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Iwahori's question for affine Hecke algebras

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Abstract. In this paper we show that an affine Hecke algebra H_q over complex numbers field with parameter $q \neq 1$ is not isomorphic to the group algebra over complex numbers field of the corresponding extend affine Weyl group if the corresponding root system has no factors of type A_1 and the order of q is different from 11 and 13 if the root system has factors of type E_8.

For a Hecke algebra (over a field) of a finite Coxeter group, it is known that the Hecke algebra is isomorphic to the group algebra over the field when both the Hecke algebra and the group algebra are semisimple. For a simple criterion of the semi-simplicity for the Hecke algebra we refer to [G]. It is natural to consider the question for affine Hecke algebra, this question is analogue to a question of Iwahori for Weyl group[I]. For affine Hecke algebras the answer is different. In an unpublished work around 1970's, Casselman proved, by using group cohomology, that the Hecke algebra of a simple group over a p-adic field k with respect to an Iwahori subgroup is not isomorphic to the group algebra of the corresponding affine Weyl group, unless the group is of k-rank one. (The authors thank the referee for pointing out this fact.) We will show that an affine Hecke algebra H_q over the complex number field with parameter $q \neq 1$ is not isomorphic to the group algebra over complex numbers field of the corresponding extend affine Weyl group if the the root system has no factors of type A_1. For technical reason, we require that the order of q is different from 11 and 13 if the root system has factors of type E_8, see Theorem 1.2.

1. Affine Hecke algebras

1.1. Let R be an irreducible root system, W_0 the Weyl group of R, $Q = \mathbb{Z}R$ the root lattice and X the weight lattice. The Weyl group W_0 acts on Q and X. Then the semidirect product $W_a = W_0 \ltimes Q$ is an affine Weyl group, which is a subgroup of the extend affine Weyl group $W = W_0 \ltimes X$. Fix a positive root system of R and denote by R^+ the set of positive roots of R. Then we have a length function $l : W \to \mathbb{N}$ given by the formula (see [**IM**])

$$l(wx) = \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^-}} |\langle x, \alpha^\vee \rangle + 1| + \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^+}} |\langle x, \alpha^\vee \rangle|,$$

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where $R^- = -R^+$. The set S of simple reflections consists of all elements in W_a with length 1. The set of dominant weights X^+ is $\{x \in X \mid l(wx) = l(w) + l(x)\}$.

Denote by H_q the Hecke algebra of (W, S) over the complex numbers field $\mathbb{C}$ with a non zero parameter $q \in \mathbb{C}^*$. By definition, H_q has a $\mathbb{C}$-basis consisting of elements $T_w,\ w \in W$, and the multiplication law is given by the relations: $(T_r - q)(T_r + 1) = 0$ if $r \in S$, $T_wT_u = T_{wu}$ if $l(wu) = l(w) + l(u)$.

The main result of this article is the following.

THEOREM 1.2. *Let $q \neq 1$ be a nonzero complex number and R an irreducible root system of rank greater than 1. Assume that the order of q is different from 11 and 13 if R is of type E_8. Then H_q is not isomorphic to $\mathbb{C}[W]$.*

REMARK. When W is of type $\tilde{A}_2$, the result was proved in [X1, 11.7]. When W is of type $\tilde{A}_1$, we know that H_q is isomorphic to $\mathbb{C}[W]$ if and only if $q \neq -1$, see loc.cit.. For type $\tilde{A}_2$, Yan showed that H_q is isomorphic to H_p if and only if $p = q$ or $pq = 1$.

The theorem should be valid for R being of type E_8 and the order of q is 11 or 13, but the authors have not been able to work out it. Recently the authors learned that this result is known to G. Lusztig for long time.

1.3. The center of H_q will play a key role in the proof. We need to recall the description of Bernstein for the center.

For each x in X, we can find y and z in X^+ such that $x = yz^{-1}$. Define $\theta_x = q^{\frac{1}{2}(l(z)-l(y))}T_yT_z^{-1}$. It is known that θ_x is independent of the choice of y and z.

(a) (Bernstein) For any x, y in X, $\theta_x\theta_y = \theta_y\theta_x = \theta_{xy}$. The elements $\theta_x,\ x \in X$ form a $\mathbb{C}$-basis of the subalgebra Θ_q of H_q generated by all $\theta_y,\ y \in X$. The Hecke algebra H_q is a free Θ_q-module with a basis $T_w,\ w \in W_0$.

For each dominant weight x in X, let O_x be the W_0-orbit of x. That is

$$O_x = \{wxw^{-1} \in X \mid w \in W_0\}.$$

For $x \in X^+$, define

$$S_x = \sum_{y \in O_x} \theta_y.$$

Let n be the rank of R and $x_1, x_2, ..., x_n$ be the fundamental dominant weights. Then we have (see [L1, Theorem 8.1] for a proof)

(b) (Bernstein) The center $Z(H_q)$ of H_q is a polynomial algebra over $\mathbb{C}$ in n-variables, generated by $S_{x_i},\ i = 1, 2, ..., n$. The elements $S_x,\ x \in X^+$, form a $\mathbb{C}$-basis of the center $Z(H_q)$ of H_q.

Note that Θ_q is a free $Z(H_q)$-module of rank $|W_0|$ (see [S]).

1.4. Let G be a simply connected simple algebraic group over $\mathbb{C}$ with root system R. Let T be a maximal torus of G. Then we can identify W_0 with $N_G(T)/T$ and identify X with $\mathrm{Hom}(T, \mathbb{C}^*)$.

Let $\mathcal{C}$ be a semisimple class of G. Choose an element s in $\mathcal{C} \cap T$. The map $\theta_x \to x(s),\ x \in X$ defines a homomorphism $\phi'_{q,s} :\ \Theta_q \to \mathbb{C}$. It is known that any algebra homomorphism from $Z(H_q)$ to $\mathbb{C}$ is the restriction of $\phi'_{q,s}$ for some $s \in T$. We shall denote by $\phi_{q,s}$ the restriction to $Z(H_q)$ of $\phi'_{q,s}$. Let $I_{q,s}$ be the two-sided ideal of H_q generated by all $S_x - \phi_{q,s}(S_x),\ x \in X^+$. Let $H_{q,s}$ be the quotient algebra

$H_q/I_{q,s}$. Then $H_{q,s}$ is a $\mathbb{C}$-algebra of dimension $|W_0|^2$. Since for any $t \in \mathcal{C} \cap T$, we have $H_{q,s} = H_{q,t}$, the algebra $H_{q,s}$ depends only on the semisimple class $\mathcal{C}$.

We shall say that the central character $\phi_{q,s}$ admits one-dimensional representations if $H_{q,s}$ has one-dimensional representations.

1.5. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. For any semisimple element s in G, define $\mathcal{N}_{s,q}$ to be the subset of $\mathcal{N}$ given by $\mathcal{N}_{s,q} = \{N \mid N \in \mathcal{N},\ \mathrm{A}d(s)N = qN\}$. For a nilpotent element N in $\mathcal{N}_{s,q}$, let $\mathcal{B}_N^s$ be the variety consisting of all Borel subalgebras of $\mathfrak{g}$ which contain N and are fixed by $\mathrm{A}d(s)$. Let $C_G(s)$ (resp. $C_G(N)$) be the centralizer of s (resp. N) in G. Denote by $A(s,N)$ the component group $C_G(s) \cap C_G(N)/(C_G(s) \cap C_G(N))^o$ of $C_G(s) \cap C_G(N)$. Then $A(s,N)$ acts on the total complex coefficient Borel-Moore homology group $H_*(\mathcal{B}_N^s)$. Let $A(s,N)^\vee$ be the set of irreducible representations of $A(s,N)$ that appear in $H_*(\mathcal{B}_N^s)$.

The group G acts on the set G_{ss} of semisimple elements of G by conjugacy and act on the variety $\mathcal{N}$ by adjoint action. So G acts on the set of the union of all $A(s,N)^\vee$. If $\sum_{w \in W_0} q^{l(w)} \neq 0$, the Deligne-Langlands-Lusztig classification says that the isomorphism classes of irreducible representations of H_q is in one-to-one correspondence to the G-orbits in the set of all triples (s, N, ρ), $s \in G_{ss}$, $N \in \mathcal{N}$, $\rho \in A(s,N)^\vee$. See [K, KL2,X2].

Let s be a semisimple element in G. The group $C_G(s)$ acts on the variety $\mathcal{N}_{s,q}$ through the adjoint action of G. So $C_G(s)$ acts on the set of all $A(s,N)^\vee$, $N \in \mathcal{N}_{q,s}$. The Deligne-Langlands-Lusztig classification is equivalent to the following assertion.

