooth real-valued functions on $J^\infty E$. In local coordinates, vector fields are formal series of the form

$$X = A_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \geq 0 \\ 1 \leq i_1 \leq \cdots \leq i_k \leq n}} B^\alpha_{i_1 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_k}}, \tag{2.12}$$

in which $A_i, B^\alpha_{i_1 \ldots i_k}$ are smooth functions on $J^\infty E$.

Now, in analogy with the finite-dimensional case, any local section $s : (x_i) \mapsto (x_i, s^\alpha(x_i))$ of E lifts to a unique local section $j^\infty(s)$ of $J^\infty E$ called the *infinite prolongation* of s. In coordinates, $j^\infty(s)$ is the section

$$j^\infty(s)(x_i) = \left(x_i, s^\alpha(x_i), \frac{\partial s^\alpha}{\partial x_{i_1}}(x_i), ..., \frac{\partial^k s^\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}(x_i), ... \right) .$$

Also, if V is a vector field on the base space M, the *infinite prolongation*, $pr^\infty V$ can be defined as in (2.5): the action of $pr^\infty V$ on a smooth function f on $J^\infty E$ at a point $\sigma = j^\infty(s)(p) \in J^\infty E$, $p \in M$, is given by

$$pr^\infty V(\sigma) \cdot f = V(p) \cdot (f \circ j^\infty(s)) . \tag{2.13}$$

Particularly important are the infinite prolongations of the partial derivatives $\partial/\partial x_j$. We obtain

$$D_j = pr^\infty \left(\frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u_{i_1 j} \frac{\partial}{\partial u^\alpha_{i_1}} + u_{i_1 i_2 j} \frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \cdots . \tag{2.14}$$

As before, the operators D_j are called *total derivative* operators. The infinite prolongation $pr^\infty V$ on $J^\infty E$ of a vector field V on E is defined exactly as in (2.6), (2.7). The same formulae hold for $pr^\infty V$, the only difference being that in this case we need to sum over all multi-indices J with $\#J \geq 0$ in (2.6).

Now we define the *Cartan connection* on $J^\infty E$: the horizontal lift of a vector field V on M —also called the *total derivative in the V direction*— is simply $pr^\infty V$. Locally, horizontal vector fields are linear combinations of the total derivatives D_j defined in (2.14).

It is possible to check, see [**30, 32**], that in contrast with finite-dimensional jet bundles, $pr^\infty[V_1, V_2] = [pr^\infty V_1, pr^\infty V_2]$ for all vector fields V_1 and V_2 on M, and therefore the Cartan connection determines, at least formally, an integrable distribution on $J^\infty E$ which we denote $\mathcal{C}$. Locally, the maximal integral manifolds of $\mathcal{C}$ are the graphs of the infinite prolongations $j^\infty(s)$ of local sections $s : M \to E$, see [**32**].

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Finally, to complete this brief description of differential geometry on $J^\infty E$, we consider differential forms: by definition, smooth differential forms on $J^\infty E$ are the pull–backs of differential forms on $J^k E$ by the projections π_k^∞. Thus, any differential k–form ω on $J^\infty E$ may be written in canonical coordinates as a finite linear combination of terms

$$A\, d\, x_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge du^{\alpha_1}_{j_1 \ldots j_{p_1}} \wedge \cdots \wedge d\, u^{\alpha_q}_{l_1 \ldots l_{p_q}}\,, \tag{2.15}$$

in which $p + q = k$ and A is a smooth function on $J^\infty E$.

2.3. Differential equations. Consider a k th order differential equation (or more generally, a k th order system of DEs)

$$\Xi\left(x_i, u^\alpha, \frac{\partial u^\alpha}{\partial x_i}, ..., \frac{\partial^k u^\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}\right) = 0\,. \tag{2.16}$$

This equation defines a locus in the jet bundle $J^k E$. We restrict this locus to a submanifold $S^{(k)}$ of $J^k E$, and we assume that the function Ξ is smooth on a neighbourhood of $S^{(k)}$. We further ask $S^{(k)}$ to be a sub–bundle of $J^k E$, so that it fibers over the space M of independent variables. This sub-bundle is called the *equation manifold* of $\Xi = 0$.

The *full equation manifold* $S^{(\infty)}$ of $\Xi = 0$ is a sub-bundle of the infinite jet space $J^\infty E$ constructed thus: The *prolongations* $S^{(k+1)}$, $S^{(k+2)}$, ... of $S^{(k)}$ are defined by total differentiation of (2.16), so that, for instance,

$$\begin{aligned} S^{(k+1)} \quad = \quad & \{j^{k+1}(s)(x_i) \in J^{k+1}E : j^k(s)(x_i) \in S^{(k)} \\ & \text{and } (D_i \Xi)(j^{k+1}(s)(x_i)) = 0 \text{ for all } i = 1, 2, \ldots, n\}\,. \end{aligned}$$

We assume that the tower $M \leftarrow S^{(k)} \leftarrow S^{(k+1)} \leftarrow \cdots$ is well defined, that is, each $S^{(l+1)}$ $(l \geq k)$ is a submanifold of $J^{l+1}E$ which fibers over $S^{(l)}$. We then define $S^{(\infty)}$ as the inverse limit of the tower $M \leftarrow S^{(k)} \leftarrow S^{(k+1)} \leftarrow \cdots$. Locally, $S^{(\infty)}$ is the set of infinite jets in $J^\infty E$ satisfying (2.16) and *all* its (total) differential consequences.

A *local solution* of Equation (2.16) is a local section of $S^{(k)}$ which is the *kth* prolongation of a local section $s : (x_i) \mapsto (x_i, u^\alpha(x_i))$ of E. A *local smooth solution* of Equation (2.16) is a local section of of $S^{(\infty)}$ which is the infinite prolongation of a local section $s : (x_i) \mapsto (x_i, u^\alpha(x_i))$ of E.

If $\sigma = j^\infty(s)(p)$ is a point in $S^{(\infty)}$, we define the horizontal subspace $\mathcal{C}_\sigma(S^{(\infty)})$ of $T_\sigma S^{(\infty)}$ as $\mathcal{C}_\sigma(S^{(\infty)}) = T_\sigma S^{(\infty)} \cap \mathcal{C}_\sigma$, in which $\mathcal{C}_\sigma$ is the horizontal subspace of the Cartan distribution on $J^\infty E$ at σ. Thus, there exists a connection on $S^{(\infty)}$ —also called the *Cartan connection*— obtained by restricting the Cartan connection on $J^\infty E$. Locally, the *horizontal vector fields* on $S^{(\infty)}$ are linear combinations of the restriction to this manifold of the total derivatives D_i defined in (2.14).

Finally, we make the important remark that vector fields and differential forms on $S^{(k)}$ and $S^{(\infty)}$ are defined via pull-back by the canonical inclusions $\iota^{(k)} : S^{(k)} \hookrightarrow J^k E$ and $\iota : S^{(\infty)} \hookrightarrow J^\infty E$. For example, the *contact ideal* of $S^{(k)}$, $\mathcal{I}(S^{(k)})$, is the contact ideal $\mathcal{I}^k(E)$ of $J^k E$ pulled-back to $S^{(k)}$ by the inclusion ι. Examples of contact ideals of equations of interest for mathematical physics appear in [**23**].

Example 2.6. Let $u_t = F(x, t, u, ..., u_{x_k})$ be an evolution equation in two independent variables. Natural coordinates on S^∞ are $(x, t, u, u_x, ..., u_{xx\ldots x}, ...)$, as

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derivatives of u with respect to t may be replaced by expressions containing only x–derivatives by using the differential consequences of $u_t = F$. The horizontal vector fields on $S^{(\infty)}$ are linear combinations of the total derivatives (2.4) restricted to $S^{(\infty)}$, that is,

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} \quad \text{and} \quad D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D_x^i(F) \frac{\partial}{\partial u_i}, \tag{2.17}$$

and the basic contact forms (2.2) restricted to $S^{(\infty)}$ become

$$\theta_J^\alpha = d\, u_J^\alpha - u_{J\,x} dx - D_J(F) dt \ ,$$

in which $J = (j_1, \ldots, j_k)$ is a multi-index, $u_{J\,x}$ abbreviates $u_{x_{j_1} x_{j_2} \ldots x_{j_k}\, x}$ where each x_{j_i} is either x or t, and $D_J = D_{x_{j_1}} D_{x_{j_2}} \ldots D_{x_{j_k}}$. These formulae are at the basis of the algebraic approach to integrability of evolutionary PDEs, see for instance [**33, 34**].

3. A glimpse of Gromov's h-principle

We generalize the geometric definition of a differential equation: a *differential relation* $\mathcal{R}$ of order k is an *arbitrary* sub-manifold of $J^k E$. In particular, we do not insist that $\mathcal{R}$ fibers over the base space M. For example, consider $M = \mathbf{R}^n$, $E = \mathbf{R}^n \times \mathbf{R}^q$, and take

$$\mathcal{R} = \left\{ (x_1, \ldots, x_n, u^1, \ldots, u^q, u^1_{x_i}, \ldots, u^q_{x_i})_{i=1,\ldots n} : \operatorname{rank} \begin{pmatrix} u^1_{x_1} & \ldots & u^1_{x_n} \\ \vdots & \vdots & \vdots \\ u^q_{x_1} & \ldots & u^q_{x_n} \end{pmatrix} = n \right\},$$

so that $\mathcal{R}$ is an open subset of $J^1 E$. Let us identify a function $f : M \to \mathbf{R}^q$, $f(x_i) = (u^\alpha(x_i))$ with the section $s_f : (x_i) \mapsto (x_i, u^\alpha(x_i))$ of E. We see that the function f is an immersion if and only if the section s_f *solves* the differential relation $\mathcal{R}$, that is —again generalizing from the previous section— if and only if the first order prolongation of s_f, $j^1(s_f)$, is contained in $\mathcal{R}$.

