

# CONTEMPORARY MATHEMATICS

563

## Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics

Jairo Charris Seminar 2010  
Universidad Sergio Arboleda  
Santa Marta, Colombia

Primitivo B. Acosta-Humánez  
Federico Finkel  
Niky Kamran  
Peter J. Olver  
Editors



American Mathematical Society  
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2010 *Mathematics Subject Classification*. Primary 12H05, 33E30, 81Q60, 81Q80, 82B23, 33E99.

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### Library of Congress Cataloging-in-Publication Data

Algebraic aspects of Darboux transformations, quantum integrable systems and supersymmetric quantum mechanics / Primitivo B. Acosta-Humánez...[et al.], editors.

p. cm. — (Contemporary mathematics ; v. 563)

Includes bibliographical references.

ISBN 978-0-8218-7584-1 (alk. paper)

1. Differential algebra.. 2. Darboux transformations. 3. Quantum theory—Mathematics. I. Acosta-Humánez, Primitivo B.,

QA247.4.A45 2012  
512'.56—dc23

2011050423

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## Preface

Jairo Charris (1939–2003) was a celebrated Colombian mathematician who worked in the field of orthogonal polynomials and special functions. Every year, as a recognition of his legacy in Colombia and the relevant contributions he made to this field, an international high-level meeting in mathematics — the *Jairo Charris seminar* — is organized in his country. The topics covered in these meetings include (but are not restricted to) those in which Jairo Charris worked during his life.

This volume corresponds to the Jairo Charris Seminar 2010 entitled “Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics” which was held at the Universidad Sergio Arboleda, in Santa Marta, Colombia in August 2010. The aim of this conference was to discuss recent developments and several approaches to the algebraic aspects of Darboux transformations, quantum integrable systems and supersymmetric quantum mechanics.

Some papers in this volume are based on the talks delivered by the authors in the meeting, while the rest of the contributions are invited papers.

The contribution by Brezhnev treats the classical spectral problem  $\Psi'' - u(x)\Psi = \lambda\Psi$  and its finite-gap potentials as exactly solvable in quadratures by the Picard–Vessiot approach, without involving special functions. He shows that the duality between spectral and quadrature approaches is realized through the Weierstrass permutation theorem for a logarithmic Abelian integral. From this standpoint, he inspects known facts and obtain new ones: an important formula for the  $\Psi$ -function and the  $\Theta$ -function extensions of Picard–Vessiot fields. In particular, extensions by Jacobi’s  $\theta$ -functions lead to the (quadrature) algebraically integrable equations for the  $\theta$ -functions themselves.

In their contribution, Dutta and Roy study information theoretic measures of uncertainty for Darboux transformed partner potentials of linear and radial harmonic type. In particular, they evaluate Shannon, Renyi and Fisher lengths for the isospectral partner potentials whose solutions are given in terms of exceptional orthogonal polynomials, and the results are compared with the corresponding ones for the harmonic oscillator.

Gómez-Ullate, Kamran and Milson survey some recent developments in the theory of orthogonal polynomials defined by differential equations. The key finding is that there exist orthogonal polynomials defined by 2nd order differential equations



that fall outside the classical families of Jacobi, Laguerre, and Hermite polynomials. Unlike the classical families, these new examples, called exceptional orthogonal polynomials, feature non-standard polynomial flags; the lowest degree polynomial has degree  $m > 0$ . In this contribution the authors review the classification of codimension  $m = 1$  exceptional polynomials, and give a novel, compact proof of the fundamental classification theorem for codimension 1 polynomial flags. They also describe the mechanism or rational factorizations of 2nd order operators as the analogue of the Darboux transformation in this context. The authors include an example of a higher codimension generalization of the classical Jacobi polynomials, and perform the complete analysis of the values of the parameter for which these families have non-singular weights.

The contribution by Gonzalez Leon, de la Torre Mayado, Mateos Guilarte and Senosiain deals with the analysis of two planar supersymmetric quantum mechanical systems built around the quantum integrable Kepler/Coulomb and Euler/Coulomb problems. The supersymmetric spectra of both systems are unveiled, profiting from symmetry operators not related to the invariance with respect to rotations. It is shown analytically how the first problem arises at the limit of zero distance between the centers of the second problem. It appears that the supersymmetric modified Euler/Coulomb problem is a quasi-isospectral deformation of the supersymmetric Kepler/Coulomb problem.

