

Hyperfinite actions on countable sets and probability measure spaces

Miklós Abért and Gábor Elek

ABSTRACT. We introduce the notion of hyperfiniteness for permutation actions of countable groups and give a geometric and analytic characterization, similar to the known characterizations for amenable actions. We also answer a question of van Douwen on actions of the free group on two generators on countable sets.

1. Introduction

Let Γ be a countable group acting on a countably infinite set X by permutations. An invariant mean on X is a Γ -invariant, finitely additive map μ from the set of subsets of X to $[0, 1]$ satisfying $\mu(X) = 1$. Von Neumann [NEU] initiated the study of invariant means of group actions.

We say that a group action of Γ on X is *amenable* if X admits a Γ -invariant mean. The group Γ is amenable, if the right action of Γ on itself is amenable. Over the decades, amenability of groups has become an important subject, with connections to various areas in mathematics, like combinatorial group theory, probability theory, ergodic theory and harmonic analysis.

All actions of amenable groups are amenable and for free actions, this trivially goes the other way round as well, but in general, one has to assume certain faithfulness conditions to make the notion meaningful. Even when making the natural assumption that the action is transitive, the general picture is that for most sets of conditions, one can construct corresponding amenable actions of groups that are very far from being amenable themselves.

In particular, van Douwen [DOU] constructed a transitive amenable action of the free group on two generators such that every nontrivial element of the group fixes only finitely many points. We call this condition *almost freeness*. Further examples of amenable actions of non-amenable groups were given by Glasner and Monod [GN] and by Moon [MO1], [MO2].

For probability measure preserving (p.m.p.) actions, the notion that mostly corresponds to amenability is hyperfiniteness. Let Γ act on a probability measure space, preserving the measure. The action is called *hyperfinite* if the measurable equivalence relation generated by the action is up to measure zero an ascending

2010 *Mathematics Subject Classification*. Primary 37A15.

Key words and phrases. Hyperfinite equivalence relations, amenability.

Supported by OTKA Grant 69062 and TAMOP Grant 4.2.1/B-09/1/KMR-2010-003.

union of finite measurable equivalence relations. As before, all p.m.p. actions of amenable groups are hyperfinite, and for free actions, there is equivalence, but in general, very large groups can act hyperfinitely. In particular, Grigorchuk and Nekhrashevych [GRIG] constructed an ergodic, faithful, hyperfinite p.m.p. action of a non-amenable group. More generally, for a hyperfinite p.m.p. action of a group Γ , the action of Γ on almost all orbits is amenable. For the other direction, Kaimanovich [KAI] presented a counterexample. The main goal of this paper is to introduce and analyze the notion of hyperfiniteness for permutation actions of countable groups. If Γ acts on a countably infinite set X , then the action always extends to the Stone-Cech compactification βX . This connection establishes a bijection between invariant means on X and invariant measures on βX . In particular, an action is amenable if and only if the extended action preserves a regular Borel-probability measure. This suggests the following definition.

DEFINITION 1.1. Let the countable group Γ act on the set X by permutations. We say that the action is *hyperfinite* if βX admits a regular Borel probability measure that is invariant under the extended action and for which this action is hyperfinite.

In particular, every hyperfinite action is automatically amenable. Our first result gives a combinatorial and a geometric characterization of hyperfiniteness for actions of finitely generated groups. Let G_n be a sequence of graphs with an absolute bound on the degrees of vertices in G_n . We say that (G_n) is *hyperfinite*, if for all $\varepsilon > 0$, there exists $Y_n \subseteq V(G_n)$ and $K > 0$ such that $|Y_n| < \varepsilon |G_n|$ and every connected component of the subgraph induced on $V(G_n) \setminus Y_n$ has size at most K ($n \geq 1$). This notion was introduced in [ECOST].

THEOREM 1. Let Γ be a group generated by the finite symmetric set S , acting on the countably infinite set X by permutations. Let S_Γ denote the Schreier graph of this action with respect to S . Then the following are equivalent:

- 1) The action is hyperfinite;
- 2) There exists a hyperfinite Følner-sequence in S_Γ ;
- 3) There exists an invariant mean μ on X , such that for all $\varepsilon > 0$, there exists $Y \subseteq X$ with $\mu(Y) < \varepsilon$ and $K > 0$ such that the connected components of the induced subgraph of S_Γ on $X \setminus Y$ have size at most K .

In [DOU] van Douwen asked the following question. Let H be a countable infinite amenable group. Is there an almost free transitive action of F_2 , the free group of two generators, on H such that every invariant mean on H is F_2 -invariant? We will show that the answer is negative, however, it is true if we change the almost freeness condition to faithfulness.

THEOREM 2.

- (1) There exists no almost-free transitive action of F_2 on a finitely generated amenable group H which preserves all H -invariant means.
- (2) For any finitely generated amenable group H , there exists a faithful, transitive action of F_2 on H which preserves all the H -invariant means.

Finally, we show the following.

THEOREM 3. There exists an ergodic, faithful p.m.p. profinite action of a non-amenable group that is hyperfinite but topologically free.

Note that this answers a question of Grigorchuk, Nekrashevich and Sushchanskii [GRNS]. Note that Bergeron and Gaboriau [BG] also constructed an ergodic, faithful p.m.p profinite action which is not free, but topologically free.

Acknowledgement

The authors thank Nicolas Monod for helpful comments on a previous version of this paper.