Now, we can also consider *formal solutions* to a given differential relation $\mathcal{R} \subseteq J^k E$. We simply state that a formal solution to $\mathcal{R}$ is a section $s : M \to J^k E$ such that $s(M) \subset \mathcal{R}$. We now define, after [**20**]:

Definition 3.1. A differential relation $\mathcal{R} \subseteq J^k E$ satisfies Gromov's h-principle if every formal solution s of $\mathcal{R}$ is homotopic (within $\mathcal{R}$) to a genuine solution to $\mathcal{R}$, that is, if there exists a one-parameter family of sections $s_t : M \to J^k E$, $0 \le t \le 1$, such that $s_t(M) \subset \mathcal{R}$ for all t, $s_0 = s$, and $s_1 : M \to J^k E$ is of the form $s_1 = j^k(s')$ for a section s' of E.

A section of $J^k E$ such as s_1 in Definition 3.1 is called an *holonomic section* of $J^k E$. Note that in the special case of a kth order differential (system of) equation(s)

$$\Xi \left(x_i, u^\alpha, \frac{\partial u^\alpha}{\partial x_i}, ..., \frac{\partial^k u^\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}} \right) = 0, \tag{3.1}$$

a formal solution would be simply a section $s : M \to S^{(k)}$, and we would say that (3.1) satisfies Gromov's h-principle if there exists a one-parameter family of sections $s_t : M \to S^k$, $0 \le t \le 1$, such that $s_0 = s$ and s_1 is a local solution of Equation (3.1).

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A very readable exposition of Gromov's h-principle appears in Y. Eliashberg and N. Mishachev's book [**20**]. As stated in this reference, it is not usually the case that non-trivial PDEs satisfy the h-principle, but on the other hand, there are some geometrically significative differential relations which do obey it. Eliashberg and Mishachev proved in 2001 (see [**20**] and references therein) an approximation theorem for sections of jet bundles from which several "h-principle results" follow. We state it below, after making a few notational remarks:

Let us fix a metric $dist : J^kE \times J^kE \to \mathbf{R}$ on J^kE. (For instance, we could prescribe a Riemannian metric tensor on J^kE and use it to define a metric function). We define an *isotopy* $\psi_t : M \to M$, $0 \leq t \leq 1$, as a one-parameter family of diffeomorphisms of M such that $\psi_0 = id_M$, and we say that the isotopy ψ_t is *δ-small* if $dist(\psi_t(p), p) < \delta$ for all $p \in M$ and $0 \leq t \leq 1$. We also recall that a *sub-polyhedron* $P \subset M$ is a subcomplex of a triangulation of M, and that P is *proper* if no simplex of P is of dimension equal to the dimension of M. (Triangulations of not necessarily compact manifolds are discussed in Whitney's treatise [**51**]).

THEOREM 3.2 (Holonomic approximation theorem, [**20**]). *Let P be a proper sub-polyhedron of M and let $s : U \to J^kE$ be a local section, in which U is an open neighborhood of P. For arbitrary $\delta, \epsilon > 0$ there exists a δ-small isotopy $\psi_t : M \to M$, $0 \leq t \leq 1$, and a holonomic section $\tilde{s} : \widetilde{U} \to J^kE$, in which $\widetilde{U}$ is an open neighborhood of the image $\psi_1(P)$, such that $\psi_1(P)$ is contained in the domain of the original section s and*

$$dist\,(\tilde{s}(p)\,,\, s(p)) < \epsilon \quad \textit{for all } p \in \widetilde{U}\ .$$

It is usually assumed that M is compact, but this restriction is not necessary, see [**20**]. In order to apply this result to differential equations, let us agree that for an *ϵ-approximate solution* to $S^{(k)}$ we mean a holonomic section σ of J^kE such that σ solves the open differential relation $\mathcal{R}_\epsilon$, in which $\mathcal{R}_\epsilon$ is an open neighborhood of $S^{(k)}$ satisfying $dist(p, S^{(k)}) < \epsilon$ for all $p \in \mathcal{R}_\epsilon$. We have [**20**]:

COROLLARY 3.1. *Let $S^{(k)} \subset J^kE$ be a closed hypersurface, $P \subset M$ a proper sub-polyhedron of M and assume that $s : U \to S^{(k)} \subset J^kE$ is a formal solution to $S^{(k)}$, in which U is an open neighborhood of P. For arbitrary $\delta, \epsilon > 0$ there exist a δ-small isotopy $\psi_t : M \to M$, $0 \leq t \leq 1$, and an ϵ-approximate solution $j^k(\tilde{s})$ to $S^{(k)}$, such that $\psi_1(P)$ is contained in the domain of s and*

$$dist\left(j^k(\tilde{s})(p)\,,\, s(p)\right) < \epsilon \quad \textit{for all } p \in \widetilde{U},$$

in which $\widetilde{U}$ is an open neighborhood of $\psi_1(P)$ contained in the domain of $j^k(\tilde{s})$.

In other words, given a formal solution to $S^{(k)}$ with domain $D \subseteq M$, it is always possible to deform D slightly to a domain $D' \subseteq M$, and to find an ϵ-approximate solution to $S^{(k)}$ with domain D':

EXAMPLE 3.3. Let $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$ be a scalar first order equation with $(x_1, \ldots, x_n)$ in an open set $U \subset \mathbf{R}^n$, and suppose we wish to solve the Cauchy problem with initial data $u|_\Gamma = g$, in which Γ is a subset of ∂U. This problem is treated in classical books such as [**21**]. First of all, we use the data $u|_\Gamma = g$ to construct admissible initial data on a subset of $J^1(\mathbf{R}^n \times \mathbf{R})$: we fix a point $x^0 \in \Gamma$, and we determine the values of the derivatives u_{x_i} for $y \in \Gamma$ near x^0 [**21**]. We obtain an $(n-1)$-dimensional submanifold Γ' contained in the equation manifold $S^{(1)}$ of $F = 0$. We can now take an open neighborhood of Γ' in $S^{(1)}$ and

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any section $\tilde{s}$ extending a section s determined by Γ'. That such a section $\tilde{s}$ exists is guaranteed by the smooth Tietze extension theorem [**1**, p. 380]. This section is a formal solution to the equation $F = 0$, which extends the initial data Γ'. The last corollary implies that for any $\epsilon > 0$ there exists an ϵ-approximate solution to $F = 0$ arbitrarily near a "small perturbation" of the original set Γ.

4. Symmetries of differential equations

Let us assume for a moment that $\Xi(x^i, u, \dots) = 0$ is a scalar partial differential equation. Intuitively, a symmetry of $\Xi = 0$ is any function G such that for any solution $u(x^i)$ to $\Xi = 0$, the deformed function $u(x^i) + \tau G(u(x^i))$ is also a solution, to first order in τ. Symmetries may be local (*i.e.*, G may be a smooth function on some finite or infinite-dimensional jet bundle) or nonlocal (*i.e.*, G may depend for instance on integrals of u). Local symmetries of differential equations are classical [**5, 10, 36, 37**], and therefore we introduce them very quickly, just to fix our notation. We then concentrate in nonlocal symmetries following [**45, 44, 25, 26**].

4.1. Local symmetries.

4.1.1. *Classical symmetries.* Let $\Xi(x_i, u^\alpha, \dots, u^\alpha_J) = 0$ be an nth order (system of) partial differential equation(s). A vector field V on the space E of independent and dependent variables,

$$V = \sum_{i=1}^{n} \xi^i(x_j, u^\beta) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \varphi^\alpha(x_j, u^\beta) \frac{\partial}{\partial u^\alpha},$$

is a *classical symmetry* of $\Xi = 0$ if the equations

$$pr^{(n)} V (\Xi) = 0 \tag{4.1}$$

hold whenever $u^\alpha(x_i)$ is a solution to $\Xi = 0$, see [**5, 10, 32, 36**].

The flow of the vector field V on the space E is determined by the system of equations

$$\frac{d\,x_i}{d\,\tau} = \xi^i(x_j, u^\beta), \qquad \frac{d\,u^\alpha}{d\,\tau} = \varphi^\alpha(x_j, u^\beta), \tag{4.2}$$

and it can be rigorously proven [**36**] that if Equation (4.1) holds and the system $\Xi = 0$ satisfies some mild technical conditions, then for each value of the parameter τ the transformation $x_i \mapsto x_i(\tau)$, $u^\alpha \mapsto u^\alpha(\tau)$ sends solutions of $\Xi = 0$ to solutions of the same system.

