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Homomorphisms between neighboring G_1T-Verma modules

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Let G be a reductive group over an algebraically closed field $\Bbbk$ of positive characteristic p, B a Borel subgroup of G, T a maximal torus of B, G_1 (resp. B_1) the Frobenius kernel of G (resp. B), and Λ the character group of B. Let $\widehat{\nabla}$ denote the induction functor from the category of B_1T-modules to the category of G_1T-modules. We will rework Koppinen's results [**Kop**] on homomorphisms between neighboring Weyl modules for neighboring $\widehat{\nabla}(\lambda)$'s, $\lambda \in \Lambda$. By their infinitesimal nature, given [**D, AK, J**], the arguments are considerably easier than in [**Kop**].

As any 1-dimensional representation of a unipotent group is trivial, Λ may also be thought of as the character group of T. Thus let $R \subset \Lambda$ be the root system of G relative to T. We choose a positive system R^+ of R such that the roots of B are $-R^+$, and denote the set of simple roots by R^s. Let W be the Weyl group of G and $W_p = W \ltimes p\mathbb{Z}R$, acting on Λ via the dot action: $x \bullet \lambda = x(\lambda+\rho)-\rho$, $x \in W_p$, $\lambda \in \Lambda$, $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. Given $\lambda, \mu \in \Lambda$ we modify Koppinen's notion of a small interval between μ and λ to suit our need, and show that if the interval is small, the set of G_1T-homomorphisms from $\widehat{\nabla}(\lambda)$ to $\widehat{\nabla}(\mu)$ is nonzero.

Precisely, we define the interval $[\mu, \lambda]_r$, r for "restricted", from μ to λ to be the set $\{\nu \in \Lambda \mid \mu \uparrow\uparrow \nu \uparrow\uparrow \lambda\} \setminus \{\nu - pn\beta \mid \nu \uparrow\uparrow \lambda, \beta \in R^+, p|\langle \nu+\rho, \beta^\vee\rangle, n \in \mathbb{N}^+\}$ with respect to the strong linkage relation $\uparrow\uparrow$ but disallowing reflections in the hyperplanes parallel to the ones containing the weights in question. We say $[\mu, \lambda]_r$ is *small* iff it contains no consecutive weights reflected in the same directions, and show that there is a nonzero G_1T-homorphism from $\widehat{\nabla}(\lambda)$ to $\widehat{\nabla}(\mu)$ in case $[\mu, \lambda]_r$ is small. As a corollary, assuming Lusztig's conjecture on the irreducible characters for G [**L**], we determine the socle level of $\widehat{\nabla}(0)$ on which the simple factor $\hat{L}(w \bullet 0)$, $w \in W$, of highest weight $w \bullet 0$ appears. Here and elsewhere 0 in $\widehat{\nabla}(0)$ denotes 1-dimensional B_1T-module induced by $0 \in \Lambda$, not the zero module. It verifies in general an observation made in [**KY**] for G of rank at most 2 and is compatible with [**K**]. Thanks to [**KT, KL, L94, AJS**], or more recently, [**F**], the Lusztig conjecture is now a theorem for large p, and hence we also obtain positivity of the

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constant terms of certain periodic inverse Kazhdan-Lusztig polynomials.

1°. Intertwining homomorphisms of G_1T-Verma modules

(1.1) We will let $\mathcal{C}$ denote the category of G_1T-modules, and $\mathcal{C}(P,Q)$ the set of G_1T-linear homomorphisms from a G_1T-module P to another G_1T-module Q. Let us first recall from [**D, AK**]/[**J**, II.9] some basics of intertwining homomorphisms of G_1T-Verma modules.

For each $w \in W$ let $\widehat{\nabla}_w$ be the induction functor from the category of $(wBw^{-1})_1T$-modules to $\mathcal{C}$. For each $\nu \in \Lambda$ put $\nu\langle w\rangle = \nu + (p-1)(w \bullet 0)$ after [**AJS, J**]. The formal character of $\widehat{\nabla}_w(\nu\langle w\rangle)$ is the same as that of $\widehat{\nabla}(\nu)$. Let $\alpha \in R^s$, $s_\alpha \in W$ the reflection associated to α, $w \in W$, and write $\langle \nu+\rho, w\alpha^\vee\rangle \equiv d$ mod p with $d \in [0,p[= \{n \in \mathbb{Z} \mid 0 \le n < p\}$. If $d > 0$, we have 2 exact sequences

$$(1)\quad \widehat{\nabla}_w((\nu+(p-d)w\alpha)\langle w\rangle) \to \widehat{\nabla}_w(\nu\langle w\rangle) \to \widehat{\nabla}_w((\nu-dw\alpha)\langle w\rangle) \to \widehat{\nabla}_w((\nu-pw\alpha)\langle w\rangle),$$

$$(2)\quad \widehat{\nabla}_{ws_\alpha}((\nu-dw\alpha)\langle ws_\alpha\rangle) \to \widehat{\nabla}_{ws_\alpha}(\nu\langle ws_\alpha\rangle) \to \widehat{\nabla}_w(\nu\langle w\rangle) \to \widehat{\nabla}_w((\nu-dw\alpha)\langle w\rangle)$$

while for $d = 0$

$$(3)\quad \widehat{\nabla}_w(\nu\langle w\rangle) \simeq \widehat{\nabla}_{ws_\alpha}(\nu\langle ws_\alpha\rangle).$$