2. The Stone-Cech compactification

Let X be a countably infinite set and βX be its Stone-Cech compactification. The elements of βX are the ultrafilters on X and the set X itself is identified with the principal ultrafilters. For a subset $A \subseteq X$, let U_A be the set of ultrafilters $\omega \in \beta X$ such that $A \in \omega$. Then $\{U_A\}_{A \subseteq X}$ forms a base of the compact Hausdorff topology of βX . It is well-known that the Banach-algebra of continuous functions $C(\beta X)$ can be identified with the Banach-algebra $l^\infty(X)$. Let μ be a finitely additive measure on X . Then one can associate a regular Borel measure $\widehat{\mu}$ on βX , by taking

$$\widehat{\mu}(U_A) = \mu(A).$$

Indeed, let $f \in l^\infty(X)$ be a bounded real function on X . Then the continuous linear transformation

$$T(f) := \int_X f d\mu$$

is well-defined. Thus, by the Riesz representation theorem

$$T(f) = \int_{\beta X} f d\widehat{\mu}$$

for some regular Borel-measure $\widehat{\mu}$. Since $T(\chi_{U_A}) = \mu(A)$, the equality $\mu(A) = \widehat{\mu}(U_A)$ holds.

In fact, there is a one-to-one correspondance between the regular Borel-measures and the finitely additive measures on X , since the integral $\int_X f d\mu$ is completely defined by μ . If $s : X \rightarrow X$ is a bijection, then it extends to a map $\widehat{s} : \beta X \rightarrow \beta X$ by

$$\widehat{s}(\omega) = \bigcup_{A \in \omega} s(A).$$

Since $\widehat{s}(U_A) = U_{s(A)}$, the map \widehat{s} is a continuous bijection.

Thus if Γ is a countable group acting on X , then we have an extended action on βX . The following lemma is due to Blümlinger [BLU, Lemma 1]

LEMMA 2.1. *There is a one-to-one correspondance between the Γ -invariant means on X and the Γ -invariant regular Borel probability measures on X .*

Proof. Observe that the set of Γ -invariant regular measures is the annihilator of the set

$$\{f - \gamma(f) \mid f \in C(\beta X), \gamma \in \Gamma\},$$

and the set of Γ -invariant means is the annihilator of the set

$$\{f - \gamma(f) \mid f \in l^\infty(X)\}.$$

Therefore an action of Γ is amenable if and only if the corresponding action on βX has an invariant probability measure.

3. Geometrically hyperfinite actions

Let Γ be a finitely generated group acting on X preserving the mean μ . Let S be a finite, symmetric generating set for Γ and S_Γ be the Schreier graph of the action. That is

- $V(S_\Gamma) = X$.
- $(x, y) \in E(S_\Gamma)$ if $x \neq y$ and there exists $s \in S$ such that $s(x) = y$

Note that we do not draw loops in our Schreier-graphs.

Let T be a subgraph of S_Γ . The edge measure of T is defined as

$$\mu_E(T) = \frac{1}{2} \int_X \deg_T(x) d\mu(x),$$

where $\deg_T(x)$ is the degree of x in T .

We say that the action is *geometrically hyperfinite* if for any $\varepsilon > 0$, there exists $K_\varepsilon > 0$ and a subgraph $G_\varepsilon \in S_\Gamma$ such that $V(G_\varepsilon) = X$ and

$$\mu_E(S_\Gamma \setminus G_\varepsilon) < \varepsilon$$

and all the components of G_ε have size at most K_ε . It is easy to see that geometrical hyperfiniteness does not depend on the choice of the generating system. Note however, that the geometric hyperfiniteness and the hyperfiniteness of an action do depend on the choice of the invariant measure.

It is possible that for some invariant mean μ_1 the action is hyperfinite and for another invariant mean μ_2 the action is not hyperfinite, only amenable. The hyperfiniteness of a family of finite graphs was introduced in [ECOST]. Let $\mathcal{G} = \{G_n\}$ be a family of finite graphs with vertex degree bound d . Then \mathcal{G} is called hyperfinite if for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that for any $n \geq 1$ one can delete $\varepsilon|V(G_n)|$ edges from G_n to obtain a graph of maximum component size at most K_ε .

PROPOSITION 3.1. *Let S_Γ be the Schreier graph of an action of the finitely generated group Γ on X . Then the following two statements are equivalent.*

- (1) S_Γ contains a hyperfinite Følner-sequence.
- (2) The action is geometrically hyperfinite with respect to some Γ -invariant mean μ .

Recall that a Følner-sequence of S_Γ is sequence of induced subgraphs $\{F_n\}_{n=1}^\infty$, where the isoperimetric constant $i(F_n)$ tends to zero as n tends to infinity. The isoperimetric constant of a finite subgraph is the number of outgoing edges divided by the number of vertices.

Proof. Suppose that S_Γ has a hyperfinite Følner-sequence $\{F_n\}_{n=1}^\infty$. Let $G_n \subseteq F_n$ be induced subgraphs such that $\lim_{n \rightarrow \infty} \frac{|V(G_n)|}{|V(F_n)|} = 1$. Then clearly $\{G_n\}$ is a hyperfinite Følner-sequence as well. Therefore, we can suppose that $\{F_n\}_{n=1}^\infty$ are vertex-disjoint subgraphs. Indeed, let F_{n_1} be an element of the Følner-sequence such that

$$\frac{|V(F_{n_1} \setminus F_1)|}{|V(F_{n_1})|} > 1 - \frac{1}{10}.$$

Then let F_{n_2} be an element such that

$$\frac{|V(F_{n_1} \setminus (F_1 \cup F_{n_1}))|}{|V(F_{n_2})|} > 1 - \frac{1}{100}.$$

Inductively, we can construct a hyperfinite Følner-sequence consisting of vertex-disjoint subgraphs. Now let ω be an ultrafilter on \mathbb{N} and \lim_ω be the corresponding ultralimit $\lim_\omega : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$. Let

$$\mu(A) := \lim_\omega \frac{|A \cap V(F_n)|}{|V(F_n)|}.$$

Then μ is an invariant mean and the action is geometrically hyperfinite with respect to μ . Now let us suppose that μ is a Γ -invariant mean on X and the action is geometrically hyperfinite with respect to μ .