Example 4.1. In [**36**, pp. 117–120], P.J. Olver computes all classical symmetries of the heat equation $u_t = u_{xx}$. One of them is

$$V = 4t^2 \frac{\partial}{\partial t} + 4tx \frac{\partial}{\partial x} - (x^2 + 2t) u \frac{\partial}{\partial u}\ .$$

The flow of this vector field is calculated via Equations (4.2). We obtain that if $u(x, t)$ solves the heat equation, so does the function

$$u'(x, t) = \frac{1}{\sqrt{1 + 4\tau t}} \exp\left(-\tau x^2/(4\tau t + 1)\right) u\left(\frac{x}{1 + 4\tau t}, \frac{t}{1 + 4\tau t}\right)\ . \tag{4.3}$$

In particular, if the "seed" solution u is a non-zero constant, $u(x, t) = c$, we conclude that the function

$$u'(x, t) = \frac{c}{\sqrt{1 + 4\tau t}} \exp\left(\frac{-\tau x^2}{4\tau t + 1}\right) \tag{4.4}$$

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also solves the heat equation. As observed in [**36**], this is, up to translations and constants, the fundamental solution to the heat equation. This interesting observation may seem slightly counterintuitive at first: it looks like we are "creating" heat (the solution $u'(x,t)$) from equilibrium (the solution $u(x,t)=c$) as we move along τ! But of course this is not so, because τ is not "physical time".

P. Olver's observation has been the starting point of several investigations. Notably, M. Craddock and A. Dooley [**13**] and then M. Craddock and E. Platen [**14**] have characterized all functions $f(x)$ for which the equations $u_t = u_{xx} + f(x)u_x$ and $u_t = xu_{xx} + f(x)u_x$ admit point symmetries carrying constant solutions to fundamental solutions.

4.1.2. *Generalized symmetries.* We consider first order differential operators on $J^\infty E$ of the special form

$$V = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \varphi^\alpha \frac{\partial}{\partial u^\alpha}, \tag{4.5}$$

in which ξ^i and φ^α are smooth functions on $J^\infty E$. Geometrically, a differential operator such as (4.5) is a vector field *along the projection* π_E^∞, that is, a smooth map $V : J^\infty E \to TM$ such that for all $\sigma \in J^\infty E$, $V(\sigma) \in T_{\pi_E^\infty(\sigma)}E$.

Following established terminology, [**5, 36, 37**], we call V a *generalized vector field.* V is a *generalized symmetry* of the system $\Xi = 0$ if and only if $pr^\infty V\,(\Xi) = 0$ whenever $u^\alpha(x_i)$ is a solution of $\Xi = 0$, in which $pr^\infty V$ is the infinite prolongation of V defined in Section 2.2. We can split $pr^\infty V$ using the Cartan connection of $J^\infty E$ in horizontal and vertical parts, $pr^\infty V = V_H + Vert(V)$. Since V_H is a finite linear combination of total derivatives, it *always* satisfies $V_H(\Xi) = 0$ on (local smooth) solutions, and therefore it is enough to consider generalized symmetries V such that $pr^\infty V$ is a vertical vector field. Such a V must be of the form [**5, 30, 32, 36**]

$$V = \sum_{\alpha=1}^{m} G^\alpha \frac{\partial}{\partial u^\alpha}, \tag{4.6}$$

for some m-tuple of differential functions $G = (G^\alpha)$, so that its infinite prolongation is simply

$$pr^\infty V = \sum_{\alpha=1}^{m} \sum_{\#J\geq 0} D_J G^\alpha \frac{\partial}{\partial u_J^\alpha},$$

where $D_J = D_{j_1} D_{j_2} \dots D_{j_k}$ for $J = (j_1, j_2, \dots j_k)$. The symmetry condition $pr_V^\infty(\Xi) = 0$ for generalized vector fields such as (4.6) says precisely that the "infinitesimal deformation" $u^\alpha(x_i) \mapsto u^\alpha(x_i) + \tau G^\alpha(u^\alpha(x_i))$ is —to first order in τ— a solution of $\Xi = 0$ whenever $u^\alpha(x_i)$ is a solution of $\Xi = 0$.

At least at a formal level, a generalized symmetry V_G of a system of equations $\Xi = 0$ transforms solutions into solutions: if $u_0^\alpha(x_i)$ is a solution to $\Xi = 0$, we consider the Cauchy problem for the "integral curve" of the generalized vector field $V = \sum_{\alpha=1}^m G^\alpha\, \partial/\partial u^\alpha$ through $u_0^\alpha(x_i)$, namely,

$$\frac{\partial u^\alpha}{\partial \tau} = G^\alpha\ ; \qquad u^\alpha(x_i, 0) = u_0^\alpha(x_i)\ . \tag{4.7}$$

Then as proven in [**36**], for any value of the parameter τ, a solution $u^\alpha(x_i, \tau)$ to the Cauchy problem (4.7) —assuming it exists, see [**5, 10, 36**]— is also a solution to the original system $\Xi = 0$.

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4.1.3. *Internal symmetries.* An infinitesimal contact transformation is a vector field X on J^kE such that $X(\mathcal{I}^k(E)) \subseteq \mathcal{I}^k(E)$, that is, the Lie derivative of any contact form with respect to X is again a contact form. An infinitesimal contact transformation X is an *external* symmetry of a system of differential equations $\Xi = 0$ if $X(\Xi) = 0$ on solutions to $\Xi = 0$. An *internal symmetry* of a system of equations $\Xi = 0$ of order k is a vector field W on $S^{(k)}$ (so that the condition $W(\Xi) = 0$ holds) such that $W(\mathcal{I}(S^{(k)})) \subseteq \mathcal{I}(S^{(k)})$. These symmetries were first considered by E. Cartan, see [**5, 23, 32, 37**] and references therein.

Certainly, if W is an external symmetry, then the restriction of W to $S^{(k)}$ is an internal symmetry. On the other hand, the following basic result holds [**5**]:

THEOREM 4.2. *Let $\Xi = 0$ be a system of equations of order k and let W be a vector field on $S^{(k)}$. If W is an internal symmetry and $V = \pi(W)$ is the projection of W to the bundle E, then V is a generalized symmetry of the system $\Xi = 0$, and moreover, $W = (pr^{(k)}V)|_{S^{(k)}}$.*

The converse to Theorem 4.2 is not necessarily true. In order for a generalized symmetry V to determine an internal symmetry, we must check the contact condition $pr^{(k)}V(\mathcal{I}(S^{(k)})) \subseteq \mathcal{I}(S^{(k)})$, which is not trivial since $pr^{(k)}V$ may depend on derivatives of order $(k+1)$ or higher. A subtle result proven in [**5**] states that if a system of partial differential equations $\Xi = 0$ satisfies a technical condition (dubbed the "descent property") then it is indeed true that every internal symmetry of $\Xi = 0$ arises from a generalized symmetry.

Now, one of the important observations of [**5**] is that the descent property is actually a common property, and so it would seem that genuine internal symmetries are quite hard to come by. However, as it will be briefly discussed in Section 4.3, nonlocal symmetries of partial differential equations are a natural source of "generalized" internal symmetries.

4.2. Nonlocal symmetries.

4.2.1. *Basic definitions.* Intuitively, a function G depending on x, t, u, a finite number of x–derivatives of u, *and for example integrals of* u, is a "nonlocal symmetry" of a scalar equation $\Xi(x, t, u, \dots) = 0$ if for any solution $u(x, t)$ of $\Xi = 0$, the function $u(x, t) + \tau G(u(x, t))$ is also a solution to first order in τ, precisely as in the generalized symmetry case.

These symmetries are indeed important for differential equations. For instance, G. Bluman and his collaborators have used them to find nonlocal transformations relating nonlinear PDEs (a recent summary of their work appears in [**6**]) and Sokolov and Svinolupov [**48**] have used them to classify integrable second order evolutionary PDEs. They have been also used to derive Bäcklund/Darboux transformations [**46, 45, 25**], and to obtain highly nontrivial explicit solutions to nonlinear PDEs [**22, 45, 44, 46**].

A completely general (*i.e.*, without *a priori* restrictions on the form nonlocal objects may take, as it happens for instance in the recent textbook [**10**] on applied aspects of symmetry theory) geometric formulation of nonlocal symmetries has been developed by J. Krasil'shchik and A. Vinogradov, see [**49, 30, 31, 32**] and references therein. In this section we present a version of their theory following [**45, 44, 25, 26**]. We remark that other approaches to nonlocal symmetries (of formal, algebraic or heuristic nature) are possible, see for instance [**2, 3, 6, 10, 38, 39**],

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but we believe the Krasil'shchik-Vinogradov theory is the most satisfying approach to nonlocalities available today (some comments along these lines are also in [**39**]).

We begin with a classical example appearing already in [**49**]:

EXAMPLE 4.3. We consider Burgers' equation $u_t = u_{xx} + uu_x$. The expression

$$G = (2S_x - uS)\,\mathrm{e}^{-\frac{1}{2}\int u\,dx}, \tag{4.8}$$

in which S is any solution to the heat equation $S_t = S_{xx}$, satisfies the following condition: if u is a solution to Burgers' equation, the "deformation" $u + \tau G$ is also a solution to first order in τ, that is, G formally satisfies the linearized Burgers equation. We make this observation rigorous by considering an extra dependent variable γ such that $\gamma_x = u$ and $\gamma_t = u_x + (1/2)\,u^2$. Then, we can write (4.8) as $G = (2S_x - uS)\,\exp(-\frac{1}{2}\gamma)$, so that G becomes "local" and could be perhaps considered as the characteristic of a local symmetry for the "augmented" system

$$u_t = u_{xx} + uu_x\ ; \qquad \gamma_x = u\ ; \qquad \gamma_t = u_x + (1/2)\,u^2\ . \tag{4.9}$$

But then, in order to formalize *this* idea, we also need to consider the infinitesimal variation of γ as u changes to $u + \tau G$! This observation led Vinogradov and Krasil'shchik to develop their theory of *coverings* of PDEs.