Regardless of d we have

$$(4)\quad \mathcal{C}(\widehat{\nabla}_{ws_\alpha}(\nu\langle ws_\alpha\rangle), \widehat{\nabla}_w(\nu\langle w\rangle)) \simeq \Bbbk$$

and that if $w_0 = s_1 s_2 \dots s_N$ is a reduced expression of the longest element w_0 of W with $s_i = s_{\alpha_i}$, $\alpha_i \in R^s$, and if $\phi_i \in \mathcal{C}(\widehat{\nabla}_{s_1\dots s_{i-1}s_i}(\nu\langle s_1\dots s_{i-1}s_i\rangle), \widehat{\nabla}_{s_1\dots s_{i-1}}(\nu\langle s_1\dots s_{i-1}\rangle)) \setminus 0$, then

$$(5)\quad \operatorname{im}(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_N) = \hat{L}(\nu).$$

Note also that for SL_2, in case $d = 0$, $\widehat{\nabla}(\nu)$ is irreducible, and hence $\mathcal{C}(\widehat{\nabla}(\nu), \widehat{\nabla}(\nu - pn\alpha)) = 0$ $\forall n \neq 0$.

(1.2) Let us denote by $\uparrow\uparrow$ the strong linkage relation on Λ generated by the pairs $(s_{\beta,np} \bullet \lambda = s_\beta \bullet \lambda + np\beta, \lambda)$, $\beta \in R^+$, $n \in \mathbb{Z}$, such that $np \le \langle \lambda+\rho, \beta^\vee\rangle$. We will write $\lambda \xrightarrow{\beta} \nu$ iff $p \nmid \langle \lambda+\rho, \beta^\vee\rangle$ and $\nu = s_{\beta,np} \bullet \lambda$ for some $n \in \mathbb{Z}$ such that $np < \langle \lambda+\rho, \beta^\vee\rangle$. Let $\uparrow\uparrow_r$ denote the partial order generated by these $\lambda \xrightarrow{\beta} \nu$. Thus $\uparrow\uparrow$ and $\uparrow\uparrow_r$ coincide on p-regular weights, i.e., on those λ such that $p \nmid \langle \lambda+\rho, \beta^\vee\rangle$ $\forall \beta \in R^+$.

For a G_1T-module M we will write $[M : \hat{L}(\nu)]$ for the multiplicity of the simple module $\hat{L}(\nu)$ of highest weight ν in a composition series of M. If $[\widehat{\nabla}(\lambda) : \hat{L}(\nu)] \neq 0$, one has $\nu \uparrow\uparrow \lambda$ by the strong linkage principle [**D, D89**], and, in fact, we have $\nu \uparrow\uparrow_r \lambda$; if $\nu < \lambda$, then by (1.1.5) and by (1.1.2) $\exists i \in [1,N]$ with $d_i > 0$:

$$0 \neq [\widehat{\nabla}_{s_1\dots s_i}((\lambda - d_i\beta_i)\langle s_1\dots s_i\rangle) : \hat{L}(\nu)] = [\widehat{\nabla}(\lambda - d_i\beta_i) : \hat{L}(\nu)],$$

where $\beta_i = s_1 \dots s_{i-1}\alpha_i$ and $\langle \lambda+\rho, \beta_i^\vee\rangle \equiv d_i \mod p$ with $d_i \in [0,p[$.

(1.3) Let us also note that [**Kop**, Lemma 6.2] carries over to our setup; $\forall\lambda\in\Lambda$, $\mathrm{soc}\widehat{\nabla}(\lambda) = \hat{L}(\lambda)$, and $\forall\nu\in\Lambda$, $\forall i\in\mathbb{N}$, $\mathrm{Ext}^i_{\mathcal{C}}(\widehat{\nabla}_{w_0}(\nu\langle w_0\rangle), \widehat{\nabla}(\lambda)) \simeq \delta_{i0}\delta_{\nu\lambda}\Bbbk$, where $w_0\in W$ is such that $w_0R^+ = -R^+$.

Lemma: Let $\lambda\in\Lambda$ and let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4$ be an exact sequence in $\mathcal{C}$.

(i) If $[M_4 : \hat{L}(\lambda)] = 0$, then $\mathcal{C}(f_2, \widehat{\nabla}(\lambda)) : \mathcal{C}(M_3, \widehat{\nabla}(\lambda)) \to \mathcal{C}(M_2, \widehat{\nabla}(\lambda))$ is injective.

(ii) If $[M_1 : \hat{L}(\lambda)] = 0 = [M_4 : \hat{L}(\eta)]$ $\forall\eta\in\Lambda$ with $\lambda \upuparrows_r \eta$, then $\mathcal{C}(f_2, \widehat{\nabla}(\lambda))$ is bijective.