Let $\{G_\varepsilon\}_{\varepsilon>0}$ be the subgraphs of S_Γ as in the definition of hyperfiniteness. We need the following lemma.

LEMMA 3.2. *Let $R \subseteq S_\Gamma$ be a subgraph of components of size at most C .*

Suppose that the edge-density (number of edges divided by the number of vertices) in each component is at least α . Then $\alpha\mu(V(R)) \leq \mu_E(R)$.

Proof. We can write R as a vertex-disjoint union $R = \cup_{i=1}^k R_i$, where all the components of R_i are isomorphic, having l_i vertices and m_i edges. Let $S_i \subset V(R_i)$ be a set containing exactly one vertex from each component. We can even suppose that under the isomorphisms of the components we always choose the same vertex.

Thus by the invariance of the mean, we have a partition

$$V(R_i) = \cup_{j=1}^{l_i} S_i^j,$$

where $S_i^1 = S_i$, $\mu(S_i^j) = \frac{1}{l_i} \mu(V(R_i))$, and S_i^j also has the property that it contains one vertex from each component and under the isomorphisms of the components, it always contains the same vertex. Then

$$\mu_E(R_i) = \frac{1}{2} \sum_{j=1}^{l_i} d_i^j \mu(S_i^j),$$

where d_i^j is the degree in a component of R_i at a vertex of S_i^j . This yields $\mu_E(R_i) = m_i \mu(V(R_i)) / l_i$. Therefore

$$\mu_E(R) = \sum_{i=1}^k \mu_E(R_i) = \sum_{i=1}^k \frac{m_i}{l_i} \mu(V(R_i)) \geq \alpha \mu(V(R)).$$

Now, pick a sequence $\varepsilon_1 \geq \varepsilon_2 \geq \dots$ such that

$$(3.1) \quad \sum_{i=1}^{\infty} \sqrt{\varepsilon_i} < 1.$$

Let $\delta > 0$ be a real number and G_δ be a subgraph as above.

Let S_i^δ be the union of components of G_δ in which the edge density of $S_\Gamma \setminus G_{\varepsilon_i}$ is at least $\sqrt{\varepsilon_i}$. By the previous lemma, we have

$$\mu(V(S_i^\delta)) \sqrt{\varepsilon_i} \leq \mu_E(S_\Gamma \setminus G_{\varepsilon_i}).$$

Hence $\mu(V(S_i^\delta)) \leq \sqrt{\varepsilon_i}$. By (3.1), for any $n \geq 1$, there exists $G'_\delta \subset G_\delta$, having the following properties.

- G'_δ is a union of components of G_δ .
- $\mu(V(G'_\delta)) > 0$.

- If Z is a component of G'_δ then the edge-density of $S_\Gamma \setminus G_{\varepsilon_i}$ inside Z is less than $\sqrt{\varepsilon_i}$, for any $1 \leq i \leq n$. That is, we can remove $\sqrt{\varepsilon_i}|V(Z)|$ edges from Z to obtain a graph of maximum component size K_{ε_i} .

For $\varepsilon > 0$ let $W_\delta^\varepsilon \subset G_\delta$ be the union of components H such that the isoperimetric constant of H is less than ε . By our previous lemma, it is easy to see that for any fixed $\varepsilon > 0$ we have

$$\lim_{\delta \rightarrow 0} \mu(W_\delta^\varepsilon) = 1.$$

Therefore, for any $n \geq 1$ there exists δ_n and a component H_n of G_{δ_n} such that

- the isoperimetric constant of H_n is less than $\frac{1}{n}$,
- for any $1 \leq i \leq n$ one can remove $\sqrt{\varepsilon_i}|V(H_n)|$ edges from H_n to obtain a graph of maximum component size K_{ε_i} .

This implies that $\{H_n\}_{n=1}^\infty$ is a hyperfinite Følner-sequence in S_Γ .

4. Graphs and graphings

Let T be a countable graph of vertex degree bound d , such that $V(T) = X$. Then there exists an action of a finitely generated group Γ such that T is the (loopless) Schreier graph of the action. Indeed, one can label the edges of T with finitely many labels $\{c_1, c_2, \dots, c_n\}$ in such a way that incident edges are labeled differently. This way we obtain the Schreier graph of the n -fold free product of C_2 . If μ is a Γ -invariant mean on X such that T is the Schreier graph of the action with respect to a finite symmetric generating set $S \subset \Gamma$, then μ is an H -invariant mean for any other action by a finitely generated group H with the same Schreier graph. Indeed, if $h \in H$ is a generator of H and $A \subseteq X$, then A can be written as a disjoint union

$$A = \bigcup_{s \in S} A_s \cup A_1,$$

where $h(x) = s(x)$ on A_s and $h(x) = x$ on A_1 . Therefore,

$$\mu(h(A)) = \sum_{s \in S} \mu(s(A_s)) + \mu(A_1) = \mu(A).$$

Thus if T is a graph on X with bounded vertex degree, we can actually talk about T -invariant means on X .

Let us consider a Γ -action on X preserving the mean μ and the extended Γ -action on βX preserving the associated probability measure $\hat{\mu}$.

Let \mathcal{G} be the graphing of this action on βX (see [KECH]) associated to a finite symmetric generating set S , that is the Borel graph on βX such that $(x, y) \in E(\mathcal{G})$ if $x \neq y$ and $s(x) = y$ for some generator s .