DEFINITION 4.4. Let N be a non-zero integer. An N-dimensional covering π of a system of partial differential equations $\Xi = 0$, is a pair

$$\pi = \big(\{\gamma^b : b = 1, \dots, N\}\,; \{X_{ib} : b = 1, \dots, N;\ i = 1, \dots, n\}\big) \tag{4.10}$$

of variables γ^b and smooth functions X_{ib} depending on x^i, u^α, γ^b and a finite number of partial derivatives of u^α, such that the equations

$$\frac{\partial \gamma^b}{\partial x^i} = X_{ib} \tag{4.11}$$

are compatible whenever $u^\alpha(x^i)$ is a solution to $\Xi = 0$.

We usually write $\pi = (\gamma^b\,;\ X_{ib})$ instead of (4.10). Generalizing Example 4.3, we consider the variables γ^b as new dependent variables, the "nonlocal variables" of the theory, and we interpret Equations (4.11) as the equations stating how the variables γ^b relate to the original variables u^α, as in (4.9).

Our definition of nonlocal symmetries is the following (Compare [**32**, pp. 249-250]):

DEFINITION 4.5. Let $\Xi = 0$ be a system of partial differential equations, and let $\pi = (\gamma^b;\ X_{ib})$ be a covering of $\Xi = 0$. A nonlocal π–symmetry of $\Xi = 0$ is a generalized symmetry

$$X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\alpha \phi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_b \varphi^b \frac{\partial}{\partial \gamma^b}$$

of the augmented system

$$\Xi = 0\ ; \qquad \frac{\partial \gamma^b}{\partial x^i} = X_{ib}\ . \tag{4.12}$$

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Thus, in order to find nonlocal π-symmetries, we proceed exactly as in the local case: we check the conditions

$$pr^{\infty} X\,(\Xi) = 0 \quad \text{and} \quad pr^{\infty} X \left(\frac{\partial \gamma^b}{\partial x^i} - X_{ib} \right) = 0 \tag{4.13}$$

whenever $u^\alpha(x^i)$ and $\gamma^b(x^i)$ are solutions to the augmented system (4.12). As we recalled in 4.1.2, in order to capture all generalized symmetries of a given system of equations it is enough to consider evolutionary vector fields X of the form

$$X = \sum_{\alpha=1}^{m} G^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{b=1}^{N} H^b \frac{\partial}{\partial \gamma^b} \ . \tag{4.14}$$

Note that the first equation in (4.13) depends only on the system $\Xi = 0$ *and* on the "nonlocal vector field" $G^\alpha\, \partial/\partial u^\alpha$. J. Krasil'shchik and A. Vinogradov [**30, 31, 32, 49**] call $G = (G^\alpha)$ the *π–shadow* of the nonlocal π–symmetry (4.14). Interestingly, most (if not all) work on nonlocal symmetries carried out outside the Krasil'shchik–Vinogradov framework is *only* about shadows.

Proposition 4.1. *If $u_0^\alpha(x^i)$ and $\gamma_0^b(x^i)$ are solutions to the augmented system* (4.12)*, the solution to the Cauchy problem*

$$\frac{\partial u^\alpha}{\partial \tau} = G^\alpha, \qquad \frac{\partial \gamma^b}{\partial \tau} = H^b \ ;$$

$$u^\alpha(x^i, 0) = u_0^\alpha(x^i), \qquad \gamma_b(x^i, 0) = \gamma_0^b(x^i) \ ;$$

is a one–parameter family of solutions to the augmented system (4.12)*. In particular, nonlocal π–symmetries send solutions to the system $\Xi = 0$ to solutions of the same system.*

Several nontrivial examples of nonlocal symmetries appear in [**49, 22, 24, 32, 42, 43, 45, 44, 46, 25, 26**] and works cited therein. Computations are generally quite demanding, but at least we are allowed to replace all derivatives of γ_b appearing in (4.13) by means of Equations (4.11) and their differential consequences (see [**36**, p. 292]), so that we can assume without loss of generality that the coefficients G^α and H_b of the vector field (4.14) depend *only* on x^i, u^α, finite numbers of derivatives of u^α, and the new variables γ^b.

4.2.2. *Reconstructing shadows.* We begin with an example:

Example 4.6. We complete Example 4.3 on Burgers' equation $u_t = u_{xx} + u u_x$. Recall that we have considered an extra dependent variable γ such that $\gamma_x = u = X_{11}$ and $\gamma_t = u_x + (1/2)\, u^2 = X_{21}$. These two equations are of course compatible on solutions to Burgers'equation, and so we have a one-dimensional covering. Now set $G = (2S_x - uS)\, \exp(-(1/2)\gamma)$, and determine the infinitesimal variation H_1 of γ as u changes to $u + \tau G$: $\gamma_{\tau x} = \gamma_{x\tau} = u_\tau = (2S_x - uS)\, \mathrm{e}^{-\frac{1}{2}\gamma} = (2S\, \mathrm{e}^{-\frac{1}{2}\gamma})_x$, so that $\gamma_\tau = 2S \exp(-\frac{1}{2}\gamma)$, and therefore we can define $H = 2S \exp(-\frac{1}{2}\gamma)$. We can check that the vector field

$$X = G \frac{\partial}{\partial u} + H \frac{\partial}{\partial \gamma}$$

is a generalized symmetry of the equations

$$u_t = u_{xx} + u u_x, \qquad \gamma_x = u, \qquad \gamma_t = u_x + (1/2)\, u^2,$$

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and therefore X is a nonlocal symmetry for Burgers' equation. Following [**49**], we note that solutions $u(x,t)$ which are invariant under this nonlocal symmetry satisfy $G(u(x,t)) = 0$, and so the equation

$$2S_x - uS = 0 \tag{4.15}$$

must hold. This is the Cole–Hopf transformation between the nonlinear Burgers equation $u_t = u_{xx} + uu_x$ and the linear heat equation $S_t = S_{xx}$!

In some cases, see for instance [**45**], (4.14) is a classical symmetry of the augmented system (4.12), but there exist examples for which the vector field (4.14) is a genuine generalized symmetry, see [**22, 42, 43, 44, 25, 26**] and Section 4.3 below.

Shadows can be *always* completed into genuine nonlocal symmetries as we did in Example 4.6, if we extend the foregoing theory to infinite-dimensional coverings as in [**30, 31, 32**]:

PROPOSITION 4.2. *Let $\pi = (\gamma_b;\ X_{ib})$ be a covering of the system $\Xi = 0$. For any π-shadow G there exists a further covering π' and a π'-nonlocal symmetry which extends the π-shadow G.*

PROOF. This proof is modelled after [**30**]. We know that $u^\alpha_\tau = G^\alpha$ and that $\gamma^b_{x^i} = X_{ib}$. As in Examples 4.3 and 4.6, we need to find the infinitesimal variations of the nonlocal variables γ^b. We simply set

$$\frac{\partial \gamma_b}{\partial \tau} = \gamma^b_1,$$

in which γ^b_1 are new symbols for nonlocal variables. Since once we add these new symbols we need to find infinitesimal variations for them, we consider an infinite set of variables γ^b_l, $l = 0, 1, 2, \dots$, and we set

$$\frac{\partial \gamma^b_l}{\partial \tau} = \gamma^b_{l+1},$$

with the convention that $\gamma^b_0 = \gamma^b$. Now it remains to find the relations among the nonlocal variables γ^b_l and the local dependent variables u^α. We compute formally "on solutions":

$$\frac{\partial \gamma^b_l}{\partial x^i} = \frac{\partial \gamma^b_{l-1,\tau}}{\partial x^i} = \cdots = \frac{\partial \gamma^b_{\tau\dots\tau}}{\partial x^i} = \frac{\partial^l X_{ib}}{\partial \tau^l},$$

and this last derivative can be found by applying l times the τ-derivative operator

$$\widetilde{D}_\tau = \sum D_J G^\alpha \frac{\partial}{\partial u^\alpha_J} + \sum \gamma^b_{l+1} \frac{\partial}{\partial \gamma^b_l}$$

to X_{ib}. We conclude that the infinite compatible system $\Xi = 0$, $\gamma^b_{l,x^i} = \widetilde{D}^l_\tau(X_{ib})$ possesses the "generalized symmetry"

$$X = \sum_{\alpha=1}^m G^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{b,l} \gamma^b_{l+1} \frac{\partial}{\partial \gamma^b_l}\ .$$

The vector field X is a *bona fide* nonlocal π'-symmetry of $\Xi = 0$ defined on the infinite-dimensional covering

$$\pi' = \Big(\{\gamma^b_l : b = 1, \dots, N; l \geq 0\}; \{\widetilde{D}^l_\tau(X_{ib}) : i = 1, \dots, n; b = 1, \dots, N; l \geq 0\}\Big)$$

of $\Xi = 0$ which extends the shadow G. □

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Other reconstruction results have been obtained by N. Khor'kova and K. Kiso, see [**30, 32**] and references therein. Of course this last proposition, while theoretically satisfactory, is only of relative practical value. It is quite interesting that in explicit examples shadows can be reconstructed into nonlocal symmetries on finite-dimensional coverings, so that the Krasil'shchik–Vinogradov theory does have immediate applicability.