2°. Small intervals

(2.1) For each λ and $\mu\in\Lambda$ we define an *interval* from μ to λ to be

$$[\mu,\lambda]_r = \{\nu\in\Lambda \mid \mu \upuparrows_r \nu \upuparrows_r \lambda\}.$$

We say it is *small* iff it contains no sequence of form $\nu_1 \xrightarrow{\beta} \nu_2 \xrightarrow{\beta} \nu_3$, $\beta\in R^+$. Thus the sequence $\nu_1 \xrightarrow{\beta} \nu_2$ belongs to a small interval only if $\nu_2 = s_{\beta,np}\bullet\nu_1$ with $\langle \nu_1+\rho, \beta^\vee\rangle - pn \in]0,p[= \{n\in\mathbb{Z} \mid 0<n<p\}$.

(2.2) Fix a reduced expression $w_0 = s_1s_2\dots s_N$ of the longest element w_0 of W with $s_i = s_{\alpha_i}$, $\alpha_i\in R^s$. Put $w_i = s_{i+1}\dots s_{N-1}s_N$, $i\in[0,N]$, with $w_N = e$, $z_i = s_1s_2\dots s_i$ with $z_0 = e$, and $\beta_{i+1} = z_i\alpha_{i+1}$. Thus $R^+ = \{\beta_i \mid i\in[1,N]\}$. Let $[\mu,\lambda]_r$ be a small interval and put

$$I_r(\mu,\lambda) = \{i\in[1,N] \mid \exists\beta_i \text{ and } \nu\in[\mu,\lambda]_r : \lambda \xrightarrow{\beta_i} \nu\}.$$

One checks that [**Kop**, Proposition 5.1] carries over to the present setup; the verification is reduced to the rank 2 cases, where one can check an analogue of [**Kop**, Lemma 5.2] by exhausting the cases.

Proposition: Let $[\mu,\lambda]_r$ be a small interval with $\mu<\lambda$. Let $m = \min I_r(\mu,\lambda)$ with $\lambda \xrightarrow{\beta_m} \nu\in[\mu,\lambda]_r$. If $\mu<\nu$, then $m<\min I_r(\mu,\nu)$.

(2.3) We now prove an analogue of [**Kop**, Theorem 6.1].

Theorem: Assume $[\mu,\lambda]_r$ is small with $\mu<\lambda$ and keep the notations of (2.2). For each $i\in[0,\min I_r(\mu,\lambda)]$

$$\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \simeq \mathcal{C}(\widehat{\nabla}_{z_i}(\lambda\langle z_i\rangle), \widehat{\nabla}(\mu)) \neq 0.$$

Proof: For each $j\in[1,N]$ write $\langle\lambda+\rho,\beta_j^\vee\rangle \equiv d_j \mod p$ with $d_j\in[0,p[$. If $d_j\neq 0$, we have from (1.1.1) and (1.1.2) exact sequences

$$(1)\qquad \widehat{\nabla}_{z_{j-1}}(\lambda\langle z_{j-1}\rangle) \to \widehat{\nabla}_{z_{j-1}}((\lambda-d_j\beta_j)\langle z_{j-1}\rangle) \to \widehat{\nabla}_{z_{j-1}}((\lambda-p\beta_j)\langle z_{j-1}\rangle),$$

$$(2)\qquad \widehat{\nabla}_{z_j}((\lambda-d_j\beta_j)\langle z_j\rangle) \to \widehat{\nabla}_{z_j}(\lambda\langle z_j\rangle) \to \widehat{\nabla}_{z_{j-1}}(\lambda\langle z_{j-1}\rangle) \to \widehat{\nabla}_{z_{j-1}}((\lambda-d_j\beta_j)\langle z_{j-1}\rangle).$$

In case $d_j = 0$ we have from (1.1.3)

$$\widehat{\nabla}_{z_{j-1}}(\lambda\langle z_{j-1}\rangle) \simeq \widehat{\nabla}_{z_j}(\lambda\langle z_j\rangle). \tag{3}$$

Let $j \notin I_r(\mu,\lambda)$. If $d_j = 0$, then by (3)

$$\mathcal{C}(\widehat{\nabla}_{z_j}(\lambda\langle z_j\rangle), \widehat{\nabla}(\mu)) \simeq \mathcal{C}(\widehat{\nabla}_{z_{j-1}}(\lambda\langle z_{j-1}\rangle), \widehat{\nabla}(\mu)). \tag{4}$$

If $d_j \neq 0$, then $[\widehat{\nabla}_{z_j}((\lambda - d_j\beta_j)\langle z_j\rangle) : \hat{L}(\mu)] = 0$; otherwise $[\widehat{\nabla}(\lambda - d_j\beta_j) : \hat{L}(\mu)] \neq 0$, and hence $\mu \upuparrows_r \lambda - d_j\beta_j \overset{\beta_j}{\leftarrow} \lambda$ by (1.2), contrary to the choice of j. If $\mu \upuparrows_r \eta$ with $[\widehat{\nabla}_{z_{j-1}}((\lambda - d_j\beta_j)\langle z_{j-1}\rangle) : \hat{L}(\eta)] \neq 0$, then $\mu \upuparrows_r \eta \upuparrows_r \lambda - d_j\beta_j \overset{\beta_j}{\leftarrow} \lambda$, contrary to the choice of j again. It follows from (1.3) that (4) remains to hold.