(a)If $\sum_{w \in W_0} q^{l(w)} \neq 0$, then the isomorphism classes of irreducible representations of $H_{q,s}$ is in one-to-one correspondence to the $C_G(s)$-orbits of the pairs (N, ρ) $(N \in \mathcal{N}_{q,s}, \rho \in A(s,N)^\vee)$.

Since the variety $\mathcal{B}_N^s$ is not empty for any $N \in \mathcal{N}_{q,s}$ [KL2], by (a) we get

(b) The number of isomorphism classes of irreducible representations of $H_{q,s}$ is not less than the number of the $C_G(s)$-orbits in $\mathcal{N}_{q,s}$.

1.6. For a root α in R, let e_α be a non zero element in the root subspace $\mathfrak{g}_\alpha$. Let $\alpha_1, ..., \alpha_n$ be the simple roots of R. We sometimes use e_i for e_{α_i}.

For a root α in R, let U_α be the corresponding one parameter subgroup of G and $u_\alpha : \mathbb{C} \to U_\alpha$ be an isomorphism such that $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$ for any $c \in \mathbb{C}$ and $t \in T$.

2. Group algebra $\mathbb{C}[W]$ of W

In this section we consider the group algebra $\mathbb{C}[W]$, which is identified with H_1.

LEMMA 2.1. *Let s be a semisimple element in T.*
(a) If $H_{1,s}$ has a one-dimensional representation, then s is in the center of G, i.e. $\alpha(s) = 1$ for simple roots α in R.
(b) If s is in the center of G, then $H_{1,s}$ has a one-dimensional representation.

PROOF. (a) The following argument is provided by the referee. Suppose that $\pi : H_1 \to \mathbb{C}$ is a one-dimensional representation of H_1 (and of $H_{1,s}$). This π determines a character of X, hence an element t of T conjugate to s, by $\pi(x) = x(t)$

for $x \in X$. Since $\pi(wxw^{-1}) = \pi(x)$ for $x \in X$, $w \in W_0$, we see that $w(t) = t$ for any $w \in W_0$. Hence t is in the center of G and $s = t$, (a) is true.

(b) Let Ω be the subgroup of W consisting of elements of length 0. Then Ω is a finite abelian group and is isomorphic to the center of G. We identify the character group Ω^* of Ω with the center of G. If s is an element in the center of G (which is identified with Ω^*), then the one-dimensional representation $\chi : H_1 \to \mathbb{C}$ defined by $\chi(w) = 1$ for all $w \in W_a$ and $\chi(\omega) = s(\omega)$ for any $\omega \in \Omega$ has in fact central character $\phi_{1,s}$. Hence, $H_{1,s}$ has a one-dimensional representation. □

The lemma is proved.

COROLLARY 2.2. *The number of central characters $\phi_{1,s}$ $(s \in T)$ which admit one-dimensional representations is equal to the cardinality of the center of G.*

2.3. Assume that s is in the center. Then $A(s, N)$ is just the component group $A(N) = C_G(N)/C_G(N)^\circ$ of $C_G(N)$ for any nilpotent element N in $\mathfrak{g}$. Any Borel subalgebra of $\mathfrak{g}$ is fixed by $\mathrm{Ad}(s)$. So the variety of $\mathcal{B}_N^s$ is just the variety $\mathcal{B}_N$ of all Borel subalgebra of $\mathfrak{g}$ containing N. It is known that an irreducible representation of $A(N)$ appears in the total Borel-Moore homology of $\mathcal{B}_N$ if and only if it appears in the top homology of the $\mathcal{B}_N$. Combining the Deligne-Langlands-Lusztig classification for H_1 (see 1.5 (a)) and Springer's correspondence for W_0 we get the following fact.

(a) If s is in the center of G, then the isomorphism classes of irreducible representations of $H_{1,s}$ is in one-to-one correspondence to the isomorphism classes of irreducible complex representations of W_0.

The number $|\mathrm{Irr}(W_0)|$ of isomorphism classes of irreducible complex representations of W_0 is well known. For convenience, we list them as follows:

Type A_n: the number of all partitions of $n + 1$,

Type B_n $(n \geq 2)$ and C_n $(n \geq 3)$: the number of ordered pairs (ξ, η) of partitions ξ, η with $|\xi| + |\eta| = n$,

Type D_n $(n \geq 4)$: the number of unordered pairs (ξ, η) of partitions ξ, η with $|\xi| + |\eta| = n$, and any pairs (ξ, η) with $\xi = \eta$ and $|\xi| + |\eta| = n$ is counted twice.

Type E_6, E_7, E_8: 25, 60, 112, respectively

Type F_4: 25,

Type G_2: 6.

3. Some particular semisimple elements in T

In this section we assume that $q \neq 1$. We are interested in the quotient algebras $H_{q,s}$ which have one-dimensional representations.

LEMMA 3.1. *Assume that $H_{q,s}$ has a one-dimensional representation on which all T_r $(r \in S)$ act by the same scalar multiplication. Then in the conjugacy class of s we can find an element $t \in T$ such that $\alpha(t) = q$ for all simple roots α.*

PROOF. The scalar is q or -1. Using the explicit formulas for reduced expressions of fundamental weights in [L2] we can see that the lemma is true. We illustrate the argument by using type $\tilde{A}_n$.

Assume that R is of type A_n. We number the simple reflections $r_0, r_1, ..., r_n$ as usual and simply write T_i for T_{r_i}. Let $\alpha_1, ..., \alpha_n$ be the simple roots and $x_1, ..., x_n$

be the corresponding fundamental weights of the weight lattice X. According to [L2], for $1 \le i \le n$, we have

$$T_{x_i} = T_{\tau^{n+1-i}}(T_{n+1-i}T_{n+2-i}\cdots T_n)(T_{n-i}T_{n+1-i}\cdots T_{n-1})\cdots(T_1T_2\cdots T_i),$$

where $\tau \in W$ has length 0 and $\tau r_0 = r_1\tau$, $\tau r_1 = r_2\tau$,..., $\tau r_n = r_0\tau$. Note that $\tau^{n+1} = e$ is the neutral element of W.

In X we set $x_0 = x_{n+1} = 0$. Then the length of x_i is $i(n+1-i)$. By definition, for $1 \le i \le n$, we have

$$\theta_{\alpha_i} = (q^{-i(n+1-i)}T_{x_i}^2)(q^{-(i+1)(n-i)/2}T_{x_{i+1}})^{-1}(q^{-(i-1)(n+2-i)/2}T_{x_{i-1}})^{-1}.$$

If all T_i act on a one-dimensional representation of $H_{q,s}$ by scalar q, then for all $1 \le i \le n$, θ_{α_i} act on the one-dimensional representation of $H_{q,s}$ by scalar q. So the required t exists in this case.

If all T_i act on a one-dimensional representation of $H_{q,s}$ by scalar -1, then for all $1 \le i \le n$, θ_{α_i} act on the one-dimensional representation of $H_{q,s}$ by scalar q^{-1}. This means that in the conjugacy class $\mathcal{C}$ of s, there exists $t' \in \mathcal{C} \cap T$ such that $\alpha_i(t') = q^{-1}$ for all $i = 1, 2, ..., n$. Let w_0 be the longest element of W_0 and Set $t = w_0(t')$. Then t is in $\mathcal{C} \cap T$ and $\alpha_i(t) = q$ for all $i = 1, 2, ..., n$. The lemma is proved for type $\tilde{A}$. The proof for other types is similar. □

LEMMA 3.2. *Assume that $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$ and $H_{q,s}$ has a one-dimensional representation on which some T_r ($r \in S$) act by scalar multiplication of q and some T_r ($r \in S$) act by scalar multiplication of -1. Then in the conjugacy class of s there is no element $t \in T$ such that $\alpha(t) = q$ for all simple roots α.*

PROOF. The root system R must be one of the following types: B_n $(n \ge 2)$, C_n $(n \ge 3)$, F_4, G_2. We prove the lemma case by case. Let $t \in T$ be such that $\alpha(t) = q$ for all simple roots α. □

Type B_n $(n \ge 2)$**.** There exist $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ in $\mathrm{Hom}(T, \mathbb{C}^*)$ such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, ..., n-1$ and $\alpha_n = \varepsilon_n$. The maximal exponent of R is $2n-1$ and $\sum_{w\in W_0} q^{l(w)} = (q-1)^{-n}\prod_{i=1}^{n}(q^{2i}-1)$.