LEMMA 4.1. *The graphing \mathcal{G} depends only on T , assuming T is the Schreier graph for the action.*

Proof. Let H be another finitely generated group with generating system S' , such that the Schreier graph of this action is T as well. It is enough to prove that for any $\omega \in \beta X$ and $s' \in S'$ either $s'(\omega) = \omega$ or $s'(\omega) = s(\omega)$ for some $s \in S$. Observe that there exists $s \in S$ or $s = 1$ such that

$$B = \{n \in X \mid s(n) = s'(n)\} \in \omega.$$

Let $A \in \omega$. Then $s(A \cap B) = s'(A \cap B)$. Hence $s(A) \in s'(\omega)$. Thus, $s(\omega) = s'(\omega)$.

We denote the graphing associated to T by $\mathcal{G}(T)$. Note that if S is a subgraph of T , such that $V(S) = A \subseteq X$ then $\mathcal{G}(S) \subseteq \mathcal{G}(T)$ and all the edges of $\mathcal{G}(S)$ are in between points of U_A .

Let us briefly recall the local statistics for graphings [EFIN]. A rooted, finite graph of radius r is a graph H , with a distinguished vertex x such that

$$\max_{y \in V(H)} d_H(x, y) = r.$$

Let U_d^r denote the finite set of rooted finite graphs of radius r with vertex degree bound d (up to rooted isomorphism).

If T is a countable graph as above, let $A(T, H) \subseteq X$ be the set of points x such that the r -neighborhood of x on T is rooted isomorphic to H .

Similarly, let $A(\mathcal{G}(T), H) \in \beta X$ be the set of points $\omega \in \beta X$ such that the r -neighbourhood of $\omega \in \mathcal{G}(T)$ is rooted isomorphic to H .

We need the labeled version of the above setup as well. Let $U_d^{r,n}$ denote the finite set of rooted finite graphs of radius r with vertex degree bound d , edge-labeled by the set $[n] = \{1, 2, \dots, n\}$. Now let us label the edges of T by the set $[n]$ in such a way that incident edges are labeled differently. Then this labeling induces a Schreier graph of T and thus a labeling of $\mathcal{G}(T)$ as well. Again, for $\tilde{H} \in U_d^{r,n}$ let $A(T, \tilde{H}) \subseteq X$ be the set of points n such that the r -neighborhood of n on T is rooted isomorphic to \tilde{H} .

Similarly, let $A(\mathcal{G}(T), \tilde{H}) \in \beta X$ be the set of points $\omega \in \beta X$ such that the r -neighbourhood of $\omega \in \mathcal{G}(T)$ is rooted isomorphic to \tilde{H} . For $\tilde{H} \in U_d^{r,n}$, we denote by $[H]$ the underlying unlabeled rooted graph in U_d^r .

The following proposition states that the local statistics of T and $\mathcal{G}(T)$ coincide.

PROPOSITION 4.2. *For any $r \geq 1$ and $H \in U_d^r$*

$$\mu(A(T, H)) = \hat{\mu}(A(\mathcal{G}(T), H)).$$

Proof. Let us partition $A(T, H)$ into finitely many parts

$$A(T, H) = \cup_{\tilde{H}, [\tilde{H}] = H} A(T, \tilde{H})$$

LEMMA 4.3. *The r -neighborhood of any ω in $U_{A(T, \tilde{H})}$ is rooted-labeled isomorphic to \tilde{H} .*

Proof. Let γ and δ be elements in the n -fold free product of C_2 with word-length at most r .

Suppose that $\gamma(x) = \delta(x)$ if $x \in A(T, \tilde{H})$. Then $\gamma(A) = \delta(A)$ if $A \subset A(T, \tilde{H})$, thus $\gamma(\omega) = \delta(\omega)$. Now suppose that $\gamma(x) \neq \delta(x)$ if $x \in A(T, \tilde{H})$. Then $\gamma\delta^{-1}(A(T, \tilde{H})) \cap A(T, \tilde{H}) = \emptyset$. Therefore, $\gamma(\omega) \neq \delta(\omega)$. Therefore, the rooted-labeled r -ball around ω is isomorphic to \tilde{H} .

By the lemma,

$$\hat{\mu}(A(\mathcal{G}(T), H)) \geq \sum_{\tilde{H}, [\tilde{H}] = H} \hat{\mu}(U_{A(T, \tilde{H})}) = \sum_{\tilde{H}, [\tilde{H}] = H} \mu(A(T, \tilde{H})) = \mu(A(T, H)).$$

Since

$$\sum_{H \in U_d^r} \mu(A(T, H)) = \sum_{H \in U_d^r} \hat{\mu}(A(\mathcal{G}(T), H)) = 1$$

the proposition follows.

The following lemma is an immediate consequence of our proposition.

LEMMA 4.4. *Let $W \subseteq T$ be a subgraph. Then for any $l > 0$ the μ -measure of points that are in some components of size l is exactly the $\widehat{\mu}$ -measure of points that are in some components of $\mathcal{G}(W)$ of size l .*

Now we prove the main result of this section.

THEOREM 4.5. *Let Γ be a finitely generated group acting on X preserving the mean μ . Then the Schreier graph S_Γ is hyperfinite if and only if the extended action on βX is hyperfinite. That is, the notions of geometric hyperfiniteness and hyperfiniteness of Γ -actions are equivalent.*

Proof. First let us suppose that S_Γ is hyperfinite. Then $S_\Gamma = W \cup Z$, where all the components of W have size at most K and $\mu_E(Z) \leq \varepsilon$. Then $\mathcal{G}(S_\Gamma) = \mathcal{G}(W) \cup \mathcal{G}(Z)$, where all the components of $\mathcal{G}(W)$ have size at most K and $\widehat{\mu}_E(\mathcal{G}(Z)) \leq \varepsilon$. Hence if S_Γ is hyperfinite, then the extended action is hyperfinite. Now let us suppose that the extended action on βX is hyperfinite. Let T be the Schreier graph of the Γ -action on X and $\varepsilon > 0$ be a real number. A basic subgraph of T is a graph (A, B, t) , where

- A and B are disjoint subsets of X ,
- t is a bijection between A and B with graph contained in T ,
- $V(A, B, t) = X$,
- $E(A, B, t) = \bigcup_{x \in A} (x, t(x))$.