Put $m = \min I_r(\mu,\lambda)$ and $\lambda(m) = \lambda - d_m\beta_m \overset{\beta_m}{\leftarrow} \lambda$. Applying (1.3) with $j = m$ to (1) yields

$$\mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda(m)\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)) \leq \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)); \tag{5}$$

if $[\widehat{\nabla}_{z_{m-1}}((\lambda - p\beta_m)\langle z_{m-1}\rangle) : \hat{L}(\mu)] \neq 0$, then by (1.2) again $\mu \upuparrows_r \lambda - p\beta_m \overset{\beta_m}{\leftarrow} \lambda - d_m\beta_m \overset{\beta_m}{\leftarrow} \lambda$, contradicting the smallness of $[\mu,\lambda]_r$. Thus, together with (4), one obtains

$$\begin{aligned} \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) &\simeq \mathcal{C}(\widehat{\nabla}_{z_1}(\lambda\langle z_1\rangle), \widehat{\nabla}(\mu)) \simeq \mathcal{C}(\widehat{\nabla}_{z_2}(\lambda\langle z_2\rangle), \widehat{\nabla}(\mu)) \simeq \dots \\ &\simeq \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)) \geq \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda(m)\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)). \end{aligned} \tag{6}$$

Finally, if $\lambda(m) = \mu$, recall from the proof of [**AK**, 2.2.i] that $\mathcal{C}(\widehat{\nabla}_w(\mu\langle w\rangle), \widehat{\nabla}(\mu)) \simeq \Bbbk \ \forall w \in W$. If $\lambda(m) \neq \mu$, $0 \leq m-1 < m < \min I_r(\mu,\lambda(m))$ by (2.2), and hence the assertion will follow by induction on the size of $[\mu,\lambda]_r$.

(2.4) **Corollary:** *Let $[\mu,\lambda]_r$ be a small interval with $\mu < \lambda$. Let $m = \min I_r(\mu,\lambda)$ and put $\nu = s_{\beta_m,np} \bullet \lambda \in [\mu,\lambda]_r$ with $\langle \lambda+\rho, \beta_m^\vee\rangle - np \in]0,p[$. Then there is $f \in \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\nu))$ such that $\mathcal{C}(f, \widehat{\nabla}(\mu)) : \mathcal{C}(\widehat{\nabla}(\nu), \widehat{\nabla}(\mu)) \to \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu))$ is injective.*

Proof: In the notation of (2.3) we have $\nu = \lambda(m)$. We may also assume that $\mu < \lambda(m)$. If $f_1 : \widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle) \to \widehat{\nabla}_{z_{m-1}}(\lambda(m)\langle z_{m-1}\rangle)$ from (2.3.1), we have

$$\mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda(m)\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)) \xhookrightarrow{\mathcal{C}(f_1,\widehat{\nabla}(\mu))} \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)).$$

As in (2.3.6) the composite $f_2 : \widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle) \to \cdots \to \widehat{\nabla}_{z_1}(\lambda\langle z_1\rangle) \to \widehat{\nabla}(\lambda)$ induces a bijection

$$\mathcal{C}(f_2, \widehat{\nabla}(\mu)) : \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \to \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle), \widehat{\nabla}(\mu)).$$

As $0 \leq m-1 < m < \min I_r(\mu,\lambda(m)) = \min I_r(\mu,\nu)$ by (2.2), the composite $f_3 : \widehat{\nabla}_{z_{m-1}}(\nu\langle z_{m-1}\rangle) \to \cdots \to \widehat{\nabla}_{z_1}(\nu\langle z_1\rangle) \to \widehat{\nabla}(\nu)$ induces likewise a bijection $\mathcal{C}(f_3, \widehat{\nabla}(\mu)) : \mathcal{C}(\widehat{\nabla}(\nu), \widehat{\nabla}(\mu)) \to \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\nu\langle z_{m-1}\rangle), \widehat{\nabla}(\mu))$. Thus $\forall h \in \mathcal{C}(\widehat{\nabla}(\nu), \widehat{\nabla}(\mu))$, $\exists! h' \in \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu))$: $h \circ f_3 \circ f_1 = h' \circ f_2$.

On the other hand, as $m = \min I_r(\nu,\lambda)$, f_2 induces as in (2.3.6) again a bijection $\mathcal{C}(f_2, \widehat{\nabla}(\nu)) : \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\nu)) \to \mathcal{C}(\widehat{\nabla}_{z_{m-1}}(\lambda\langle z_{m-1}\rangle), \widehat{\nabla}(\nu))$. Thus $\exists! f \in$

$\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\nu)) : f_3 \circ f_1 = f \circ f_2$. Then $h' \circ f_2 = h \circ f_3 \circ f_1 = h \circ f \circ f_2$, and hence $h' = h \circ f$ by the injectivity of $\mathcal{C}(f_2, \widehat{\nabla}(\nu))$. It follows that $\mathcal{C}(f, \widehat{\nabla}(\mu)) = \mathcal{C}(f_2, \widehat{\nabla}(\mu))^{-1} \circ \mathcal{C}(f_3 \circ f_1, \widehat{\nabla}(\mu)) = \mathcal{C}(f_2, \widehat{\nabla}(\mu))^{-1} \circ \mathcal{C}(f_1, \widehat{\nabla}(\mu)) \circ \mathcal{C}(f_3, \widehat{\nabla}(\mu)) : \mathcal{C}(\widehat{\nabla}(\nu), \widehat{\nabla}(\mu)) \to \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu))$ is an injection.