Let r_i be the simple reflection corresponding to α_i and r_0 the simple reflection out of W_0. Assume $T_{r_1}, T_{r_2}, ..., T_{r_{n-1}}$ act on the one-dimensional representation by scalar q and T_{r_0}, T_{r_n} act on it by scalar -1. Using the explicit formula for θ_{x_i} in [L2] we see that $\alpha_i(s) = q$ for $i = 1, 2, ..., n-1$ and $\alpha_n(s) = q^{-1}$. Therefore $C_G(s)$ contains $T, U_{\pm\varepsilon_{n-1}}, U_{\pm(\varepsilon_{n-2}+\varepsilon_n)}$.

If the order $o(q)$ is greater than the maximal exponent $2n-1$, then t is regular and $C_G(t) = T$. In this case s and t are not conjugate in G.

Now assume that $o(q) \le 2n-1$. Our assumption on q implies that $o(q)$ is odd and $n+1 \le o(q) \le 2n-1$. Let $o(q) = n+i$ for some $1 \le i \le n-1$. Since $o(q) = n+i$, The centralizer $C_G(t)$ of t is generated by T and all U_α with $\alpha(t) = 1$. It is easy to see that $\alpha(t) = 1$ if and only if α is one of the following roots: $\pm(\varepsilon_j + \varepsilon_{n+2-j-i})$, $1 \le j < \frac{n+2-i}{2}$. Noting that all the roots $\varepsilon_j + \varepsilon_{n+1-j-i}$ are long roots and ε_{n-1} is short root, we see that $C_G(s)$ and $C_G(t)$ are not conjugate in G, so s and t are not conjugate in G.

Assume $T_{r_1}, T_{r_2}, ..., T_{r_{n-1}}$ act on the one-dimensional representation by scalar -1 and T_{r_0}, T_{r_n} act on it by scalar q. Again using the explicit formula for θ_{x_i} in [L2] we see that $\alpha_i(s) = q^{-1}$ for $i = 1, 2, ..., n-1$ and $\alpha_n(s) = q$. Let $g \in N_G(T)$ be a representative of the longest element of W_0 and $s' = gsg^{-1}$. Then $\alpha_i(s') = q$

for $i = 1, 2, ..., n-1$ and $\alpha_n(s') = q^{-1}$. By the above argument we see that s' and t are not conjugate in G.

Type C_n ($n \geq 3$). The argument is similar to that for type B_n.

Type F_4. Let $V = \mathbb{R}^4$ and $\varepsilon_1, ..., \varepsilon_4$ the standard basis of V. We may assume that R consists of the following elements:

$\pm\varepsilon_i$ $(1 \leq i \leq 4)$, $\quad \pm\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq 4)$,

$\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$.

The maximal exponent of R is 11 and

$$\sum_{w \in W_0} q^{l(w)} = (q-1)^{-3}(q+1)(q^6-1)(q^8-1)(q^{12}-1).$$

Let r_i be the simple reflection corresponding to α_i and r_0 is the reflection out of W_0. Then $r_0r_4r_0 = r_4r_0r_4$. Assume T_{r_1}, T_{r_2} act on the one-dimensional representation by scalar q and $T_{r_3}, T_{r_4}, T_{r_0}$ act on it by scalar -1.

According to [L2, p.646], in H_q we have

$$\theta_{x_4} = q^{-8} T_{r_0}T_{r_4}T_{r_3}T_{r_2}T_{r_1}T_{r_3}T_{r_4}T_{r_2}T_{r_3}T_{r_2}T_{r_4}T_{r_3}T_{r_1}T_{r_2}T_{r_3}T_{r_4},$$

$$\theta_{x_3}\theta_{x_4}^{-1} = qT_{r_4}^{-1}\theta_{x_4}T_{r_4}^{-1},$$

$$\theta_{x_2}\theta_{x_3}^{-1} = qT_{r_3}^{-1}\theta_{x_3}\theta_{x_4}^{-1}T_{r_3}^{-1},$$

$$\theta_{x_1}\theta_{x_2}^{-1}\theta_{x_3} = qT_{r_2}^{-1}\theta_{x_2}\theta_{x_3}^{-1}T_{r_2}^{-1}.$$

So we have $\alpha_1(s) = \alpha_2(s) = q$ and $\alpha_3(s) = \alpha_4(s) = q^{-1}$. The centralizer $C_G(s)$ of s contains T, $U_{\pm(\alpha_2+\alpha_3)}$, $U_{\pm(\alpha_1+\alpha_2+2\alpha_3)}$, $U_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$, $U_{\pm(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

If the order $o(q)$ is greater than 11, then t is regular and $C_G(s) = T$, so s and t are not conjugate.

Now assume that $o(q) \leq 11$. Our assumption on q implies that $o(q) = 5, 7, 9, 10, 11$. If $o(q) = 11$, then $C_G(t)$ is generated by T and $U_{\pm\beta}$, where β is the highest root in R. If $o(q) = 10$, then $C_G(t)$ is generated by T and $U_{\pm(\beta-\alpha_1)}$. If $o(q) = 9$, then $C_G(t)$ is generated by T, $U_{\pm(\beta-\alpha_1-\alpha_2)}$. If $o(q) = 7$, then $C_G(t)$ is generated by T, $U_{\pm(\beta-\alpha_1-\alpha_2-2\alpha_3)}$. $U_{\pm(\beta-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}$. If $o(q) = 5$, then $C_G(t)$ is generated by T, $U_{\pm(\alpha_1+2\alpha_2+2\alpha_3)}$, $U_{\pm(\alpha_2+2\alpha_3+2\alpha_4)}$, $U_{\pm(\beta-\alpha_1)}$. In all cases $C_G(s)$ and $C_G(t)$ are not conjugate in G. Hence s and t are not conjugate in G.

Assume T_{r_1}, T_{r_2} act on the one-dimensional representation by scalar -1 and $T_{r_3}, T_{r_4}, T_{r_0}$ act on it by scalar q. One can check as above that $\alpha_1(s) = \alpha_2(s) = q^{-1}$ and $\alpha_3(s) = \alpha_4(s) = q$. Let $g \in N_G(T)$ be a representative of the longest element of W_0 and $s' = gsg^{-1}$. Then $\alpha_1(s') = \alpha_2(s') = q$ and $\alpha_3(s') = \alpha_4(s') = q^{-1}$. By the above discussion we know that s' and t are not conjugate in G.

Type G_2. We number the simple reflections r_0, r_2, r_2 so that $r_0r_2 = r_2r_0$. In W_0 we have $x_1 = r_0r_1r_2r_1r_2r_1$ and $x_2 = r_0r_1r_2r_1r_2r_0r_1r_2r_1r_2$. In the weight lattice we have $x_1 = 2\alpha_1 + \alpha_2$ and $x_2 = 3\alpha_1 + 2\alpha_2$, where α_i are the simple roots. If T_{r_1} and T_{r_0} act on the one-dimensional representation by scalar q and T_{r_2} acts it by scalar -1, then both θ_{x_1} and θ_{x_2} act on it by scalar q. So $\alpha_1(s) = q$ and $\alpha_2(s) = q^{-1}$. Note that $(\alpha_1 + \alpha_2)(s) = 1$. So $C_G(s)$ contains T and $U_{\pm(\alpha_1+\alpha_2)}$.

If the order $o(q)$ of q greater than 5, then $C_G(s) = T$. So s and t are not conjugate. If $o(q) \leq 5$, then $o(q) = 5, 4$ since $\sum_{w \in W_0} q^{l(w)} \neq 0$. In these cases $C_G(s)$ is generated by T and $U_{\pm(\alpha_1+\alpha_2)}$, $C_G(t)$ is generated by T and $U_{\pm(3\alpha_1+\alpha_2)}$ if $o(q) = 4$ or $U_{\pm(3\alpha_1+2\alpha_2)}$ if $o(q) = 5$. Clearly $C_G(s)$ and $C_G(t)$ are not conjugate, so s and t are not conjugate.

If T_{r_1} and T_{r_0} act on the one-dimensional representation by scalar -1 and T_{r_2} acts it by scalar q, then $\alpha_1(s) = q^{-1}$ and $\alpha_2(s) = q$. Let $g \in N_G(T)$ be a representative of the longest element of W_0 and $s' = gsg^{-1}$. Then $\alpha_1(s') = q$ and $\alpha_2(s') = q^{-1}$. By the above discussion we know that s' and t are not conjugate in G.

The lemma is proved.