Clearly, T can be written as an edge-disjoint union

$$T = \bigcup_{i=1}^n (A_i, B_i, t_i).$$

Then $\mathcal{G}(T) = \bigcup_{i=1}^n (U_{A_i}, U_{B_i}, \widehat{t}_i)$ where \widehat{t}_i is the extension of t_i to βX . Since $\mathcal{G}(T)$ is hyperfinite, there exists $W \subset \mathcal{G}(T)$, such that

$$W = \bigcup_{i=1}^n (L_i, M_i, \widehat{t}_i),$$

where

- $L_i \subset U_{A_i}$ is Borel for any $1 \leq i \leq n$,
- $\sum_{i=1}^n \widehat{\mu}(U_{A_i} \setminus L_i) \leq \frac{\varepsilon}{2}$,
- $\widehat{t}(L_i) = M_i$.
- all the components of W have size at most K .

LEMMA 4.6. *For any $\delta > 0$, there exist sets $N_i \subseteq A_i$ such that*

$$\sum_{i=1}^n \widehat{\mu}(L_i \Delta U_{N_i}) < \delta.$$

Proof. Since $\widehat{\mu}$ is regular, there exist compact sets $\{C_i\}_{i=1}^n$ and open sets $\{U_i\}_{i=1}^n$ such that

- $C_i \subset L_i \subset U_i \subset U_{A_i}$,
- $\sum_{i=1}^n \widehat{\mu}(U_i \setminus C_i) < \delta$.

Cover C_i by finitely many base sets $U_{A_{i,j}}$ that are contained in U_i . Since the finite union of base sets is still a base set the lemma follows.

The following lemma is straightforward.

LEMMA 4.7. *Let W be as above and let $\{W_k\}_{k=1}^\infty$ be a sequence of subgraphings of $\mathcal{G}(T)$ such that $\lim_{k \rightarrow \infty} \widehat{\mu}_E(W_k \triangle W) = 0$. Then $\lim_{k \rightarrow \infty} \widehat{\mu}(Bad_K^{W_k}) = 0$, where $Bad_K^{W_k}$ is the set of points that are in a component of W_k of size larger than K .*

By the two lemmas, we have a sequence of subgraphings

$$W'_k = \bigcup_{i=1}^n (U_{N_{i,k}}, U_{t_i(N_{i,k})}, \widehat{t}_i)$$

such that

$$\lim_{k \rightarrow \infty} \widehat{\mu}(Bad_K^{W'_k}) = 0$$

and

$$\lim_{k \rightarrow \infty} \sum_{i=1}^n \widehat{\mu}(L_i \triangle U_{N_{i,k}}) = 0.$$

Now let us consider the subgraphs $H_k \subset T$

$$H_k = \bigcup_{i=1}^n (N_{i,k}, t_i(N_{i,k}), t_i).$$

Then by Lemma 4.4,

$$\lim_{k \rightarrow \infty} \mu(Bad_K^{H_k}) = 0.$$

and

$$\limsup_{k \rightarrow \infty} \mu_E(T \setminus H_k) \leq \varepsilon.$$

This immediately shows that T is geometrically hyperfinite.

We finish this section with a proposition that further underlines the relation between hyperfinite p.m.p actions and hyperfinite actions on countable sets.

PROPOSITION 4.8. *Let Γ be a finitely generated group acting hyperfinitely p.m.p on a probability measure space. Then for almost all orbits, the corresponding actions are hyperfinite as well.*

Proof. We use the same idea as in the proof of Proposition 3.1. Let $\varepsilon_1 > \varepsilon_2 > \dots$ be real numbers such that

$$(4.1) \quad \sum_{n=1}^\infty \sqrt{\varepsilon_n} < \frac{1}{2}.$$

We may suppose that the graphing \mathcal{G} of our action on the probability measure space (Z, μ) is the ascending union of subgraphings $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ such that for all $n \geq 1$

- $\mu_E(\mathcal{G} \setminus \mathcal{G}_n) < \varepsilon_n$
- all the components of \mathcal{G}_n are finite.

We can also suppose that the action is ergodic, since in the ergodic decomposition $\mu = \int \mu_t d\nu(t)$, almost all the μ_t are hyperfinite actions as well [KECH]. Let $X_n \subset Z$ be the set of points z such that the isoperimetric constant of the component of z is greater than $\sqrt{\varepsilon_n}$. Then using the same estimate as in Lemma 3.2, one can immediately see that

$$(4.2) \quad \mu(X_n) \leq \sqrt{\varepsilon_n}.$$

Now let $Y_n^k \subset Z$ be the set of points y in Z such that the edge density of $\mathcal{G} \setminus \mathcal{G}_k$ in the component of \mathcal{G}_n containing y is greater than $\sqrt{\varepsilon_k}$. Then

$$(4.3) \quad \mu(Y_n^k) \leq \sqrt{\varepsilon_k}.$$

For $k \geq 1$ we define the set $A_k \subset Z$ the following way. The point z is in A_k if

- The component of \mathcal{G}_{k+1} containing z has isoperimetric constant not greater than $\sqrt{\varepsilon_{k+1}}$;
- For any $1 \leq i \leq k$ the edge-density of $\mathcal{G} \setminus \mathcal{G}_i$ in the component of \mathcal{G}_{k+1} containing z is not greater than $\sqrt{\varepsilon_i}$.