4.2.3. *Nonlocal symmetries and exterior differential systems.* The main observation to be explored here is that the system of compatible equations (4.11) defines a completely integrable Pfaffian system $\mathcal{I} = \{\Gamma^b\}$ by means of

$$\Gamma^b = d\gamma^b - \sum_{i=1}^{n} X_{ib}\, dx^i \ . \tag{4.16}$$

More precisely, compatibility of the system

$$\frac{\partial \gamma^b}{\partial x^i} = X_{ib} \tag{4.17}$$

is equivalent to the complete integrability condition

$$d\,\Gamma^b \equiv 0 \bmod \{\Gamma^1, \ldots, \Gamma^N\}$$

whenever $u^\alpha(x^i)$ is a solution to the system $\Xi = 0$. Motivated by this remark, we generalize our nonlocal setting as follows:

Definition 4.7. An N-dimensional $\mathcal{I}$-covering of a system of partial differential equations $\Xi = 0$ is an exterior differential system $\mathcal{I}$ over $\mathcal{N} = J^\infty E \times \mathbf{R}^N$ such that $\mathcal{I}$ is completely integrable whenever $u^\alpha(x^i)$ is a solution to the system $\Xi = 0$.

We can also generalize our notion of a nonlocal symmetry:

Definition 4.8. Let $\mathcal{I}$ be an N-dimensional $\mathcal{I}$-covering of a system of partial differential equations $\Xi = 0$ defined on the manifold $\mathcal{N} = J^\infty E \times \mathbf{R}^N$. An $\mathcal{I}$-nonlocal symmetry of $\Xi = 0$ is a generalized vector field X over $\mathcal{N}$ such that

$$pr^\infty X(\Xi) \equiv 0 \bmod \mathcal{I} \tag{4.18}$$

and

$$\mathcal{L}_{pr^\infty X}\mathcal{I} \equiv 0 \bmod \mathcal{I}, \tag{4.19}$$

whenever $u^\alpha(x^i)$ is a solution to the system $\Xi = 0$, in which $\mathcal{L}_{pr^\infty X}$ indicates Lie derivative with respect to $pr^\infty X$.

Example 4.9. The Degasperis-Procesi (DP) equation [**16**] is

$$m + u_{xx} - u = 0 \qquad m_t + m_x\, u + 3\, m\, u_x = 0 \ . \tag{4.20}$$

This equation, together with the Camassa-Holm equation [**9**] to be reviewed in the following section, exhaust the list of integrable equations in the family

$$u_t - u_{xxt} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}, \qquad b \in \mathbf{R},$$

see [**16, 17**]. We can extract an $\mathcal{I}$-covering of Equation (4.20) from the Degasperis-Holm-Hone integrability analysis appearing in [**17**]. Indeed, in that paper is shown

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that the system of equations

$$-\gamma+\gamma_{xx}+\lambda\, m+\gamma^3+3\,\gamma\,\gamma_x = 0 \tag{4.21}$$

$$-\gamma_t-\gamma\, u_x-\gamma_x\, u-\frac{2\,\gamma_x\gamma}{\lambda}+u_{xx}-\frac{\gamma_{xx}}{\lambda} = 0 \tag{4.22}$$

$$-\delta_x+\gamma = 0 \tag{4.23}$$

$$-\delta_t+u_x-u\,\gamma-\frac{\gamma_x}{\lambda}-\frac{\gamma^2}{\lambda} = 0 \tag{4.24}$$

is compatible whenever m and u solve (4.20). The function γ can be considered as a "pseudo-potential" for the Degasperis-Procesi equation, but it is important to note that it falls outside the usual definition of a Wahlquist-Eastbrook pseudo-potential (see [**50**] and the definitions of pseudo-potentials appearing for instance in [**24, 25, 26**]). It follows that (4.21)–(4.24) does not determine a covering in the sense of Vinogradov and Krasil'shchik, see Definition 4.4 and [**32**, Chapter 5]. It does determine an $\mathcal{I}$-covering, since it is a standard fact [**8**] that a compatible system such as (4.21)–(4.24), can be written as a completely integrable exterior differential system. $\mathcal{I}$-nonlocal symmetries of the DP equation are

$$\frac{\partial}{\partial x}+F_2(\lambda)\frac{\partial}{\partial\delta}\,; \qquad \frac{\partial}{\partial t}+F_1(\lambda)\frac{\partial}{\partial\delta}\,; \qquad \frac{\partial}{\partial\delta}, \tag{4.25}$$

in which F_1, F_2 are arbitrary functions of λ. It remains as an open problem to find further nonlocal symmetries of the DP equation depending on α and δ, since the ones appearing in (4.25), while confirming that Definition 4.8 is not vacuous, are of limited practical utility.

Remark 4.10. There exists a deep theory of symmetries of integrable equations (the adjective "integrable" is understood here as meaning that the equation at hand is the integrability condition of a one-parameter family of overdetermined linear problems) based on loop groups, see R. Palais' well-known review article [**40**]. It would appear that the precise relationship between the symmetries discussed by Palais and nonlocal symmetries has not been considered in detail as yet. We do mention, however, the interesting paper [**2**], in which V.E. Adler considers (shadows of) nonlocal symmetries for a general class of integrable equations.

4.3. The Camassa-Holm and associated Camassa-Holm equations. In this section we consider nonlocal symmetries of the Camassa-Holm (CH) equation [**9**]

$$2\,u_x\,u_{xx}+u\,u_{xxx}=u_t-u_{xxt}+3\,u_x\,u,$$

which we write as a system of equations for two dependent variables m and u,

$$m=u_{xx}-u, \qquad m_t=-m_x\,u-2\,m\,u_x, \tag{4.26}$$

and of the associated Camassa-Holm (ACH) equation [**47**]

$$p_T=-p^2u_y, \qquad u=\frac{-p^2}{2}-\left(\frac{p_T}{p}\right)_y p\,. \tag{4.27}$$

We will explain the precise relation between (4.26) and (4.27) momentarily. We recall from Section 1 that nonlocal symmetries will be used in Section 5: they will allow us to find Darboux transformations and explicit solutions. Other applications appear in [**24, 26, 45**].

Results reported herein are mainly joint work with R. Hernández Heredero (Universidad Politécnica de Madrid).

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PROPOSITION 4.3. *The following systems of equations are compatible on solutions to the Camassa-Holm equation* (4.26)*:*

$$\gamma_x = m - \frac{1}{2\lambda}\gamma^2 + \frac{1}{2}\lambda, \qquad \gamma_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right)_x ; \tag{4.28}$$

$$\delta_x = \gamma, \qquad \delta_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right) ; \tag{4.29}$$

$$\beta_x = m\,\mathrm{e}^{\delta/\lambda}, \qquad \beta_t = \mathrm{e}^{\delta/\lambda}\left[-(1/2)\,\gamma^2 + (1/2)\lambda^2 - u\,m\right] . \tag{4.30}$$

This proposition is proven in [**42**] using a geometric approach explained briefly in Remark 4.11 below, but of course it can be checked directly. Its importance for us rests on the fact that the compatible system of equations (4.28)–(4.30) yields a covering of the Camassa-Holm equation.

REMARK 4.11. Equations (4.28) for the pseudo-potential γ yield the standard scalar Lax pair for the Camassa-Holm equation, [**9**], if we set $\gamma = 2\,\lambda\,\ln(\psi)_x$, see for instance [**24**]. From a geometric point of view, quadratic pseudo-potentials such as (4.28) arise thus [**42, 24**]: the Camassa-Holm equation describes pseudo-spherical surfaces, that is, generic solutions to CH determine Riemannian metrics of Gaussian curvature $K = -1$ on (open subsets of) their domains. Let $u(x,t)$ be a solution to CH, and assume that the pseudo-spherical metric $g(u(x,t))$ is determined by a moving coframe $\{\omega^1(u(x,t)), \omega^2(u(x,t))\}$ and that the corresponding connection 1-form is $\omega^3(u(x,t))$. Then, as explained by S.S. Chern and K. Tenenblat [**12**], there exists a new coframe $\{\theta^1, \theta^2\}$ and a new connection form θ^3 such that

$$d\theta^1 = 0, \qquad d\theta^2 = \theta^2 \wedge \theta^1, \qquad \text{and} \qquad \theta^3 + \theta^2 = 0 .$$

The forms θ^α are related to the forms ω^α (we suppress the explicit dependence on solutions for simplicity of notation) via

$$\theta^1 = \omega^1 \cos\rho + \omega^2 \sin\rho, \qquad \theta^2 = -\omega^1 \sin\rho + \omega^2 \cos\rho, \qquad \theta^3 = \omega^3 + d\rho,$$

in which ρ is a solution to the Pfaffian system

$$\omega^3 + d\rho - \omega^1 \sin\rho + \omega^2 \cos\rho = 0, \tag{4.31}$$

and, if $\{v_1, v_2\}$ is the orthonormal moving frame dual to the moving coframe $\{\theta^1, \theta^2\}$, then the integral curves of the vector fields v_1 and v_2 are, respectively, geodesics and horocycles of M (see [**12**]). It follows, therefore, that the function ρ satisfying Equation (4.31) is precisely the angle connecting an arbitrary orthonormal framing with an orthonormal framing along geodesics. The key observation we make now is that if we perform the change of variables $\gamma = \tan(\rho/2)$ (or, $\gamma = \cot(\rho/2)$) the Pfaffian system (4.31) becomes a pair of Riccati equations. This is precisely how the quadratic pseudo-potential γ given by (4.28) arises from geometry.

REMARK 4.12. The above remark allows us to conclude that quadratic pseudo-potentials are more fundamental from a geometric point of view than scalar Lax pairs. However, it should be also noted that no geometric interpretation similar to the one above appears to have been found for *cubic* pseudo-potentials such as (4.21) and (4.22).