Corollary 3.3. *Assume that the root system R is not simply laced. If $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$, then the number of central characters $\phi_{q,s}$ $(s \in T)$ which admit one-dimensional representations is twice the cardinality of the center of G.*

Proof. Let $s \in T$ be such that $\alpha(s) = q$ for all simple roots α. It is easy to check that the map $T_r\theta_x \to qx(s)$ for any simple reflection r in W_0 and $x \in X$ defines an algebra homomorphism $H_q \to \mathbb{C}$. Hence, the central character $\phi_{q,s}$ admits one-dimensional representations of H_q on which all T_i act by the same scalar. Since $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$, we have $q \neq -1$. So we can find $t \in T$ such that the central character $\phi_{q,t}$ admits one-dimensional representations of H_q on which some T_i act by scalar q and some T_i act by scalar -1. By the calculations in the proof of Lemma 3.2, it is no harm to assume that $\alpha(t) = q$ for short simple roots α and $\alpha(t) = q^{-1}$ for long simple roots α.

By Lemma 3.2, s and t are not conjugate in G if $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$.

For types F_4 and G_2, the center of G is trivial and the weight lattice equals the root lattice. Hence there are exactly two central characters of H_q which admit one-dimensional representations if $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$.

Now assume that R is of type B_n $(n \geq 2)$ or C_n $(n \geq 3)$. Let $s' \in T$. Assume that the central character $\phi_{q,s'}$ admits one-dimensional representations of H_q on which all T_i act by the same scalar. By Lemma 3.1, in the conjugacy class of s' we can find $s'' \in T$ such that $\alpha(s'') = q$ for all simple roots α. Then $s''s^{-1} = c$ is in the center of G. We need show that s'' and s are not conjugate in G if c is non-trivial and $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$.

First consider type B_n. In this case $G = Spin_{2n+1}(\mathbb{C})$ and $SO_{2n+1}(\mathbb{C})$ is its quotient group. Clearly, s'' and s have the same image in $SO_{2n+1}(\mathbb{C})$. It is no harm to assume the image is

$$\mathrm{diag}(1, q^n, ..., q^2, q^1, q^{-n}, ..., q^{-2}, q^{-1}).$$

If s'' and s are conjugate in G, we must have $q = q^i$ for some $2 \leq i \leq n$ or $q = q^{-j}$ for some $1 \leq j \leq n$. If $j \neq n$, then $(q-1)\sum_{w\in W_0} q^{l(w)} = 0$. If $j = n$, then $q^{n+1} = 1$. We also have $q^{\frac{n(n+1)}{2}} = -1$ in this case since c is non-trivial. This forces that $n+1$ is even and $\sum_{w\in W_0} q^{l(w)} = 0$. So s'' and s are not conjugate if $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$.

Now assume that R is of type C_n. Then $G = Sp_{2n}(\mathbb{C})$. If c is non-trivial, then we may assume that s, s'' are the following two elements:

$$\mathrm{diag}(q^{\frac{2n-1}{2}}, ..., q^{\frac{3}{2}}, q^{\frac{1}{2}}, q^{-\frac{2n-1}{2}}, ..., q^{-\frac{3}{2}}, q^{-\frac{1}{2}})$$

$$\mathrm{diag}(-q^{\frac{2n-1}{2}}, ..., -q^{\frac{3}{2}}, -q^{\frac{1}{2}}, -q^{-\frac{2n-1}{2}}, ..., -q^{-\frac{3}{2}}, -q^{-\frac{1}{2}}).$$

If s'' and s are conjugate, then $q^{\frac{1}{2}} = -q^{\frac{1}{2}+i}$ for some $n-1 \geq i \geq 1$ or $q^{\frac{1}{2}} = -q^{-\frac{1}{2}-j}$ for some $n-1 \geq j \geq 0$. So, $q^i = -1$ or $q^{1+j} = -1$. This contradicts that $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$. Hence, s'' and s are not conjugate in G.

Assume that the central character $\phi_{q,t'}$ admits one-dimensional representations of H_q on which some T_i act by the scalar q and some T_i act by the scalar -1. By the calculations in the proof of Lemma 3.2, in the conjugacy class of t' we can find $t'' \in T$ such that $\alpha(t'') = q$ for all simple short roots α and $\alpha(t'') = q^{-1}$ for all simple long roots α. Then $t''t^{-1} = c$ is in the center of G. As above, we can show that t'' and t are not conjugate in G if c is non-trivial and $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$. □

The corollary is proved.

LEMMA 3.4. *Assume that the order of q is greater than the maximal exponent of the root system R. Let $s \in T$ be such that $\alpha(s) = q$ for all simple roots α in R. Then*
(a) The number of $C_G(s)$-orbits of the variety $\mathcal{N}_{q,s}$ is equal to 2^n.
(b) For any $N \in \mathcal{N}_{q,s}$, the group $A(s,N)$ acts on $H_*(\mathcal{B}_N^s)$ trivially, so $A(s,N)^\vee$ contains only the trivial representation of $A(s,N)$.
(c) The number of the isomorphism classes of irreducible representations of $H_{q,s}$ is 2^n.

PROOF. Since the order of q is greater than the maximal exponent of the root system R, we see that s is regular, so $C_G(s) = T$. Moreover, we have $\mathcal{N}_{q,s} = \{\sum_{1\le i\le n} a_i e_i \,|\, a_i \in \mathbb{C}\}$. For a subset I of $\{1,2,...,n\}$, set $e_I = \sum_{i\in I} e_i$. (We understand that $e_I = 0$ if I is the empty set.) Then any element in $\mathcal{N}_{q,s}$ is $C_G(s)$-conjugate to some e_I. Clearly if $I \neq J$, the e_I and e_J are not in the same $C_G(s)$-orbit. Part (a) is proved.

Since s is regular, the variety $\mathcal{B}^s$ of Borel subalgebras of $\mathfrak{g}$ fixed by $\mathrm{Ad}(s)$ is exactly the variety $\mathcal{B}^T$ of Borel subalgebras of $\mathfrak{g}$ fixed by $\mathrm{Ad}T$. Now $C_G(s) = T$, so for any $N \in \mathcal{N}_{q,s}$, the group $C_G(s) \cap C_G(N)$ acts trivially on $\mathcal{B}_N^s$. Part (b) follows. □

Part (c) follows from (a) and (b). The lemma is proved.

3.5. In the rest part of this section R is assumed to be simply laced and $H_{q,s}$ is required to satisfy the assumption of Lemma 3.1. We hope to know the number of the $C_G(s)$-orbits in $\mathcal{N}_{q,s}$, provided that $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$ and the order $o(q)$ of q is not greater than the maximal exponent of R. The following fact of Lusztig [L3] will be useful.

(a) Assume $x \in \mathfrak{g}$ and $\mathrm{Ad}(s)(x) = qx$. If $(q-1)\sum_{w\in W_0} q^{l(w)} \neq 0$, then $x \in \mathcal{N}_{q,s}$.

Since $\alpha(s) = q$ for all simple roots α in R, by (a) we get

(b) $\mathcal{N}_{q,s}$ is a linear space spanned by all e_β with $\beta(s) = q$ and is naturally a $C_G(s)$-module.

Since $T \subset C_G(s)$, any $C_G(s)$-submodule of $\mathcal{N}_{q,s}$ is spanned by some e_β with $\beta(s) = q$. Since $C_G(s)$ is reductive, $\mathcal{N}_{q,s}$ is the direct sum of some irreducible $C_G(s)$-submodules of $\mathcal{N}_{q,s}$.

We discuss, on a case by case basis, the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$. To do this, we first decompose $\mathcal{N}_{q,s}$ into the direct sum of some irreducible $C_G(s)$-submodules. Let $\beta \in R$ be such that $\beta(s) = 1$ and $u_\beta(\xi)$ ($\xi \in \mathbb{C}$, see 1.6 for definition) an element in the one parameter subgroup U_β of G. Then we may assume that $u_\beta(\xi)$ acts on $\mathfrak{g}$ (hence $\mathcal{N}_{q,s}$) through $\exp(\xi \mathrm{ad}(e_\beta))$. This idea is useful in determining the submodule structure of $\mathcal{N}_{q,s}$. Recall that we identify the weight lattice X of R with $\mathrm{Hom}(T, \mathbb{C}^*)$.

Type A_n. The maximal exponent is n and $\sum_{w\in W_0} q^{l(w)} = (q-1)^{-n}\prod_{i=2}^{n+1}(q^i-1)$. There are no complex numbers $q \neq 1$ such that its order is not greater than n and $\sum_{w\in W_0} q^{l(w)} \neq 0$.

Type D_n ($n \geq 4$). There exist $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ in $\mathrm{Hom}(T, \mathbb{C}^*)$ such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, ..., n-1$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. The maximal exponent of R is $2n-3$ and $\sum_{w\in W_0} q^{l(w)} = (q-1)^{-n}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$. The assumption on q implies that $o(q)$ is odd and $n+1 \leq o(q) \leq 2n-3$. Let $o(q) = n+i$ for some $1 \leq i \leq n-3$.