By (4.1), the measure of A_k is not zero. Therefore by ergodicity, almost every point of Z has an orbit containing a point from each A_k . Let $z \in Z$ be such a point. Let F_k be the component of \mathcal{G}_{k+1} of a point in the orbit of z . Then by the two conditions above $\{F_k\}$ is a hyperfinite Følner-sequence.

5. On faithfulness

If Γ acts on the countably infinite set preserving the mean μ , faithfulness means that for any $1 \neq \gamma \in \Gamma$ the fixed point set of γ is not the whole set. Glasner and Monod proved that for any countable group Γ the free product $\Gamma \star \mathbb{Z}$ can act on a countable set in an amenable, transitive and faithful manner. One can see however, that in their construction, if an element γ is in the group Γ , then the fixed point set of γ has mean one. We call a group action preserving a mean μ *strongly faithful* if the fixed point set of any non-unit element has μ -measure less than one.

PROPOSITION 5.1. *If a countable group Γ admits an amenable, strongly faithful action on countable set, then the group is sofic.*

(see [EHYP] for the definition of soficity). *Proof.* Recall that such an action is called *essentially free*, if the fixed point set of any non-unit element has μ -measure zero. It is proved in [EHYP, Corollary 4.2] that any countable group with an amenable essentially free action is sofic. Hence, the only thing remaining is to show the following lemma.

LEMMA 5.2. *Let Γ be a countable group acting amenably and strongly faithfully on a countably infinite set X , preserving the mean μ . Then Γ admits an amenable, essentially-free action on a countably infinite set.*

Proof. Let $K = \bigcup_{n=1}^{\infty} X^n$. Let us consider the product action of Γ on X^n . Define the mean μ_2 the following way. If $A \subset X \times X$ let

$$\mu_2(A) = \int_X \mu(\pi_1((A \cap (X, z)))) d\mu(z),$$

where π_1 is the projection to the first coordinate.

Clearly, μ_2 is preserved by the Γ -action. Inductively, we can construct invariant means $\{\mu_n\}_{n=1}^{\infty}$ on the sets $\{X^n\}_{n=1}^{\infty}$. Now, let ω be a non-principal ultrafilter on \mathbb{N} . Let us define a mean on K the following way.

$$\nu(B) = \lim_{\omega} \mu_n(B \cap X^n).$$

Then μ is a Γ -invariant mean on K .

Let $1 \neq \gamma \in \Gamma$ and F be the fixed point set of γ in X . The fixed point set of γ in X^n is exactly F^n , and obviously,

$$\mu_n(F^n) = (\mu(F))^n.$$

Hence, $\nu(\cup_{n=1}^\infty F^n) = 0$ and the lemma follows.

This finishes the proof of our proposition.

6. On a problem of van Douwen

In [DOU] van Douwen asked the following question [Question 1.4]: If H is any countable infinite amenable group, then is there an almost free transitive action of F_2 (the free group of two generators) on H such that every invariant mean on H is F_2 -invariant ?

THEOREM 6.1.

- (1) *There exists no almost-free transitive action of F_2 on a finitely generated amenable group H which preserves all H -invariant means.*
- (2) *For any finitely generated amenable group H , there exists a faithful, transitive action of F_2 on H which preserves all the H -invariant means.*

Proof. Let $Cay(H, S)$ be the Cayley-graph of the finitely generated group H with respect to a symmetric generating system S . Suppose that F_2 acts almost freely on H . We separate two cases for the action. **Case 1** There exists a Følner-sequence F_1, F_2, \dots in $Cay(H, S)$ for which the following holds.

- $\{sF_n \cup tF_n \cup s^{-1}F_n \cup t^{-1}F_n \cup F_n\}_{n=1}^\infty$ are disjoint subsets, where s, t are generators of F_2 .
- There exists $\varepsilon > 0$ such that for any $n \geq 1$, $\frac{|(sF_n \cup tF_n \cup s^{-1}F_n \cup t^{-1}F_n) \setminus F_n|}{|F_n|} > \varepsilon$.

We define the H -invariant mean μ by

$$\mu(A) := \lim_\omega \frac{|A \cap F_n|}{|F_n|},$$

where ω is a nonprincipal ultrafilter on \mathbb{N} and \lim_ω is the corresponding ultralimit.

We claim that μ is not preserved by the F_2 -action. Observe that for any $n \geq 1$ at least one of the following four inequalities hold: $|sF_n \setminus F_n| \geq \frac{\varepsilon}{4}|F_n|$, $|tF_n \setminus F_n| \geq \frac{\varepsilon}{4}|F_n|$, $|s^{-1}F_n \setminus F_n| \geq \frac{\varepsilon}{4}|F_n|$, $|t^{-1}F_n \setminus F_n| \geq \frac{\varepsilon}{4}|F_n|$. Hence we can assume that for the set A defined by

$$A = \{n \mid |sF_n \setminus F_n| \geq \frac{\varepsilon}{4}|F_n|\},$$

$A \in \omega$. Therefore

$$\mu(s \cup_{n=1}^\infty F_n) < 1 \quad \text{and} \quad \mu(\cup_{n=1}^\infty F_n) = 1.$$

Therefore μ is not preserved by the F_2 -action. **Case 2** If Følner-sequences described in **Case 1** do not exist, then any Følner-sequence is almost invariant under the F_2 -action, that is

$$(6.1) \quad \lim_{n \rightarrow \infty} \frac{|(sF_n \cup tF_n \cup s^{-1}F_n \cup t^{-1}F_n) \setminus F_n|}{|F_n|} = 0.$$

Now let us fix a Følner-sequence $\{G_n\}_{n=1}^\infty$ in $Cay(H, S)$.