We can classify *all* first order nonlocal π-symmetries of the CH equation. Note that Equations (4.28)–(4.30) depend explicitly on the parameter λ. In order to

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perform a complete classification, we find it necessary to assume that λ is also affected by symmetry transformations. In other words, we extend the augmented system (4.26), (4.28)–(4.30) with the equations

$$\lambda_x = 0, \qquad \lambda_t = 0, \tag{4.32}$$

and we consider the four-dimensional covering π of CH determined by γ, δ, β and λ.

THEOREM 4.13. *The first order generalized symmetries of the augmented CH system* (4.26), (4.28)–(4.32), *represented by vector fields*

$$V = G^1 \frac{\partial}{\partial m} + G^2 \frac{\partial}{\partial u} + H^1 \frac{\partial}{\partial \gamma} + H^2 \frac{\partial}{\partial \delta} + H^3 \frac{\partial}{\partial \beta} + K \frac{\partial}{\partial \lambda} \tag{4.33}$$

in which G^α, H^b, and K are functions of x, t, m, u, their first derivatives, γ, δ, β, and λ are

(4.34)

$$\begin{aligned} V_1 = {} & (2mu_x + um_x) \frac{\partial}{\partial m} - u_t \frac{\partial}{\partial u} + \left(\frac{\lambda^2}{2} - \frac{\lambda u}{2} + um - \frac{\gamma^2}{2} - \frac{u\gamma^2}{2\lambda} + \gamma u_x \right) \frac{\partial}{\partial \gamma} \\ & + (\lambda\gamma + u\gamma - \lambda u_x) \frac{\partial}{\partial \delta} - \frac{1}{2} \mathrm{e}^{\delta/\lambda} \left(\lambda^2 - 2um - \gamma^2 \right) \frac{\partial}{\partial \beta} , \end{aligned}$$

(4.35)

$$V_2 = m_x \frac{\partial}{\partial m} + u_x \frac{\partial}{\partial u} + \left(\frac{\lambda}{2} + m - \frac{\gamma^2}{2\lambda} \right) \frac{\partial}{\partial \gamma} + \gamma \frac{\partial}{\partial \delta} + \mathrm{e}^{\delta/\lambda} m \frac{\partial}{\partial \beta} ,$$

(4.36)

$$V_3 = \frac{\partial}{\partial \delta} + \frac{\beta}{\lambda} \frac{\partial}{\partial \beta},$$

(4.37)

$$V_4 = \frac{\partial}{\partial \beta},$$

(4.38)

$$V_5 = \mathrm{e}^{\delta/\lambda} \left(\frac{2m\gamma}{\lambda} + m_x \right) \frac{\partial}{\partial m} + \mathrm{e}^{\delta/\lambda} \gamma \frac{\partial}{\partial u} + \mathrm{e}^{\delta/\lambda} m \frac{\partial}{\partial \gamma} + \beta \frac{\partial}{\partial \delta} + \left(\mathrm{e}^{2\delta/\lambda} m + \frac{\beta^2}{2\lambda} \right) \frac{\partial}{\partial \beta},$$

(4.39)

$$V_6 = A(\lambda) \left(m \frac{\partial}{\partial m} + u \frac{\partial}{\partial u} + \gamma \frac{\partial}{\partial \gamma} + \delta \frac{\partial}{\partial \delta} + \beta \frac{\partial}{\partial \beta} + \lambda \frac{\partial}{\partial \lambda} - t\, V_1 \right) ,$$

where $A(\lambda)$ is an arbitrary function. Consequently, these vector fields are nonlocal π-symmetries of the CH equation.

This theorem appears in [**25, 26**]. The function $A(\lambda)$ is included in V_6 since it affects the way in which λ varies with the infinitesimal symmetry transformation (4.39) and, in fact, it is of importance for the Lie algebra structure of nonlocal π-symmetries [**25, 26**]. We also note that V_1 and V_2 are simply the generators of shifts with respect to the independent variables: they are equivalent to $\partial/\partial t$ and $-\partial/\partial x$ respectively.

COROLLARY 4.1. *The commutation table of the six nonlocal symmetries* (4.34)–(4.39) *of the CH equation is that of Figure* 1, *whenever $u, m, \gamma, \delta, \beta, \lambda$ solve the augmented CH system* (4.26), (4.28)–(4.32).

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	V_1	V_2	V_3	V_4	V_5	${V_6}^*$
V_1						$-B(\lambda)\,V_1$
V_2						
V_3				$-\frac{1}{\lambda}V_4$	$\frac{1}{\lambda}V_5$	$B(\lambda)\,V_3$
V_4			$\frac{1}{\lambda}V_4$		V_3	$B(\lambda)\,V_4$
V_5			$-\frac{1}{\lambda}V_5$	$-V_3$		
V_6	$A(\lambda)\,V_1$		$-A(\lambda)\,V_3$	$-A(\lambda)\,V_4$		$-[A'(\lambda)B(\lambda)-A(\lambda)B'(\lambda)]\frac{\lambda}{A(\lambda)}V_6$

FIGURE 1. Commutation table of the CH nonlocal symmetry algebra. In ${V_6}^*$ we have used $B(\lambda)$ instead of $A(\lambda)$.

This corollary implies that the nonlocal π-symmetries (4.34)–(4.38) generate a five-dimensional Lie algebra $\mathcal{G}_5$ isomorphic to the direct sum of $sl(2)$ and the commutative Lie algebra generated by V_1 and V_2. In fact, if instead of V_3, V_4, V_5 we use

$$e=-\sqrt{2\lambda}\,V_4, \qquad f=\sqrt{2\lambda}\,V_5, \qquad h=-2\lambda V_3,$$

we find the commutators $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$. Now, if in addition to $V_1,\dots,V_5$ we consider also V_6, we can check, following [**26**], that the infinite-dimensional Lie algebra generated by the nonlocal symmetries $V_1,\dots,V_6$ contains a semidirect sum of the loop algebra over $sl(2,\mathbf{R})$ and the centerless Virasoro algebra:

PROPOSITION 4.4. *Let us define the vector fields*

$$T^1_n=-2\lambda^{n+1}V_3, \quad T^2_n=-\sqrt{2\lambda}\,\lambda^n V_4, \quad T^3_n=\sqrt{2\lambda}\,\lambda^n V_5, \quad \textit{and} \;\; W_n=-\lambda^n L,$$

in which $n\in\mathbf{Z}$ *and* $L=V_6$ *with* $A(\lambda)=1$. *Then, the following commutation relations hold:*

(4.40)

$$[T^1_m,T^2_n]=2\,T^2_{m+n}, \qquad [T^1_m,T^3_n]=-2\,T^3_{m+n}, \qquad [T^2_m,T^3_n]=T^1_{m+n}\;;$$

(4.41)

$$[W_m,W_n]=(m-n)\,W_{m+n}\;;$$

(4.42)

$$[T^1_m,W_n]=m\,T^1_{m+n}, \quad [T^2_m,W_n]=(m-\frac{1}{2})\,T^2_{m+n}, \quad [T^3_m,W_n]=(m+\frac{1}{2})\,T^3_{m+n}\;.$$

REMARK 4.14. We stress the fact that the commutation relations of Corollary 4.1 are valid *only on shell*. For example, the commutator of V_5 and V_6 is (we write A, A', A'', instead of $A(\lambda)$, $A'(\lambda)$, $A''(\lambda)$):

$$\begin{aligned}(4.43)\quad [V_5,V_6]=&-\mathrm{e}^{\delta/\lambda}A'\lambda_x(m+tm_t)\frac{\partial}{\partial m}\\ &-\mathrm{e}^{\delta/\lambda}(A''u\lambda_x^2+2A'\lambda_x u_x+A''t\lambda_x^2u_t+2A't\lambda_x u_{xt}+\lambda_{xx}A'u+\lambda_{xx}A't\,u_t)\frac{\partial}{\partial\gamma},\end{aligned}$$

so that $[V_5,V_6]=0$ *only after* the augmented system of equations (4.26), (4.28)–(4.32) is imposed. Moreover, we note that $[V_5,V_6]$ *is not* in the linear span of $V_1,\dots,V_6$, unless we work on shell. This observation connects the foregoing theory with internal symmetries: internal symmetries of a kth order equation form a Lie algebra on the equation manifold $S^{(k)}\subset J^kE$, but there is no reason why they should form a Lie algebra on the whole jet space J^kE (see [**5**]). As (4.43) shows, the

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same phenomenon appears in the realm of nonlocal symmetries. This is a reflection of the fact that they are defined on coverings of the full equation manifold $S^{(\infty)}$ of the differential equation being considered, and not on some "universal" jet space as it happens with local symmetries [**39**]. Thus, nonlocal symmetries really generalize the class of *internal* symmetries.

REMARK 4.15. In [**29**], S. Kouranbaeva proved that the group of diffeomorphisms of the line (or the circle) appears as a configuration space for the Camassa-Holm equation (4.26). More precisely, Equation (4.26) is the geodesic spray of the weak H^1 Riemannian metric on this group. Proposition 4.4 above explains why this group of diffeomorphisms can be so naturally associated to CH.