By assumption, $\alpha_i(s) = q$. Since $o(q) = n+i$, we see the other roots satisfying $\alpha(s) = q$ are the following: $\varepsilon_j + \varepsilon_{n-1-j-i}$, $-\varepsilon_k - \varepsilon_{n+1-k-i}$, with $1 \leq j < \frac{n-i-1}{2}$ and $1 \leq k < \frac{n+1-i}{2}$.

It is easy to check that $\alpha(s) = 1$ if and only if α is one of the following roots: $\pm(\varepsilon_j + \varepsilon_{n-j-i})$, $1 \leq j < \frac{n-i}{2}$.

If $i = n-3$, then $C_G(s)$ is generated by T and $U_{\pm(\varepsilon_1+\varepsilon_2)}$. The linear space $\mathcal{N}_{q,s}$ is spanned by $e_{\varepsilon_1-\varepsilon_2}$, $e_{\varepsilon_2-\varepsilon_3}$,,..., $e_{\varepsilon_{n-1}-\varepsilon_n}$, $e_{\varepsilon_{n-1}+\varepsilon_n}$ and $e_{-\varepsilon_1-\varepsilon_3}$. It is easy to see that $e_{\varepsilon_2-\varepsilon_3}$ and $e_{-\varepsilon_1-\varepsilon_3}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$. For any $1 \leq j \neq 2 \leq n-1$, the element $e_{\varepsilon_j-\varepsilon_{j+1}}$ spans a one-dimensional $C_G(s)$-submodule M_j of $\mathcal{N}_{q,s}$. The space $\mathcal{N}_{q,s}$ is the direct sum of the submodules $M_1, M_2, ..., M_n$ and each M_j has two $C_G(s)$-orbits. Therfore the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is 2^n .

Now assume that $i \leq n-5$. The following elements span a $C_G(s)$-submodule M_j of $\mathcal{N}_{q,s}$: $e_{\varepsilon_j-\varepsilon_{j+1}}$, $e_{-\varepsilon_{j+1}-\varepsilon_{n-j-i}}$, $e_{\varepsilon_j+\varepsilon_{n-j-1-i}}$, $e_{\varepsilon_{n-j-1-i}-\varepsilon_{n-j-i}}$ for $j = 1, 2, ..., \frac{n-i-3}{2}$. The elements $e_{\varepsilon_j-\varepsilon_{j+1}}$ and $e_{-\varepsilon_{j-1}-\varepsilon_{j+1}}$ span a two-dimensional $C_G(s)$-submodule M_j' of $\mathcal{N}_{q,s}$ for $j = \frac{n-i+1}{2}$. The element $e_{\varepsilon_j-\varepsilon_{j+1}}$ spans a one-dimensional $C_G(s)$-submodule M_j' of $\mathcal{N}_{q,s}$ for $j = \frac{n-i-1}{2}, n-i, n-i+1, ..., n-1$. Also the element $e_{\varepsilon_{n-1}+\varepsilon_n}$ spans a one-dimensional $C_G(s)$-submodule M_n' of $\mathcal{N}_{q,s}$ Clearly $\mathcal{N}_{q,s}$ is the direct sum of all the submodules M_j, M_k'. One may check easily that each M_j has three $C_G(s)$-orbits and each M_k' has two $C_G(s)$-orbits. So the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than $2^{i+3} \cdot 3^{\frac{n-i-3}{2}}$.

Type E_6. We number the simple roots as in [B]. One has

$$\sum_{w\in W_0} q^{l(w)} = \frac{(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)}{(q-1)^6}.$$

The maximal exponent is 11. The assumption on q implies that $o(q) = 7, 10, 11$.

If $o(q) = 7$, then $C_G(s)$ is generated by $T, U_{\pm\beta_i}$, $i = 1, 2, 3$, here $\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$, $\beta_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$. If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots: $\gamma_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$, $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $-\gamma_3 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$, $-\gamma_4 = -(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$, $-\gamma_5 = -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$. It is easy to check the following facts:

(1) e_{α_1}, e_{γ_2}, e_{α_5}, $e_{-\gamma_4}$ span a $C_G(s)$-submodule M_1 of $\mathcal{N}_{q,s}$,

(2) e_{α_3}, e_{γ_1}, e_{α_6}, $e_{-\gamma_3}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(3) $e_{\alpha_4}, e_{-\gamma_5}$ span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$,

(4) e_{α_2} spans a $C_G(s)$-submodule M_4 of $\mathcal{N}_{q,s}$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. Noting that M_i has three $C_G(s)$-orbits for $i = 1, 2$ and M_j has two $C_G(s)$-orbits for $j = 3, 4$, we see that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than 36.

If $o(q) = 10$ or 11, then we can see easily that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is 2^6.

Type E_7**.** We number the simple roots as in [B]. One has

$$\sum_{w \in W_0} q^{l(w)} = \frac{(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)^7}.$$

The maximal exponent is 17. The assumption on q implies that $o(q) = 11, 13, 15, 16, 17$.

If $o(q) = 11$, then $C_G(s)$ is generated by $T, U_{\pm\beta_i}$, $i = 1, 2, 3$, here $\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$, $\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$. If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots: $\gamma_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$, $\gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$, $-\gamma_3 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$, $-\gamma_4 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)$, $-\gamma_5 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)$. It is easy to check the following facts:

(5) e_{α_2}, e_{γ_2}, e_{α_7}, $e_{-\gamma_4}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(6) e_{α_4}, e_{γ_1}, e_{α_6}, $e_{-\gamma_4}$ span a $C_G(s)$-submodule M_4 of $\mathcal{N}_{q,s}$,

(7) $e_{\alpha_3}, e_{-\gamma_5}$ span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$,

(8) For $i = 1$ or 5, the element e_{α_i} spans a $C_G(s)$-submodule M_i of $\mathcal{N}_{q,s}$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. Noting that M_i has three $C_G(s)$-orbits for $i = 2, 4$ and M_j has two $C_G(s)$-orbits for $j = 1, 3, 5$, we see that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than $2^3 \times 3^2 = 72$.

If $o(q) = 13$, then $C_G(s)$ is generated by $T, U_{\pm\sigma_i}$, $i = 1, 2$, here $\sigma_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, $\sigma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$. If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots: $\tau_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, $-\tau_2 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)$, $-\tau_3 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)$. It is easy to check the following facts:

(9) e_{α_2}, e_{τ_1}, e_{α_5}, $e_{-\tau_2}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(10) $e_{\alpha_6}, e_{-\tau_3}$ span a $C_G(s)$-submodule M_6 of $\mathcal{N}_{q,s}$,

(11) The element e_{α_i} spans a $C_G(s)$-submodule M_i of $\mathcal{N}_{q,s}$ for $i = 1, 3, 4, 7$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. Noting that M_2 has three $C_G(s)$-orbits and M_j has two $C_G(s)$-orbits for $j = 1, 3, 4, 6, 7$, we see that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than $2^5 \times 3 = 96$.

If $o(q) = 15, 16, 17$, then we can see easily that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is 2^7.

Type E_8**.** This is the most complicated case. We number the simple roots as in [B]. One has

$$\sum_{w \in W_0} q^{l(w)} = \frac{(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{(q-1)^8}.$$

The maximal exponent is 29. The assumption on q implies that $o(q) = 11, 13, 16, 17, 19, 21, 22, 23, 25, 26, 27, 28, 29$.

If $o(q) = 11$, then $C_G(s)$ is generated by $T, U_{\pm\beta_i}$, $1 \le i \le 6$, here

$\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$,

$\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$,
$\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$,
$\beta_4 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$,
$\beta_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\beta_6 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\beta_7 = \beta_1 + \beta_5 = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$\beta_8 = \beta_3 + \beta_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$.