(2.5) **Corollary:** *If* $[\mu, \lambda]_r = \{\mu, \lambda\}$ *is small, then* $[\widehat{\nabla}(\lambda) : \hat{L}(\mu)] = 1$ *and* $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \simeq \Bbbk$.

Proof: If $\widehat{Q}(\mu)$ is the injective hull of $\hat{L}(\mu)$, $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \leq \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{Q}(\mu))$ and $\dim \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{Q}(\mu)) = [\widehat{\nabla}(\lambda) : \hat{L}(\mu)]$. It is therefore enough to show that $[\widehat{\nabla}(\lambda) : \hat{L}(\mu)] \leq 1$. One has from (1.1.5 and 2)

$$[\widehat{\nabla}(\lambda) : \hat{L}(\mu)] \leq \sum_{d_i > 0} [\widehat{\nabla}(\lambda - d_i \beta_i) : \hat{L}(\mu)],$$

and hence the assertion follows by the minimality of $[\mu, \lambda]_r$.

(2.6) **Remark:** Once we knew $[\widehat{\nabla}(\lambda) : \hat{L}(\mu)] \neq 0$, we could also argue for the nonvanishing of $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu))$ as follows. Let $f \in \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{Q}(\mu)) \setminus 0$. The short exact sequence $0 \to \hat{L}(\mu) \to \mathrm{im} f \to Q \to 0$ induces a long exact sequence $\mathcal{C}(\mathrm{im} f, \widehat{\nabla}(\mu)) \to \mathcal{C}(\hat{L}(\mu), \widehat{\nabla}(\mu)) \to \mathrm{Ext}^1_{\mathcal{C}}(Q, \widehat{\nabla}(\mu))$. If $\mathcal{C}(\mathrm{im} f, \widehat{\nabla}(\mu)) \neq 0$, we will be done. Otherwise, $\mathrm{Ext}^1_{\mathcal{C}}(Q, \widehat{\nabla}(\mu)) \neq 0$. Then there would be a composition factor $\hat{L}(\nu)$ of Q, hence $\nu \uparrow\uparrow_r \lambda$ by (1.2) and $\nu < \lambda$ such that $\mathrm{Ext}^1_{\mathcal{C}}(\hat{L}(\nu), \widehat{\nabla}(\mu)) \neq 0$. Then $\nu > \mu$ by [**J**, II.9.8], and hence $\mathrm{Ext}^1_{\mathcal{C}}(\hat{L}(\nu), \widehat{\nabla}(\mu)) \simeq \mathcal{C}(\mathrm{rad}\widehat{\nabla}_{w_0}(\nu\langle w_0 \rangle), \widehat{\nabla}(\mu))$. It would follow that $[\mathrm{rad}\widehat{\nabla}_{w_0}(\nu\langle w_0 \rangle) : \hat{L}(\mu)] \neq 0$, and hence $\mu \uparrow\uparrow_r \nu$, contradicting the minimality of $[\mu, \lambda]_r$.

3°. Around special points

(3.1) Assume G is semisimple and simply connected, so $\rho \in \Lambda$. We say $\lambda \in \Lambda$ is *special* iff $\lambda \in -\rho + p\Lambda$. As tensoring with $p\eta$, $\eta \in \Lambda$, is an automorphism of our category $\mathcal{C}$, we will consider points around the special point $-\rho$. Let F be a facet for W_p containing $-\rho$ in its closure. If $\nu \in F \cap \Lambda$, the W-orbit $W \bullet \nu$ forms a small interval [**Kop**, 3.2].

Theorem: Let F be a facet containing $-\rho$ in its closure, let $\nu \in F \cap \Lambda$, and put $J = W \bullet \nu$. Let $\lambda, \mu, \eta \in J$.

(i) $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \neq 0$ iff $\mu \uparrow\uparrow_r \lambda$, in which case $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \simeq \Bbbk$.

(ii) If $f \in \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \setminus 0$ and if $h \in \mathcal{C}(\widehat{\nabla}(\mu), \widehat{\nabla}(\eta)) \setminus 0$, then $h \circ f \neq 0$.