Now let us consider the associated Camassa-Holm (ACH) equation (4.27), that is,

$$p_T = -p^2 u_y, \qquad u = \frac{-p^2}{2} - \left(\frac{p_T}{p}\right)_y p\,.$$

This equation was first obtained by J. Schiff in [**47**] by applying the transformation

$$p = \sqrt{2m}, \quad d\,y = p\,dx - p\,u\,dt, \quad \text{and} \quad d\,T = d\,t \tag{4.44}$$

to the CH equation (4.26), and it has been studied for its own sake (*i.e.* without reference to its relation with the CH equation) in [**27, 28**].

Certainly, (4.44) is "nonlocal": the new independent variable y is a "potential variable" which can be determined from x, t, p and u only through integration. In particular, (4.44) is *not* an invertible transformation, as explained in [**47**], and so CH and ACH are *not* equivalent equations. In fact, transformations such as (4.44) can change essentially the character of the nonlinearity of the initial equation, as remarked in [**33, 34**]. In particular, they can change the symmetry structure (see [**36**, pp.121, 122] for a discussion on this point in the context of Burgers' equation). Motivated by these remarks, we compute nonlocal symmetries for ACH. Proposition 4.3 yields:

LEMMA 4.16.

(1) *The associated Camassa-Holm equation* (4.27) *admits a pseudo-potential* γ *determined by the compatible equations*

$$\gamma_y = -\frac{1}{2\lambda p}\gamma^2 + \frac{p}{2} + \frac{\lambda}{2p}, \qquad \gamma_T = \frac{\gamma^2}{2} + \frac{p_T}{p}\gamma + \lambda u - \frac{1}{2}\lambda^2\,. \tag{4.45}$$

(2) *The associated Camassa-Holm equation* (4.27) *admits potentials* δ *and* β *determined by the compatible equations*

$$\delta_y = \frac{\gamma}{p}, \qquad \delta_T = \lambda\left(-\frac{p_T}{p} - \gamma\right) \tag{4.46}$$

and

$$\beta_y = \frac{p}{2}\,\mathrm{e}^{\delta/\lambda}, \qquad \beta_T = \frac{1}{2}\left(-\gamma^2 + \lambda^2\right)\mathrm{e}^{\delta/\lambda}\,. \tag{4.47}$$

And we also remark that transformation (4.44) implies that

$$\lambda_y = \lambda_T = 0\,. \tag{4.48}$$

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REMARK 4.17. The pseudo-potential equations (4.45) determine a Lax pair for the associated Camassa-Holm equation. Indeed, setting $\gamma = \psi_2/\psi_1$ and replacing into the first equation of (4.45) we find

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_y = \begin{pmatrix} 0 & \frac{1}{2\lambda p} \\ \frac{p}{2} + \frac{\lambda}{2p} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

precisely the y-part of the overdetermined linear problem for ACH obtained by Schiff in [**47**]. It appears that Schiff's linear problem for ACH has not been studied rigorously from the point of view of scattering/inverse scattering as yet.

THEOREM 4.18. *The first order generalized symmetries of the augmented ACH system* (4.27), (4.45)–(4.48) *are*

(4.49)
$$\begin{aligned} W_1 = &-p^2 u_y \frac{\partial}{\partial p} + u_T \frac{\partial}{\partial u} \\ &- \left(\frac{\lambda^2}{2} - \lambda u - \frac{\gamma^2}{2} + p\gamma u_y\right) u_x \frac{\partial}{\partial \gamma} - \lambda\left(\gamma - p u_y\right)\frac{\partial}{\partial \delta} + \frac{1}{2}\mathrm{e}^{\delta/\lambda}(\lambda^2 - \gamma^2)\frac{\partial}{\partial \beta}, \end{aligned}$$

(4.50)
$$W_2 = p_y \frac{\partial}{\partial p} + u_y \frac{\partial}{\partial u} + \left(\frac{\lambda}{2p} + \frac{p}{2} - \frac{\gamma^2}{2\lambda p}\right)\frac{\partial}{\partial \gamma} + \frac{\gamma}{p}\frac{\partial}{\partial \delta} + \frac{1}{2}\mathrm{e}^{\delta/\lambda} p \frac{\partial}{\partial \beta},$$

(4.51)
$$W_3 = \lambda \frac{\partial}{\partial \delta} + \beta \frac{\partial}{\partial \beta},$$

(4.52)
$$W_4 = \frac{\partial}{\partial \beta},$$

(4.53)
$$\begin{aligned} W_5 = &\, 2\mathrm{e}^{\delta/\lambda} p\gamma \frac{\partial}{\partial p} \\ &+ 2\mathrm{e}^{\delta/\lambda}\lambda\left(\gamma - p u_y\right)\frac{\partial}{\partial u} - \mathrm{e}^{\delta/\lambda}\left(\lambda^2 - \gamma^2\right)\frac{\partial}{\partial \gamma} - 2\lambda\left(\mathrm{e}^{\delta/\lambda}\gamma - \beta\right)\frac{\partial}{\partial \delta} + \beta^2 \frac{\partial}{\partial \beta}, \end{aligned}$$

(4.54)
$$W_6 = A(\lambda)\left(p\frac{\partial}{\partial p} + 2u\frac{\partial}{\partial u} + 2\gamma\frac{\partial}{\partial \gamma} + 2(\delta - \lambda)\frac{\partial}{\partial \delta} + 2\lambda\frac{\partial}{\partial \lambda} - 2T\frac{\partial}{\partial T} + y\frac{\partial}{\partial y}\right),$$

where $A(\lambda)$ is an arbitrary function. Consequently, these vector fields are nonlocal symmetries of the associated CH equation.

COROLLARY 4.2. *The commutator table of the six nonlocal symmetries* (4.49)–(4.54) *of the associated CH equation is that of Figure* 2 *whenever $u, p, \gamma, \delta, \beta, \lambda$ solve the augmented ACH system* (4.27), (4.45)–(4.48).

The commutator table of the nonlocal symmetries $W_1, \ldots, W_6$ is not the same as in the CH case, as it may have been expected from the comments on the non-equivalence of CH and ACH we made before. Interestingly, it is shown in [**34**] that transformations such as (4.44) *are invertible* if we consider them as transformations on the space of variables $u, u_x, u_{xx}, \cdots$. Invertibility is lost once we also consider the independent variables x, t, as pointed out in [**47**]. It may be noted that the difference

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	W_1	W_2	W_3	W_4	W_5	${W_6}^*$
W_1						$-2B(\lambda)W_1$
W_2						$B(\lambda)W_2$
W_3				$-\ W_4$	W_5	
W_4			W_4		$2W_3$	
W_5			$-W_5$	$-2W_3$		
W_6	$2A(\lambda)W_1$	$-A(\lambda)W_2$				$-2[A'(\lambda)B(\lambda) - A(\lambda)B'(\lambda)]\frac{\lambda}{A(\lambda)}W_6$

FIGURE 2. Commutation table of the ACH nonlocal symmetry algebra. In ${W_6}^*$ we have used $B(\lambda)$ instead of $A(\lambda)$.

between the commutation relations of Corollaries 3 and 4 appears precisely when we consider the scaling symmetries V_6 and W_6, and these symmetries *do depend* on the independent variables x, t, y, T!

As in the CH case, the symmetries $W_1, ..., W_5$ generate a five-dimensional Lie algebra isomorphic to the direct sum of $sl(2, \mathbf{R})$ and the two-dimensional commutative Lie algebra generated by W_1, W_2. Addition of W_6 allows us to construct infinite-dimensional Lie algebras of nonlocal symmetries to the ACH equation. Instead of Proposition 4.4, in this case we obtain the standard semidirect sum of the loop algebra over $sl(2, \mathbf{R})$ and the centerless Virasoro algebra:

PROPOSITION 4.5. *Let us define the vector fields*

$$T_n^1 = -2\lambda^n W_3, \qquad T_n^2 = -\lambda^n W_4, \qquad T_n^3 = \lambda^n W_5, \qquad \textit{and} \qquad V_n = -\frac{1}{2}\lambda^n L,$$

in which $n \in \mathbf{Z}$ *and* $L = W_6$ *with* $A(\lambda) = 1$. *Then, the following commutation relations hold:*

$$[T_m^1, T_n^2] = 2\,T_{m+n}^2, \qquad [T_m^1, T_n^3] = -2\,T_{m+n}^3, \qquad [T_m^2, T_n^3] = T_{m+n}^1\ ; \tag{4.55}$$

$$[V_m, V_n] = (m-n)\,V_{m+n}\ ; \tag{4.56}$$

$$[T_m^1, V_n] = m\,T_{m+n}^1, \qquad [T_m^2, V_n] = m\,T_{m+n}^2 \qquad [T_m^3, V_n] = m\,T_{m+n}^2\ . \tag{4.57}$$

5. Darboux transformations for the ACH equation

In this final section we construct Darboux transformations for the associated Camassa-Holm equation. Darboux transformations for the standard Camassa-Holm equation are not considered in this paper because they are rather involved and we have discussed them in detail elsewhere [**25, 26**].

First of all, we note that symmetries (4.49), (4.50) and (4.54) are extensions to the covering of ACH determined by γ, δ and β of local symmetries, and that (4.51) and (4.52) are "vertical" symmetries, in the sense that they only transform some of the nonlocal variables. Thus, it is natural to focus our attention on (4.53).