By [ESZA, Proposition 4.1], $\{G_n\}_{n=1}^\infty$ is a hyperfinite sequence. That is, for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that one can remove $\varepsilon|V(G_n)|$ vertices and the

incident edges in such a way that in the resulting graph G'_n , all components have size at most K_ε . By the counting argument applied in the proof of Proposition 3.1, it is easy to see that one can even suppose that all the remaining components have isoperimetric constant at most $\sqrt{\varepsilon}$ in G_n .

Now let us consider the following graph sequence $\{T_n\}_{n=1}^\infty$ edge-labeled by the set $\{s, t, s^{-1}, t^{-1}\}$

- $V(T_n) = V(G_n)$
- (x, y) is a directed edge labeled by s (resp. by t, s^{-1} or t^{-1}) if $s(x) = y$ (resp. $t(x) = y, s^{-1}(x) = y$ or $t^{-1}(x) = y$).

By almost-freeness and (6.1) it is clear that $\{T_n\}_{n=1}^\infty$ is a sofic approximation of F_2 (see [ESZA] for definition).

LEMMA 6.2. $\{T_n\}_{n=1}^\infty$ is a hyperfinite graph sequence.

Proof. By (6.1) there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{x \rightarrow 0} f(x) = 0$ satisfying

$$|(sL \cup tL \cup s^{-1}L \cup t^{-1}L) \setminus L| < f(\delta)|L|$$

for any finite set $L \subset H$ with isoperimetric constant less than δ . Let $\{G'_n\}_{n=1}^\infty$ be the subgraphs of $\{G_n\}_{n=1}^\infty$ obtained by removing $\varepsilon|V(G_n)|$ vertices and the incident edges such that all the components of G'_n have size at most K_ε and G_n -isoperimetric constant at most $\sqrt{\varepsilon}$.

The number of edges of T_n that are in between the components of G'_n is less than $4f(\sqrt{\varepsilon})|V(T_n)|$. Hence by removing $4f(\sqrt{\varepsilon})|V(T_n)|$ edges from T_n and all the edges that are incident to a vertex in $V(T_n) \setminus V(G'_n)$ we can obtain a graph with maximum component size at most K_ε . Therefore $\{T_n\}_{n=1}^\infty$ is hyperfinite.

Since by [ESZA, Proposition 4.1], F_2 has no hyperfinite sofic approximation, we obtain a contradiction.

Therefore F_2 has no almost-free action on H that preserves all the H -invariant means.

Now we construct a faithful and transitive F_2 -action on H that preserves all the H -invariant means.

First, fix a subset $\{i_n\}_{n=-\infty}^\infty \subset \mathbb{Z}$ such that $i_n < i_{n+1}$ for all $n \in \mathbb{Z}$. Then fix a function $f : \mathbb{Z} \rightarrow \{1, -1\}$. The action α of F_2 on \mathbb{Z} is defined the following way. Let s and t be the generators of F_2 .

- If n is odd and $i_n < j < i_{n+1}$, then let $s(j) = j$.
- If n is even and $f(i_n) = 1$, then if $i_n \leq j < i_{n+1}$, let $s(j) = j + 1$. If $j = i_{n+1}$, let $s(j) = i_n$.
- If n is even and $f(i_n) = -1$, then if $i_n < j \leq i_{n+1}$, let $s(j) = j - 1$. If $j = i_n$, let $s(j) = i_{n+1}$.
- If n is even and $i_n < j < i_{n+1}$, then let $t(j) = j$.
- If n is odd and $f(i_n) = 1$, then if $i_n \leq j < i_{n+1}$, let $t(j) = j + 1$. If $j = i_{n+1}$, let $t(j) = i_n$.
- If n is odd and $f(i_n) = -1$, then if $i_n < j \leq i_{n+1}$, let $t(j) = j - 1$. If $j = i_n$, let $t(j) = i_{n+1}$.

Note that the orbits of the F_2 -action α generated by s (we call these orbits s -orbits) resp. by t are finite cycles.

Clearly, one can define $\{i_n\}_{n=-\infty}^\infty$ and $f : \mathbb{Z} \rightarrow \{1, -1\}$ in such a way that for any $1 \neq \gamma \in F_2$ there exists $n \in \mathbb{Z}$ such that $\gamma(n) \neq n$. Now let $\phi : \mathbb{Z} \rightarrow H$ be a

bijection and $K > 0$ such that $d(\phi(n), \phi(n+1)) \leq K$ for any $n \in \mathbb{Z}$, where $d(x, y)$ is the shortest path distance in the Cayley graph of H .

Such bijection always exists from \mathbb{Z} to an infinite connected bounded degree graph G if G has one or two ends [SEW]. On the other hand, the Cayley graph of an amenable group always has one or two ends [MV]. The F_2 -action on H is given by

$$\gamma(x) := \phi(\alpha(\gamma)\phi^{-1}(x)).$$

Let μ be an H -invariant mean. We need to prove that μ is invariant under the F_2 -action above.

LEMMA 6.3. *Let $n \geq 1$ and let*

- $\Omega_n^s := \{x \in H \mid d(s(x), x) \geq Kn\}$
- $\Omega_n^{s^{-1}} := \{x \in \mathbb{Z} \mid d(s^{-1}(x), x) \geq Kn\}$
- $\Omega_n^t := \{x \in \mathbb{Z} \mid d(t(x), x) \geq Kn\}$
- $\Omega_n^{t^{-1}} := \{x \in \mathbb{Z} \mid d(t^{-1}(x), x) \geq Kn\}$

Then the μ -measure of any of these sets is less than $\frac{1}{n}$.

Proof. Clearly, each s -orbit of size at least $n+1$ contains at most one element of Ω_n^s . The other s -orbits are disjoint from Ω_n^s . We need to show that the union of s -orbits of the F_2 -action on H of size at least $n+1$ has measure at least $n\mu(\Omega_n^s)$.