If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots:

$\gamma_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$,
$\gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$,
$-\gamma_3 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$,
$-\gamma_4 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)$,
$-\gamma_5 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)$,
$-\gamma_6 = -(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$,
$-\gamma_7 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$,
$-\gamma_8 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)$,
$\gamma_9 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$,
$\gamma_{10} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\gamma_{11} = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\gamma_{12} = 2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$\gamma_{13} = 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$-\gamma_{14} = -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)$,
$-\gamma_{15} = -(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)$,

It is easy to check the following fact:

(12) The ten elements $e_{\alpha_1}, e_{\gamma_{11}}, e_{\alpha_7}, e_{\gamma_2}, e_{-\gamma_6}, e_{-\gamma_3}, e_{\alpha_2}, e_{\gamma_{12}}, e_{\gamma_{13}}, e_{-\gamma_{14}}$ span a $C_G(s)$-submodule M_1 of $\mathcal{N}_{q,s}$,

(13) The eleven elements e_{α_3}, $e_{\gamma_{10}}$, $e_{-\gamma_5}$, e_{α_6}, e_{α_8}, $e_{-\gamma_8}$, $e_{-\gamma_{15}}$, e_{γ_1}, e_{α_4}, e_{γ_9}, $e_{-\gamma_4}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(14) $e_{\alpha_5}, e_{-\gamma_7}$ span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i, but it is not easy to see the number of $C_G(s)$-orbits of $\mathcal{N}_{q,s}$ in this case.

If $o(q) = 13$, then $C_G(s)$ is generated by $T, U_{\pm\sigma_i}$, $i = 1, 2$, here

$\sigma_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$,
$\sigma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$,
$\sigma_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 + \alpha_8$,
$\sigma_4 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\sigma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\sigma_6 = \sigma_1 + \sigma_3 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$.

If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots:

$\tau_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$,
$-\tau_2 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)$,
$-\tau_3 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)$,
$-\tau_4 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)$,
$-\tau_5 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$,
$-\tau_6 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)$,
$\tau_7 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,

$\tau_8 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\tau_9 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$-\tau_{10} = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)$,
$\tau_{11} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$.

It is easy to check the following facts:

(15) The eleven elements $e_{\alpha_2}, e_{\tau_1}, e_{\alpha_5}, e_{-\tau_2}, e_{-\tau_4}, e_{\tau_7}, e_{\alpha_6}, e_{-\tau_3}, e_{\tau_{11}}, e_{\tau_{10}}, e_{\alpha_8}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(16) The element e_{α_1} spans a $C_G(s)$-submodule M_1 of $\mathcal{N}_{q,s}$,

(17) The elements $e_{\alpha_3},\ e_{-\tau_6}$ span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$.

(18) The elements $e_{\alpha_4},\ e_{-\tau_5}, e_{\tau_9}, e_{\alpha_7}$ span a $C_G(s)$-submodule M_4 of $\mathcal{N}_{q,s}$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. But it is not easy to see the number of $C_G(s)$-orbits of $\mathcal{N}_{q,s}$ in this case.

If $o(q) = 16$, then $C_G(s)$ is generated by $T, U_{\pm\xi_i}$, $i = 1, 2, 3, 4$, here
$\xi_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$,
$\xi_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$\xi_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\xi_4 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots:
$\eta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
$-\eta_2 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$,
$-\eta_3 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$,
$-\eta_4 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)$,
$-\eta_5 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)$,
$\eta_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$\eta_7 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\eta_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$.

It is easy to check the following facts:

(19) The elements e_{α_1}, e_{η_1} span a $C_G(s)$-submodule M_1 of $\mathcal{N}_{q,s}$,

(20) The elements $e_{\alpha_2}, e_{\alpha_6}, e_{\eta_6}, e_{-\eta_5}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,

(21) The elements $e_{\alpha_3}, e_{\alpha_8}, e_{\eta_8}, e_{-\eta_2}$ span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$,

(22) The elements $e_{\alpha_4}, e_{\alpha_7}, e_{\eta_7}, e_{-\eta_3}$ span a $C_G(s)$-submodule M_4 of $\mathcal{N}_{q,s}$,

(23) The elements $e_{\alpha_5},\ e_{-\eta_4}$ span a $C_G(s)$-submodule M_5 of $\mathcal{N}_{q,s}$,

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. Note that $U_{\pm\xi_i}$ act on $M_1 + M_3$ trivially for $i = 2, 4$ and $U_{\pm\xi_3}$ also act on M_1 trivially. By direct computation we see that the $C_G(s)$-submodule $M_1 + M_3$ has eight $C_G(s)$-orbits, representatives of the orbits can be chosen as follows: 0, e_{α_1}, e_{α_3}, $e_{\alpha_3} + e_{\alpha_8}$, $e_{\alpha_1} + e_{\alpha_3}$, $e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_8}$, $e_{\alpha_1} + e_{-\eta_2}$, $e_{\alpha_1} + e_{\alpha_3} + e_{-\eta_2}$. Similarly, one can see that $M_4 + M_5$ has also eight $C_G(s)$-orbits, and M_2 has three $C_G(s)$-orbits. Thus the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than $8 \times 8 \times 3 = 192$.

If $o(q) = 17$, then $C_G(s)$ is generated by $T, U_{\pm\eta_i}$, $i = 1, 6, 7, 8$, here
$\eta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
$\eta_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$,
$\eta_7 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\eta_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$.

If α is not simple root, then $\alpha(s) = q$ if and only if α is one of the following roots:
$-\xi_1 = -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$,

$-\xi_2 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)$,
$-\xi_3 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$,
$-\xi_4 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)$,
$\xi_5 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$,
$\xi_6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$,
$\xi_7 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$.
It is easy to check the following facts:
(24) The elements e_{α_2}, $e_{-\xi_2}$ span a $C_G(s)$-submodule M_2 of $\mathcal{N}_{q,s}$,
(25) The element e_{α_5} spans a $C_G(s)$-submodule M_5 of $\mathcal{N}_{q,s}$,
(26) The elements e_{α_1}, $e_{-\xi_1}$, e_{ξ_5}, e_{α_8} span a $C_G(s)$-submodule M_1 of $\mathcal{N}_{q,s}$,
(23) The elements e_{α_3}, $e_{-\xi_3}$, e_{ξ_6}, e_{α_7} span a $C_G(s)$-submodule M_3 of $\mathcal{N}_{q,s}$,
(24) The elements e_{α_4}, $e_{-\xi_4}$, e_{ξ_7}, e_{α_6} span a $C_G(s)$-submodule M_4 of $\mathcal{N}_{q,s}$.

Clearly $\mathcal{N}_{q,s}$ is the direct sum of all M_i. Noting that M_i has three $C_G(s)$-orbits for $i = 4, 5$, M_5 has two $C_G(s)$-orbits, $M_2 + M_4$ has eight $C_G(s)$-orbits, we see that the number of $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than $3 \times 3 \times 8 \times 2 = 144$.

Completely similar to the above discussions, one see that the $C_G(s)$-orbits in $\mathcal{N}_{q,s}$ is not less than 144 if $o(q)$ =19,21,22,23,25,26,27,28,29.

Proof of Theorem 1.2

In this section we prove Theorem 1.2, the main result of this article.

First assume that $\sum_{w\in W_0} q^{l(w)} \neq 0$.

Assume that $\psi : H_q \to \mathbb{C}[W] = H_1$ is an isomorphism of $\mathbb{C}$-algebras. Then ϕ induces an isomorphism from $Z(H_q) \to Z(H_1)$. Let s be a semisimple element in T. Composing the homomorphism $\phi_{1,s} : Z(\mathbb{C}[W]) \to \mathbb{C}$ with ψ, we get a homomorphism $\phi_{1,s}\psi : Z(H_q) \to \mathbb{C}$. So there exists a semisimple element t in T such that $\phi_{1,s}\psi = \phi_{q,t}$. This induces an isomorphism $H_{q,t} \to H_{1,s}$.

Assume further that $H_{1,s}$ has one dimensional representation, then $H_{q,t}$ has also one dimensional representation. So the number of the central characters $\phi_{q,t}$ ($t \in T$) which admit one-dimensional representations is equal to the number of the central characters $\phi_{1,s}$ ($s \in T$) which admit one-dimensional representations. According to Corollary 2.2 and Corollary 3.3, this is not true if R is not simply laced. So the theorem is true if $\sum_{w\in W_0} q^{l(w)} \neq 0$ and R is not simply laced.

Now assume that R is simply laced. Let s, t be as above. By 2.3 (a), the number $|\mathrm{Irr} H_{1,s}|$ of isomorphism classes of irreducible representations of $H_{1,s}$ is equal to the number $|\mathrm{Irr} W_0|$ of isomorphism classes of irreducible complex representations of the Weyl group W_0. We will show that $|\mathrm{Irr} W_0|$ is not equal to the number $|\mathrm{Irr} H_{q,t}|$ of isomorphism classes of irreducible representations of $H_{q,t}$. We do this case by case.