Proof: Write $J = [\lambda^-, \lambda^+]_r$. $\forall \lambda \in J$, using the translation functors we have

$$\mathcal{C}(\mathrm{T}^{\nu}_{-\rho} \widehat{\nabla}(-\rho), \widehat{\nabla}(\lambda)) \simeq \mathcal{C}(\widehat{\nabla}(-\rho), \mathrm{T}^{-\rho}_{\nu} \widehat{\nabla}(\lambda)) \simeq \mathcal{C}(\widehat{\nabla}(-\rho), \widehat{\nabla}(-\rho)) \simeq \Bbbk.$$

Recall from [**J**, II.9.8] that there is an epi $\mathrm{T}_{-\rho}^{\nu}\widehat{\nabla}(-\rho) \to \widehat{\nabla}(\lambda^+)$. Then $\Bbbk \simeq \mathcal{C}(\mathrm{T}_{-\rho}^{\nu}\widehat{\nabla}(-\rho), \widehat{\nabla}(\lambda)) \geq \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\lambda)) \neq 0$ by (2.3), and hence

$$\mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\lambda)) \simeq \Bbbk. \tag{1}$$

Consider a linkage $\mu \xrightarrow{\alpha} \mu'$, $\alpha \in R^s$, in J. Take a reduced expression $w_0 = s_1 \dots s_N$ such that $\alpha = \alpha_1 = \beta_1$ in the notation of (2.2). Then $\forall \lambda \in [\lambda^-, \mu']_r$, $\min I_r(\lambda, \mu) = 1$. Let $f \in \mathcal{C}(\widehat{\nabla}(\mu), \widehat{\nabla}(\mu')) \setminus 0$. As $\alpha \in R^s$, $\mathcal{C}(\widehat{\nabla}(\mu), \widehat{\nabla}(\mu')) \simeq \Bbbk$ by (2.5), and one has an injection by (2.4)

$$\mathcal{C}(f, \widehat{\nabla}(\lambda)) : \mathcal{C}(\widehat{\nabla}(\mu'), \widehat{\nabla}(\lambda)) \rightarrowtail \mathcal{C}(\widehat{\nabla}(\mu), \widehat{\nabla}(\lambda)). \tag{2}$$

Let $\lambda \in J$ be arbitrary again. As $\lambda \in W \bullet \lambda^+$, there is a chain of strong linkage $\lambda^+ = \lambda_0 \xrightarrow{\gamma_1} \lambda_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_n} \lambda_n = \lambda$ with all $\gamma_i \in R^s$. Let $\phi_i \in \mathcal{C}(\widehat{\nabla}(\lambda_{i-1}), \widehat{\nabla}(\lambda_i)) \setminus 0$ and put $\phi = \phi_n \circ \cdots \circ \phi_2 \circ \phi_1 : \widehat{\nabla}(\lambda^+) \to \widehat{\nabla}(\lambda)$. By (2) $\forall \mu \in [\lambda^-, \lambda]_r$,

$$\begin{aligned} \mathcal{C}(\phi, \widehat{\nabla}(\mu)) = \mathcal{C}(\phi_1, \widehat{\nabla}(\mu)) \circ \mathcal{C}(\phi_2, \widehat{\nabla}(\mu)) \cdots \circ \mathcal{C}(\phi_n, \widehat{\nabla}(\mu)) :& \\ \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \rightarrowtail \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\mu)).& \end{aligned} \tag{3}$$

It follows from (1) and (2.3) that $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\mu)) \simeq \Bbbk$, and (i) holds.

By the translation principle $\forall \lambda \in J$,

$$[\widehat{\nabla}(\lambda) : \hat{L}(\lambda^-)] = 1. \tag{4}$$

Taking $\mu = \lambda^-$ in (3), we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\lambda^-)) & \xrightarrow{\mathcal{C}(\phi, \widehat{\nabla}(\lambda^-))} & \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\lambda^-)). \\ & \searrow \qquad \nearrow & \\ & \mathcal{C}(\mathrm{im}\phi, \widehat{\nabla}(\lambda^-)) & \end{array}$$

As $\mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\lambda^-)) \neq 0$ by (i), $[\mathrm{im}\phi : \hat{L}(\lambda^-)] \neq 0$, and hence

$$[\mathrm{coker}\phi : \hat{L}(\lambda^-)] = 0. \tag{5}$$

Let $\xi \xrightarrow{\beta} \xi'$ be a linkage in J, $\beta \in R^+$, and $f' \in \mathcal{C}(\widehat{\nabla}(\xi), \widehat{\nabla}(\xi')) \setminus 0$. Take chains of linkages $\lambda^+ \xrightarrow{\gamma_1} \xi_1 \dots \xrightarrow{\gamma_n} \xi$ and $\lambda^+ \xrightarrow{\gamma_1'} \xi_1' \dots \xrightarrow{\gamma_m'} \xi'$ with all $\gamma_i, \gamma_j' \in R^s$, and let $\psi \in \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\xi))$, $\psi' \in \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\xi'))$ be the corresponding composites. By (3), $\mathcal{C}(\psi, \widehat{\nabla}(\xi)) : \mathcal{C}(\widehat{\nabla}(\xi), \widehat{\nabla}(\xi')) \to \mathcal{C}(\widehat{\nabla}(\lambda^+), \widehat{\nabla}(\xi'))$ is monic, and hence $f' \circ \psi \neq 0$. Then $f' \circ \psi \in \Bbbk^\times \psi'$. By (5) one has $[\mathrm{im}\psi : \hat{L}(\lambda^-)] = 1 = [\mathrm{im}\psi' : \hat{L}(\lambda^-)] = [\mathrm{im}(f' \circ \psi) : \hat{L}(\lambda^-)]$. It follows from (4) that

$$[\mathrm{im} f' : \hat{L}(\lambda^-)] = 1 \quad \text{and} \quad [\mathrm{coker} f' : \hat{L}(\lambda^-)] = 0. \tag{6}$$