We compute the flow of (4.53), so as to obtain a rule to generate solutions to the associated Camassa-Holm equation

$$p_T = -p^2 u_y, \qquad u = \frac{-p^2}{2} - \left(\frac{p_T}{p}\right)_y p\ . \tag{5.1}$$

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The flow of (4.53) is determined by the system of equations

$$\frac{\partial p}{\partial \tau} = 2\mathrm{e}^{\delta/\lambda} p\gamma, \tag{5.2}$$

$$\frac{\partial u}{\partial \tau} = 2\lambda \mathrm{e}^{\delta/\lambda}\left(\gamma - p\frac{\partial u}{\partial y}\right), \tag{5.3}$$

$$\frac{\partial \gamma}{\partial \tau} = -\,\mathrm{e}^{\delta/\lambda}\left(\lambda^2 - \gamma^2\right), \tag{5.4}$$

$$\frac{\partial \delta}{\partial \tau} = -\,2\lambda\left(\mathrm{e}^{\delta/\lambda}\gamma - \beta\right), \tag{5.5}$$

$$\frac{\partial \beta}{\partial \tau} = \beta^2, \tag{5.6}$$

with initial conditions $u(y,T,0) = u_0$, $p(y,T,0) = p_0$, $\gamma(y,T,0) = \gamma_0$, $\delta(y,T,0) = \delta_0$ and $\beta(y,T,0) = \beta_0$. The solution to this initial value problem is determined by

$$\beta(\tau) = \frac{\beta_0}{1-\tau\beta_0}, \tag{5.7}$$

$$\gamma(\tau) = \frac{\tau\left(\lambda^2\omega_0 - \omega_0\gamma_0^2 + \gamma_0\beta_0\right) - \gamma_0}{\tau\beta_0 - 1}, \tag{5.8}$$

$$\delta(\tau) = \delta_0 - \lambda \ln\left[\left(1-\tau\beta_0 + \omega_0\gamma_0\tau - \tau\lambda\omega_0\right)\left(1-\tau\beta_0 + \omega_0\gamma_0\tau + \tau\lambda\omega_0\right)\right], \tag{5.9}$$

$$p(\tau) = \frac{p_0\left(1-\tau\beta_0 + \omega_0\gamma_0\tau - \tau\lambda\omega_0\right)\left(1-\tau\beta_0 + \omega_0\gamma_0\tau + \tau\lambda\omega_0\right)}{(-1+\tau\beta_0)^2}, \tag{5.10}$$

in which $\omega_0 = \mathrm{e}^{\delta_0/\lambda}$. As shown in [**25, 26**], it is possible to obtain $u(\tau)$ directly from the ACH equation (5.1) without solving Equation (5.3). This equation is a consequence of the augmented ACH system (4.27), (4.45)–(4.48) and the flow equations (5.2) and (5.4)–(5.6):

PROPOSITION 5.1. *Assume that p, u and the functions γ, δ and β satisfy the compatible equations* (5.1), (4.45)–(4.48), (5.2) *and* (5.4)–(5.6). *Then, u satisfies Equation* (5.3), *namely,*

$$\frac{\partial u}{\partial \tau} = 2\lambda \mathrm{e}^{\delta/\lambda}\left(\gamma - p\frac{\partial u}{\partial y}\right). \tag{5.11}$$

EXAMPLE 5.1. We take $p_0 = c$, in which c is a constant different from zero. Equation (5.1) tells us that $u_0 = -c^2/2$. We compute γ_0, δ_0 and β_0 by means of (4.45)–(4.47) and obtain

$$\gamma_0(y,T) = \mu\,\lambda, \tag{5.12}$$

$$\delta_0(y,T) = \mu\,\lambda\left(\frac{y}{c} - \lambda T\right), \tag{5.13}$$

$$\beta_0(y,T) = \frac{c^2}{2\mu}\,\exp\left[\mu\left(\frac{y}{c} - \lambda T\right)\right], \tag{5.14}$$

in which $\mu = \sqrt{\lambda c^2 + \lambda^2}/\lambda$. Equation (5.10) implies that

$$p(y,T,\tau) = c\frac{\left(1+\tau(-\frac{c^2}{2\mu} + \lambda\mu + \lambda)\mathrm{e}^{\mu(\frac{y}{c}-\lambda T)}\right)\left(1+\tau(-\frac{c^2}{2\mu} + \lambda\mu - \lambda)\mathrm{e}^{\mu(\frac{y}{c}-\lambda T)}\right)}{\left(-1+\tau\dfrac{c^2}{2\mu}\,\mathrm{e}^{\mu(\frac{y}{c}-\lambda T)}\right)^2} \tag{5.15}$$

solves the associated Camassa-Holm equation (5.1).

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Now we are ready to obtain two Darboux transforms for the associated Camassa-Holm equation (5.1):

Let us consider the quadratic pseudo-potential $\gamma(y,T)$ defined by Equations (4.45), namely,

$$\gamma_y = -\frac{1}{2\lambda p}\gamma^2 + \frac{p}{2} + \frac{\lambda}{2p}, \qquad \gamma_T = \frac{\gamma^2}{2} + \frac{p_T}{p}\gamma + \lambda u - \frac{1}{2}\lambda^2 \,. \tag{5.16}$$

The first equation in (5.16) yields

$$p(y,T) = \frac{\partial}{\partial y}\gamma\,(y,T) + \sqrt{\left(\frac{\partial}{\partial y}\gamma\,(y,T)\right)^2 + \frac{\gamma\,(y,T)^2}{\lambda} - \lambda}, \tag{5.17}$$

and replacing (5.17) into the equation for γ_T, we obtain a differential equation for γ. This equation is quite long, so we will not write it here explicitly. however, it can be checked that it is invariant under the change

$$y \mapsto -y, \quad \sqrt{\lambda} \mapsto -\sqrt{\lambda}\,.$$

Equation (5.17), however, is not invariant under this change. Instead, we obtain that

$$\overline{p}(y,T) = -\frac{\partial}{\partial y}\gamma\,(y,T) + \sqrt{\left(\frac{\partial}{\partial y}\gamma\,(y,T)\right)^2 + \frac{\gamma\,(y,T)^2}{\lambda} - \lambda} \tag{5.18}$$

is also a solution to ACH. Subtracting (5.17) and (5.18) we find:

Proposition 5.2. *If $\gamma(y,T)$ is determined by Equations* (4.45)*, and $p(y,T)$ is a solution to the associated Camassa-Holm equation* (5.1)*, then so is*

$$\overline{p}(y,T) = p(y,T) - 2\,\frac{\partial\gamma}{\partial y}(y,T)\,. \tag{5.19}$$

Transformation (5.19) was originally found by J. Schiff in [**47**], using loop groups, and an alternative derivation was presented in [**27**]. The foregoing derivation using pseudo-potentials first appeared in the papers [**25, 26**] by R. Hernandez Heredero and the present author.

Formulas (5.7)–(5.10) for the flow of the vector field (4.53) can be used to obtain a second Darboux transform for ACH. We recall it following [**25, 26**]:

Proposition 5.3. *Assume that $p(y,T)$ solves the associated Camassa-Holm equation* (5.1)*, and that $\beta(y,T)$ is a solution to*

$$\frac{\beta_T}{\beta_y} = \frac{-\gamma^2 + \lambda^2}{p}, \tag{5.20}$$

in which γ is a solution to the compatible system (4.45)*. Then, the function $\overline{p}(y,T)$ given by*

$$\overline{p} = p\left(-1 + \frac{2\tau\lambda p}{-1+\tau\beta}\left(\frac{\beta_y}{p}\right)_y\right)^2 - \frac{4\,\lambda^2\tau^2}{(-1+\tau\beta)^2}\,\frac{\beta_y^2}{p} \tag{5.21}$$

also solves (5.1)*.*

Proof. Start with Equation (5.10) for $p(y,T)$ and write it in the form

$$p = p_0\left(-1 + \frac{\omega_0\gamma_0\tau}{-1+\tau\beta_0}\right)^2 - \frac{\tau^2\lambda^2}{(-1+\tau\beta_0)^2}\,p_0\,\omega_0^2\,. \tag{5.22}$$

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Now use Equations (4.46) and $\omega_0 = \mathrm{e}^{\delta_0/\lambda}$. Then, (5.22) becomes

$$p = p_0 \left(-1 + \frac{\tau\lambda p_0 (\mathrm{e}^{\delta_0/\lambda})_y}{-1+\tau\beta_0} \right)^2 - \frac{\tau^2\lambda^2}{(-1+\tau\beta_0)^2}\, p_0\, \mathrm{e}^{2\delta_0/\lambda}\,. \tag{5.23}$$

Formula (5.21) follows from this equation by using (4.47), changing p to $\overline{p}$, and then dropping the subindex $_0$, while Equation (5.20) is obtained directly from the second equation appearing in (4.47), simply using the identity $\mathrm{e}^{\delta_0/\lambda} = 2\beta_y/p$. $\square$

Now, the first equation in (5.16) implies that Equation (5.20) can be written as

$$\frac{\beta_T}{\beta_y} = -\lambda(p - 2\gamma_y),$$

and Proposition 5.2 says that $\hat{p} = p - 2\gamma_y$ is a new solution to the ACH equation. We therefore interpret Proposition 5.3 as providing us with a nonlinear superposition rule for ACH:

Corollary 5.1. *Assume that $p(y,T)$ solves the ACH equation, and that $\gamma(y,T)$ is a solution to* (5.16)*. Set $\hat{p} = p - 2\,\gamma_y$, and assume that $\beta(y,T)$ satisfies $\beta_T/\beta_y = -\lambda\,\hat{p}$. Then, $\hat{p}$ is a solution to ACH and so is $\overline{p}$ given by* (5.21)*.*

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