Type A_n ($n \geq 2$). It is known that $|\mathrm{Irr} W_0|$ is the number $p(n+1)$ of partitions of $n+1$. Since $\sum_{w\in W_0} q^{l(w)} \neq 0$, the order of q is greater than n. By Lemma 3.4, we see that $|\mathrm{Irr} H_{q,t}| = 2^n$. It suffices to show that $|\mathrm{Irr} H_{q,t}| = 2^n > p(n+1) = |\mathrm{Irr} W_0|$ if $n \geq 2$.

One can use Euler's Pentagonal Theorem to prove $2^n > p(n+1)$ for $n \geq 2$. Set $p(0) = 1$ and $p(m) = 0$ if $m < 0$. The theorem says

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} (p(n - \frac{k(3k-1)}{2}) + (p(n - \frac{k(3k+1)}{2})),$$

which implies that $p(n) \leq p(n-1) + p(n-2)$ for $n \geq 2$. One can also use induction on n to prove the result directly. When $n = 2, 3, 4, 5$, $p(n+1)$ is $3, 5, 7, 11$ respectively.

The result is true in these cases. Now assume that $k \geq 5$ and the result is true for integer n satisfying $2 \leq n \leq k$. For $1 \leq i \leq k+2$, let P_i be the number of partitions $a_1 \geq a_2 \geq \cdots \geq a_m$ of $k+2$ such that $a_1 = i$. Then $P_{k+2} = P_{k+1} = P_1 = 1$, $P_k = 2$, $P_{k-1} = 3$. By induction hypothesis, $2^{k+1-i} > P_i$ whenever $2 \leq i \leq k-2$. Hence,

$$\begin{aligned} p(k+2) &= P_{k+2} + P_{k+1} + P_k + P_{k-1} + \cdots + P_2 + P_1 \\ &= 8 + P_{k-2} + P_{k-3} + \cdots + P_3 \\ &< 8 + 2^3 + 2^4 + \cdots + 2^{k-1} \\ &= 2^k < 2^{k+1}. \end{aligned}$$

Type D_n ($n \geq 4$). It is known that $|\mathrm{Irr}W_0|$ is the number $\mathcal{P}(n)$ of unordered pairs (ξ, η) of partitions ξ, η with $|\xi| + |\eta| = n$, and any pairs (ξ, η) with $\xi = \eta$ and $|\xi| + |\eta| = n$ is counted twice, see 2.3. By 1.6(b) and the discussion in 3.5, we have $|\mathrm{Irr}H_{q,t}| \geq 2^4 \cdot 3^{\frac{n-4}{2}}$ if $n \geq 4$ is even and $|\mathrm{Irr}H_{q,t}| \geq 2^5 \cdot 3^{\frac{n-5}{2}}$ if $n \geq 5$ is odd. Define $\mathcal{D}(n)$ ($n \geq 4$) to be $2^4 \cdot 3^{\frac{n-4}{2}}$ if n is even and to be $2^5 \cdot 3^{\frac{n-5}{2}}$ if n is odd. It suffices to show that $\mathcal{D}(n) > \mathcal{P}(n)$ if $n \geq 4$.

We first show that

(∗) $p(n) \leq 2p(n-2)$ if $n \geq 8$.

Let Q_n be the set of all partitions of n. For $1 \leq k \leq n$, let $Q_{n,k}$ be the set of partitions of n with smallest term k and $p(n,k)$ be the cardinality of $Q_{n,k}$. Then $p(n,1) = p(n-1)$ and $p(n)$ is the sum of all $p(n,k)$. So $p(n) - p(n-1)$ is the sum of all $p(n,k)$ with $k \geq 2$. Since $p(n) - p(n-2) = p(n) - p(n-1) + p(n-1) - p(n-2)$, we get

$$p(n) - p(n-2) = \sum_{k=2}^{n} p(n,k) + \sum_{k=2}^{n-1} p(n-1,k).$$

We define an injection

$$\tau : \Big(\bigcup_{k=2}^{n} Q_{n,k}\Big) \cup \Big(\bigcup_{k=2}^{n-1} Q_{n-1,k}\Big) \to \bigcup_{k=1}^{n-2} Q_{n-2,k} = Q_n.$$

For a partition $a_1 \geq a_2 \geq a_3 \cdots \geq a_r > 0$ of n, we shall simply write it as $a_1 a_2 a_3 \cdots a_r$. For $A = a_1 a_2 \cdots a_{r-1} a_r \in Q_{n-1,k}$ ($k \geq 2$), define $\tau(A) = a_1 a_2 \cdots a_{r-1}(a_r - 1)$. Note that $a_{r-1} > a_r - 1 \geq 1$.

For $A = a_1 a_2 \cdots a_{r-1} a_r \in Q_{n,k}$ ($k \geq 4$), define $\tau(A) = a_1 a_2 \cdots a_{r-1} 1 \cdots 1$, where 1 appears $a_r - 2 = k - 2$ times. Note that $a_{r-1} - (a_r - 2) \geq 2$.

For $A = a_1 a_2 \cdots a_{r-2} a_{r-1} 3 \in Q_{n,3}$, set $\tau(A) = a_1 a_2 \cdots a_{r-2} 2 1 \cdots 1$, where 1 appears $a_{r-1} - 1$ times. Note that $a_{r-1} - 1 \geq 2$ and $a_{r-2} \geq 3$.

For $r \geq 3$ and $A = a_1 a_2 \cdots a_{r-3} a_{r-2} a_{r-1} 2 \in Q_{n,2}$, set $\tau(A) = a_1 a_2 \cdots a_{r-3} a_{r-2} a_{r-1}$ if $a_{r-2} = a_{r-1}$ and set $\tau(A) = a_1 a_2 \cdots a_{r-3}(a_{r-1} + 1) 1 \cdots 1$ if $a_{r-2} > a_{r-1}$, where 1 appears $a_{r-2} - 1$ times. Note that if $a_{r-2} > a_{r-1}$, then $a_{r-2} - 1 \geq 2$ and $a_{r-1} + 1 - (a_{r-2} - 1) \leq a_{r-2} - (a_{r-2} - 1) \leq 1$.

For $A = a_1 2 \in Q_{n,2}$, let $\tau(A) = (a_1 - 6)2211$. Since $n \geq 8$, $\tau(A)$ is well defined.

It is easy to check that τ is injective. Hence we have $p(n) - p(n-2) \leq p(n-2)$ if $n \geq 8$.

Now we show that $\mathcal{D}(n) > \mathcal{P}(n)$. We use induction on n. When $n = 4, 5, 6, 7, 8, 9, 10, 11, 12$, $\mathcal{P}(n)$ is 13, 18, 37, 55, 100, 150, 251, 376, 599, respectively, and $\mathcal{D}(n)$ is 16, 32, 48, 96, 144, 288, 432, 864, 1296, respectively. So the result is true in these cases.

Since $\mathcal{D}(n+2) = 3\mathcal{D}(n)$, it suffices to show that $\mathcal{P}_{n+2} < 3\mathcal{P}(n)$ for $n \geq 11$.

Assume that $n = 2k \geq 12$. Then

$$\mathcal{P}(n) = \sum_{i=0}^{k-1} p(n-i)p(i) + \frac{p(k)(p(k)+3)}{2},$$

$$\mathcal{P}(n+2) = \sum_{i=0}^{k} p(n+2-i)p(i) + \frac{p(k+1)(p(k+1)+3)}{2}.$$

It is well known that $p(k+1) \leq \frac{1}{2}(p(k+2)+p(k))$. Since $k+2 \geq 8$, by $(*)$, we get $p(k+1) \leq \frac{3}{2}p(k)$. Again by $(*)$, for $0 \leq i \leq k-1$, we have $p(n+2-i) \leq 2p(n-i)$ since $n+2-i = 2k+2-i > k+3 > 8$. Thus

$$3\mathcal{P}(n) - \mathcal{P}(n+2) \geq \sum_{i=0}^{k-1} p(n-i)p(i) + \frac{3}{8}p(k)^2 + \frac{9}{4}p(k) - p(k+2)p(k).$$

Euler's Pentagonal Theorem implies that $p(k) \leq p(k-1) + p(k-2)$. Hence

$$3\mathcal{P}(n)-\mathcal{P}(n+2) \geq \sum_{i=0}^{k-3} p(n-i)p(i) + p(k+1)p(k-1) + \frac{3}{8}p(k)^2 + \frac{9}{4}p(k) - p(k+2)p(k-1).$$

But $p(k+2) \leq \frac{3}{2}p(k+1)$, so

$$3\mathcal{P}(n) - \mathcal{P}(n+2) \geq \sum_{i=0}^{k-3} p(n-i)p(i) + \frac{3}{8}p(k)^2 + \frac{9}{4}p(k) - \frac{1}{2}p(k+1)p(k-1).$$

When $k-1 \geq 8$, we have $p(k-1) \leq 2p(k-3)$, so $p(k+3)p(k-3) > \frac{1}{2}p(k+1)p(k-1)$ if $k \geq 9$. If $6 \leq k < 9$, one may check directly that $p(k+3)p(k-3) > \frac{1}{2}p(k+1)p(k-1)$. So $3\mathcal{P}(n) > 3\mathcal{P}(n+2)$ if $n = 2k \geq 12$.