If $\xi' \xrightarrow{\beta'} \xi''$ is another linkage in J and if $f'' \in \mathcal{C}(\widehat{\nabla}(\xi'), \widehat{\nabla}(\xi'')) \neq 0$, $[\mathrm{im}(f'' \circ f') : \hat{L}(\lambda^-)] = 1$ by (6) as $[\mathrm{im}(f'') : \hat{L}(\lambda^-)] = 1$. Repeat to get $\forall \lambda' \uparrow\uparrow_r \lambda$ in J and $\forall f \in \mathcal{C}(\widehat{\nabla}(\lambda), \widehat{\nabla}(\lambda')) \neq 0$, $[\mathrm{im} f : \hat{L}(\lambda^-)] = 1$ by (i), and (ii) follows.

(3.2) Write $w_0 = s_1 s_2 \dots s_N$ and put $w_i = s_{i+1}s_{i+2}\dots s_N$ $\forall i \in [0, N]$ with $w_N = e$ as in (2.2).

Corollary: Assume $p \geq h$.

(i) $\forall i \in [1, N]$, $\mathcal{C}(\widehat{\nabla}(w_i \bullet 0), \widehat{\nabla}(w_{i-1} \bullet 0)) \simeq \Bbbk$.

(ii) If $f_i \in \mathcal{C}(\widehat{\nabla}(w_i \bullet 0), \widehat{\nabla}(w_{i-1} \bullet 0)) \setminus 0$, $f_1 \circ f_2 \circ \dots \circ f_N \neq 0$.

(3.3) **Remark:** (i) could have been obtained without appealing to the theorem, using the translation principle or (1.1.1) to see $\mathcal{C}(\widehat{\nabla}(w_i \bullet 0), \widehat{\nabla}(w_{i-1} \bullet 0)) \neq 0$ and (2.6). Also, (ii) could have been obtained as in the proof of [**AK**, 6.2] as pointed out by Henning Andersen: by the translation principle again $[\widehat{\nabla}(w_i \bullet 0) : \hat{L}(w_0 \bullet 0)] = 1$ $\forall i$. As $w_{i-1} \bullet 0 = s_i w_i \bullet 0 = w_i \bullet 0 - d_i \alpha_i$ with $d_i = \langle w_i \bullet 0 + \rho, \alpha_i^\vee \rangle = \langle \rho, w_i^{-1} \alpha_i^\vee \rangle \in]0, p[$, one has from (1.1.1) an exact sequence of G_1T-modules

$$\widehat{\nabla}(w_{i-1} \bullet 0 + p\alpha_i) \to \widehat{\nabla}(w_i \bullet 0) \to \widehat{\nabla}(w_{i-1} \bullet 0).$$

As the closure of the alcove containing $w_{i-1} \bullet 0 + p\alpha_i$ does not contain $-\rho$, $[\widehat{\nabla}(w_{i-1} \bullet 0 + p\alpha_i) : \hat{L}(w_0 \bullet 0)] = 0$ by the translation principle again. It follows that $[\mathrm{im} f_i : \hat{L}(w_0 \bullet 0)] \neq 0$ and hence that $\mathrm{T}_0^{-\rho} f_i : \widehat{\nabla}(-\rho) \to \widehat{\nabla}(-\rho)$ is invertible as $\widehat{\nabla}(-\rho)$ is irreducible. Then $\mathrm{T}_0^{-\rho}(f_1 \circ f_2 \circ \dots \circ f_N)$ remains invertible as $\mathrm{T}_0^{-\rho}(f_1 \circ f_2 \circ \dots \circ f_N) = (\mathrm{T}_0^{-\rho} f_1) \circ \dots \circ (\mathrm{T}_0^{-\rho} f_N)$ by the functoriality.

(3.4) Assume in this section $p > h$ and assume the Lusztig conjecture on the irreducible characters for G [**L**]. Then we know from [**AK**] that the Loewy length of any $\widehat{\nabla}(y \bullet 0)$, $y \in W_p$, is equal to $\ell(w_0) + 1$, and that if $\mathrm{soc}_j \widehat{\nabla}(y \bullet 0)$ denotes the j-th G_1T-socle layer of $\widehat{\nabla}(y \bullet 0)$, then the multiplicity of $\hat{L}(x \bullet 0)$, $x \in W_p$, in $\mathrm{soc}_j \widehat{\nabla}(y \bullet 0)$ is given by a periodic inverse Kazhdan-Lusztig polynomial

$$Q_{w_0 y \bullet A^+, w_0 x \bullet A^+} = \sum_j q^{\frac{\mathrm{d}(x \bullet A^+, y \bullet A^+) + 1 - j}{2}} [\mathrm{soc}_j \widehat{\nabla}(y \bullet 0) : \hat{L}(x \bullet 0)],$$

where A^+ is the bottom dominant alcove and $\mathrm{d}(x \bullet A^+, y \bullet A^+)$ is the distance from alcove $x \bullet A^+$ to alcove $y \bullet A^+$ [**L80**].