Combining recent results on rational solutions of the Riccati–Schrödinger equations for shape invariant potentials to the scheme developed by Tkachuk, Fellows and Smith in the case of the one-dimensional harmonic oscillator, Grandati and Bérard show in their contribution that it is possible to generate an infinite set of solvable rational extensions for every translationally shape-invariant potential of second category.

In his contribution, Ovsienko discusses recent results and open problems related to a very special discrete dynamical system called the pentagram map. The pentagram map acts on the moduli space  $\mathcal{C}_n$  of projective equivalence classes of  $n$ -gons in the projective plane. Its continuous limit is the famous Boussinesq equation. The most remarkable property of the pentagram map is its complete integrability recently proved for the (larger) space of twisted  $n$ -gons. Integrability of the pentagram map on  $\mathcal{C}_n$  is still an open problem. He discusses the relation of the pentagram map to the space of 2-frieze patterns generalizing that of the classical Coxeter–Conway frieze patterns. This space has a structure of cluster manifold, and also possesses a natural symplectic form.

The contribution of Reyes deals with some aspects of the geometric approach to differential equations due to preeminent mathematicians such as Sophus Lie, Gaston Darboux and Élie Cartan. He considers some basic aspects of the formulation of differential equations using jet bundles and, as a non-trivial example, he states Gromov’s  $h$ -principle and applies it to systems of differential equations. Following some recent work on holonomic approximations due to Eliashberg and Mishachev, the author presents a geometric theorem on local existence of approximate solutions to PDEs. He then reviews the theory of symmetries of differential equations,

with particular emphasis on internal and nonlocal symmetries. He advances a very natural approach to nonlocal symmetries using exterior differential systems, and he argues, motivated by calculations carried out in the context of the Camassa–Holm equation, that nonlocal symmetries can be considered as generalizations of the internal symmetries introduced by É. Cartan. Finally he explains, using the associated Camassa–Holm equation, how to derive Darboux transformations using nonlocal symmetries and pseudo-potentials.

The contribution by Schulze-Halberg is devoted to the construction of Darboux transformations for the time-dependent Schrödinger equations in arbitrary spatial dimensions. The Darboux operator that connects a pair of Schrödinger equations and the corresponding potential difference are obtained in explicit form. An example in (5+1) dimensions is presented and the representation of the Darboux operator in different coordinate systems is discussed.

The univariate elliptic beta integral was discovered by Spiridonov in 2000. Recently Bazhanov and Sergeev have interpreted it as a star-triangle relation (STR). This important observation is discussed in more detail in connection to Spiridonov’s previous work on the elliptic modular double and supersymmetric dualities. In his contribution, Spiridonov describes also a new Faddeev–Volkov type solution of STR, connections with the star-star relation, and higher-dimensional analogues of such relations. In this picture, Seiberg dualities are described by symmetries of the elliptic hypergeometric integrals (interpreted as superconformal indices) which, in turn, represent STR and Kramers–Wannier type duality transformations for elementary partition functions in solvable models of statistical mechanics.

Finally the editors would like to acknowledge the assistance of all people involved with the Jairo Charris Seminar 2010: Scientific Committee, Organizing Committee, Speakers, and also the Instituto de Matemáticas y sus Aplicaciones and the Universidad Sergio Arboleda at Santa Marta as host institutions for this Jairo Charris Seminar.

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This volume represents the 2010 Jairo Charris Seminar in Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics, which was held at the Universidad Sergio Arboleda in Santa Marta, Colombia.

The papers cover the fields of Supersymmetric Quantum Mechanics and Quantum Integrable Systems, from an algebraic point of view. Some results presented in this volume correspond to the analysis of Darboux Transformations in higher order as well as some exceptional orthogonal polynomials.

The reader will find an interesting Galois approach to study finite gap potentials.

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ISBN 978-0-8218-7584-1



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