If $n = 2k+1 \geq 11$, the argument for $3\mathcal{P}(n) > \mathcal{P}(n+2)$ is similar (and simpler). The proof for type D_n is complete.

Type E_6, E_7, E_8. The number $|\mathrm{Irr}W_0|$ is 25, 60, 112, respectively, see 2.3. Using 1.5 (b), 2.3, Lemma 3.4 and the discussion in 3.5, we see that $|\mathrm{Irr}H_{q,t}| > |\mathrm{Irr}W_0|$ if $o(q) \neq 11, 13$ when R is type E_8.

In the rest part of the proof we assume that $\sum_{w\in W_0} q^{l(w)} = 0$. Then any simple quotient module of certain standard modules of H_q is a simple constituent of some other standard modules, see [X1, Theorem 7.8, p.83]. This indicates that the standard modules of H_q and the standard modules of H_1 behave differently, so H_q and H_1 should not be isomorphic. We make this point explicit.

Let $t \in T$ be such that $H_{q,t}$ has a one-dimensional representation on which all T_r act by scalar q. Assume that H_q and H_1 were isomorphic. Then there exists $s \in T$ such that $H_{q,t}$ and $H_{1,s}$ are isomorphic. By Lemma 3.1, we may assume that $\alpha(t) = q$ for all simple roots α. By Lemma 2.1, $\alpha(s) = 1$ for all simple roots α.

Let $D = \sum_{w\in W_0} T_w \in H_q$ and $D' = \sum_{w\in W_0}(-q)^{-l(w)}T_w \in H_q$. We also use D and D' for their images in $H_{q,t}$ respectively. Let $C = \sum_{w\in W_0} w \in H_1 = \mathbb{C}[W]$ and $C' = \sum_{w\in W_0}(-1)^{-l(w)}w \in H_1$. The images in $H_{1,s}$ of C and C' will again be denoted by C and C' respectively.

According to [X3, Theorem 3.3], $CH_{1,s}C'$ is a one-dimensional two-sided ideal of $H_{1,s}$. Let $\pi : H_{1,s} \to H_{q,t}$ be an isomorphism. Then $\pi(CH_{1,s}C') = \pi(C)H_{q,t}\pi(C')$ is a one-dimensional two-sided ideal of $H_{q,t}$. Let v be a nonzero element of $\pi(CH_{1,s}C')$. Then one of the following cases must happen:

(1) $T_w v = vT_w = q^{l(w)}v$ for all $w \in W_0$,

(2) $T_w v = vT_w = (-1)^{l(w)}v$ for all $w \in W_0$,
(3) $T_w v = q^{l(w)}v,\ vT_w = (-1)^{l(w)}v$ for all $w \in W_0$,
(4) $T_w v = (-1)^{l(w)}v,\ vT_w = q^{l(w)}v$ for all $w \in W_0$.

To go further we need the following facts.

(5) Let $h \in H_q$. If $T_r h = -h$ (resp. $hT_r = -h$; $T_r h = qh$; $hT_r = qh$) for all simple reflection r in W_0, the $h \in D'H_q$ (resp. $h \in H_q D'$; $h \in H_q D$; $h \in DH_q$).

We explain the reasons for (5). For $w \in W$, let $D'_w = \sum_{y\leq w}(-q)^{-l(y)}P_{y,w}(q^{-1})T_y$, where $\leq$ stands for the Bruhat order on W, and $P_{y,w}$ are the Kazhdan-Lusztig polynomials. Then the elements $D'_w, w \in W_0$, form a basis of the subalgebra H_{q,W_0} of H_q generated by all $T_w, w \in W_0$. Let $w \in W_0$ and r be a simple reflection in W_0. Then $T_r D'_w = -D'_w$ (resp. $D'_w T_r = -D'_w$) if and only if $rw \leq w$ (resp. $wr \leq w$). Moreover, if $rw \geq w$, then $T_r D'_w = q^{\frac{3}{2}}D'_{rw} + \sum_{y\leq w} a_y D'_y$, $a_y \in \mathbb{C}$. See (2.3.a) and (2.3.c) in [KL1]. Note that the elements $D'_w\theta_x,\ w \in W_0,\ x \in X$, form a basis of H_q. Using these facts we see easily that $h \in D'H_q$ if $h \in H_q$ and $T_r h = -h$ for all simple reflections r in W_0. The arguments for other parts of (5) are similar.

The following statement is easy to check.

(6) Let r be a simple reflection in W_0 and $\theta = \sum_{x\in X} a_x\theta_x \in \Theta_q$, where $a_x \in \mathbb{C}$. If $D\theta T_r = qD\theta$ (resp. $D'\theta T_r = -D'\theta$), then $a_x = a_{r(x)}$ and $T_r\theta = \theta T_r$.

As a consequence of (5) and (6), we get (see also [L1, p.213], and note also that $DH_q D \subset DZ(H_q)$ by [L1, Proposition 8.6])

(7) Let $h \in H_q$. If $T_r h = hT_r = qh$ (resp. $T_r h = hT_r = -h$) for all simple reflections r in W_0, then h is in $DZ(H_q)$ (resp. $D'Z(H_q)$).

The following result is implicit in [L1, p.213, line -7]).

(8) Assume that $q \neq -1$. Let $h \in H_q$. If $T_r h = -h$ and $hT_r = qh$ for all simple reflections r in W_0, then h is in $D'H_q D$.

Now we argue for (8). Assume that $h \neq 0$. By (5), h is in $H_q D$. Let $\Omega = \{w \in W \mid wr \geq w$ for all simple reflections r in $W_0\}$. According to [X3, Lemma 2.2], the elements $D'_w D,\ w \in \Omega$, form a basis of $H_q D$. Let $h = \sum_{w\in\Omega}\xi_w D'_w D,\ \xi_w \in \mathbb{C}$. Assume that r is a simple reflection in W_0. Write $(T_r - q)h = \sum_{w\in\Omega}\eta_w D'_w D,\ \eta_w \in \mathbb{C}$. Note that for any $w \in W$ we have $(T_r - q)D'_w = -(1+q)D'_w$ if $rw \leq w$ and $(T_r - q)D'_w = q^{\frac{3}{2}}D'_{rw} + \sum_{y<w} a_y D'_y,\ a_y \in \mathbb{C}$ if $rw \geq w$. Moreover, $ry \leq y$ if $a_y \neq 0$ (see [KL1, (2.3.a)]. Since $D'_y D = 0$ for any $y \notin \Omega$, we must have $\eta_w = 0$ if $rw \geq w$. But $(T_r - q)h = -(1+q)h \neq 0$, so we must have $rw \leq w$ if $\xi_w \neq 0$. Therefore, $rw \leq w$ for all simple reflections r in W_0 whenever $\xi_w \neq 0$. By (5), then $D'_w \in D'H_q$ if $\xi_w \neq 0$. So h is $D'H_q D$.

There is a unique involutive automorphism $\xi \to \xi^*$ of the $\mathbb{C}$-algebra H_q such that $T_r^* = -qT_r^{-1} = q - 1 - T_r$ $(r \in S \cap W_0)$, $\theta_x^* = \theta_{x^{-1}}$ $(x \in X)$ [KL2, 2.13(d)]. Noting that $D^* = (-1)^{l(w_0)}D'$, we see that (8) is equivalent to the following result.

(9) Let $h \in H_q$. If $T_r h = qh$ and $hT_r = -h$ for all simple reflections r in W_0, then h is in $DH_q D'$.

By (7), (8) and (9), we must have $v \in \mathbb{C}D$, or $v \in \mathbb{C}D'$, or $v \in DH_{q,t}D'$, or $v \in D'H_{q,t}D$. But neither $\mathbb{C}D$ nor $\mathbb{C}D'$ is two-sided ideal of $H_{q,t}$. So cases (1) and (2) would not occur. Since $\sum_{w\in W_0} q^{l(w)} = 0$, by [X3, Theorem 3.3], $DH_{q,t}D' = D'H_{q,t}D = 0$. These contradict that $v \neq 0$. So $H_{q,t}$ and $H_{1,s}$ are not isomorphic. Therefore, H_q and H_1 are not isomorphic.

The theorem is proved.

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