Corollary: Assume the Lusztig conjecture on the irreducible characters for G. Then each $\hat{L}(w \bullet 0)$, $w \in W$, appears in $\mathrm{soc}_{\ell(w)+1} \widehat{\nabla}(0)$.

Proof: For a G_1T-module M let $\ell\ell(M)$ denote the Loewy length of M. We employ the notations from (3.3).

As each f_i annihilates $\mathrm{soc} \widehat{\nabla}_B(w_i \bullet 0) = \hat{L}(w_i \bullet 0)$, $\ell\ell(\mathrm{im}(f_{i-1} \circ f_i \circ \dots \circ f_N)) < \ell\ell(\mathrm{im}(f_i \circ f_{i+1} \circ \dots \circ f_N))$. It follows, as $\ell\ell(\widehat{\nabla}(w_i \bullet 0)) = N + 1$ $\forall i$ by [**AK**] and as $f_1 \circ f_2 \circ \dots \circ f_N \neq 0$, that $\ell\ell(\mathrm{im}(f_i \circ f_{i+1} \circ \dots \circ f_N)) = i$. Then $0 < \mathrm{rad}^i(\mathrm{im}(f_{i+1} \circ f_{i+2} \circ \dots \circ f_N)) \leq \mathrm{soc}(\mathrm{im}(f_{i+1} \circ f_{i+2} \circ \dots \circ f_N) \leq \mathrm{soc} \widehat{\nabla}(w_i \bullet 0) = \hat{L}(w_i \bullet 0)$, and hence $\hat{L}(w_i \bullet 0) = \mathrm{rad}^i(\mathrm{im}(f_{i+1} \circ f_{i+2} \circ \dots \circ f_N)) = f_{i+1} \circ f_{i+2} \circ \dots \circ f_N(\mathrm{rad}^i \widehat{\nabla}(0))$ with $f_{i+1} \circ \dots \circ f_N(\mathrm{rad}^{i+1} \widehat{\nabla}(0)) = \mathrm{rad}^{i+1}(\mathrm{im}(f_{i+1} \circ \dots \circ f_N(\widehat{\nabla}(0)))) = 0$. Thus $0 \neq [\mathrm{rad}^i \widehat{\nabla}(0) / \mathrm{rad}^{i+1} \widehat{\nabla}(0) : \hat{L}(w_i \bullet 0)] = [\mathrm{soc}_{N+1-i} \widehat{\nabla}(0) : \hat{L}(w_i \bullet 0)]$ by the rigidity of $\widehat{\nabla}(0)$ [**AK**].

As each $w \in W$ appears as some w_i in a reduced expression of w_0, the assertion follows.

(3.5) Assume $p > h$. For each $w \in W$ write $w \bullet 0 = (w \bullet 0)^0 + p(w \bullet 0)^1$ with $(w \bullet 0)^0, (w \bullet 0)^1 \in \Lambda$ such that $\langle (w \bullet 0)^0, \alpha^\vee \rangle \in [0, p[\ \forall \alpha \in R^s$. Let $L(w)$ be the simple G-module of highest weight $(w \bullet 0)^0$. Then each i-th socle layer, $i \in \mathbb{N}$, of $\widehat{\nabla}(0)$ is given by

$$\mathrm{soc}_i \widehat{\nabla}(0) = \coprod_{w \in W} L(w) \otimes G_1\mathbf{Mod}(L(w), \mathrm{soc}_i \widehat{\nabla}(0)).$$

We thus have, assuming the Lusztig conjecture [**L**], that $G_1\mathbf{Mod}(L(w), \mathrm{soc}_{\ell(w)+1} \widehat{\nabla}(0)) \neq 0\ \forall w \in W$.

More generally, let P be a standard parabolic subgroup of G, $\widehat{\nabla}_P(0) = \mathrm{ind}_{P_1T}^{G_1T}(0)$ with 0 denoting the zero weight again not the zero module, W_P the Weyl group of P, and let $W^P = \{w \in W | \ell(wx) = \ell(w) + \ell(x)\ \forall x \in W_P\}$. We have found for G of rank at most 2 in [**KY**] and in case $G = \mathrm{GL}_{n+1}$ with P such that G/P is the projective n-space [**K**] that

$$\mathrm{soc}_i \widehat{\nabla}_P(0) = \coprod_{w \in W^P} L(w) \otimes G_1\mathbf{Mod}(L(w), \mathrm{soc}_i \widehat{\nabla}_P(0)) \quad \forall i \in \mathbb{N}$$

and that $G_1\mathbf{Mod}(L(w), \mathrm{soc}_{\ell(w)+1} \widehat{\nabla}_P(0)) \neq 0\ \forall w \in W^P$. This nonvanishing is a key ingredient in our attempt to construct a Karoubian complete strongly exceptional sequence of coherent sheaves on G/P.

(3.6) Lusztig's conjecture holds for large p thanks to [**KT, KL, L94, AJS, F**].

Corollary: $\forall w \in W$, $Q_{w_0 \bullet A^+, w \bullet A^+}$ has positive constant term.

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