Let C be such an s -orbit of size t and let C_p be the unique vertex such that

$$d(s^i(C_p), s^{i+1}(c_p)) \leq K$$

for $i \leq t-2$. Consider the set

$$\bigcup_{C, |C| \geq n+1} C_p = L.$$

It suffices to prove that

$$(6.2) \quad \mu(L) = \mu(s(L)) = \dots = \mu(s^n(L))$$

Define $h_p \in H$ by $s(C_p) = h_p C_p$. Then h_p is in the K -ball around the unit element in the Cayley-graph of H . Let $L = \bigcup_{h \in B_K(1)} L^h$, where $C_p \in L^h$ if $s(C_p) = hC_p$. Then

$$\mu(s(L)) = \sum_{h \in B_K(1)} \mu(s(L^h)) = \sum_{h \in B_K(1)} \mu(L^h) = \mu(L).$$

Similarly, $\mu(s^i(L)) = \mu(L)$ if $i \leq t-1$, therefore (6.2) holds.

Now we finish the proof of the second part of our theorem. Let $A \subseteq X$. Then

$$A = \bigcup_{h \in H} A_h \quad \text{where} \quad A_h = \{x \in A \mid s(x) = hx\}$$

Obviously, $\mu(A_h) = \mu(s(A_h))$. Also, by our previous lemma,

$$\mu\left(\bigcup_{h, h \notin B_{Kn}(1)} s(A_h)\right) \leq \frac{1}{n}.$$

Therefore $\mu(s(A)) \leq \mu(A) + 1/n$. for any $n \geq 1$. Hence $\mu(s(A)) \leq \mu(A)$. However, the same way we can see that $\mu(s^{-1}(A)) \leq \mu(A)$ as well. That is, $\mu(s(A)) = \mu(A)$. Similarly, $\mu(t(A)) = \mu(A)$.

7. A topologically free, hyperfinite action of a nonamenable group

Answering a question of Grigorchuk, Nekhrashevich and Sushchanskii [GRNS] Gaboriau and Bergeron [BG] constructed a profinite, faithful, ergodic action of a nonamenable group that is not essentially free, but topologically free. Note that topological freeness of an action means that the set of points that are not in the fixed point set of any nonunit element of the group is comeager.

On the other hand, Grigorchuk and Nekrashevich constructed a profinite, ergodic action of a nonamenable group that is faithful and hyperfinite. Their construction is very far from being topologically free, in fact the set of points that are not in the fixed point set of any nonunit element is meager.

However, we prove that the two results can be combined.

THEOREM 7.1. *There exists a finitely generated non-amenable group with an ergodic, faithful, profinite action that is hyperfinite and topologically free.*

Proof. An amoeba is a finite connected graph G (with loops) having edge-labels A, B, C, D satisfying the following properties:

- G is the union of simple cycles $\{C_i\}_{i=1}^{n_G}$. Some of the cycles might be loops. We call these cycles the basic cycles.
- Any two of the basic cycles intersect each other in at most one vertex.
- Let us consider the graph T where the vertex set of T is the set of basic cycles and two vertices are connected if and only if the corresponding cycles have non-empty intersection. Then T is a tree.
- For each vertex $x \in V(G)$ and for any label A, B, C, D there exists exactly one edge (maybe a loop) incident to x having that label. Hence the degree of any vertex is 4. Note that the contribution of a loop in the degree of a vertex is 1.
- Each loop is labeled by C or D . For any vertex x there are 0 or 2 loops incident to x .

Clearly, any amoeba is a planar graph. The minimal amoeba M has 2 vertices. The edge set of M consists of a cycle of length two labeled by A and B respectively and two loops for each of the two vertices labeled by C and D . Let G be an amoeba. A *doubling* of G is a two-fold topological covering $\phi : H \rightarrow G$ by an amoeba H constructed the following way. Let $\{C_i\}_{i=1}^{n_G}$ be the set of basic cycles of G . One way to construct H is to pick a basic cycle C_i which is not a loop and consider a two-fold covering $\psi : C'_i \rightarrow C_i$. Obviously, ψ extends to a two-fold covering $\phi : H \rightarrow G$ uniquely, and H is an amoeba as well. The second way to construct H is to pick a vertex x incident to two loops C_i and C_j . Then we cover their union by a cycle of length two. Again, this covering extends to a two-fold covering in a unique way.

7.1. The cycle-elimination tower. Let start with a minimal amoeba G_1 . A *cycle-elimination tower* is a sequence of doublings

$$G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} G_3 \xleftarrow{\phi_3} \dots$$

such that there exists a sequence of vertices $\{p_n \in V(G_n)\}_{n=1}^\infty$, $\phi_n(p_{n+1}) = p_n$ with the following property. For any $k \geq 1$, there exists an integer n_k such that the k -neighborhood of p_{n_k} in the graph G_{n_k} is a tree. It is easy to see that by successively eliminating cycles, such a tower can be constructed. Let Γ be the group $C_2 \star C_2 \star C_2 \star C_2$ with free generators A, B, C, D of order 2. Note that Γ acts on the

vertices of an amoeba. Indeed, the generator A maps the vertex x to the unique vertex y such that the edge (x, y) is labeled by A . If $x = y$, that is the edge is a loop, then A fixes x . Since the covering maps commute with the Γ -actions, one can extend the Γ -action to the inverse limit space $X_\Gamma = \lim_{\leftarrow} V(G_n)$. Recall that there is a natural probability measure μ on X_Γ induced by the normalized counting measures on the vertex sets $V(G_n)$.

The Γ -action on μ preserves the measure μ and in fact this is the only Borel probability measure preserved by the action. The ergodicity of the Γ -action follows from the fact that Γ acts transitively on each vertex set $V(G_n)$.

7.2. Topological freeness. In this subsection we show that the action of Γ on X_Γ is topologically free. Let us introduce some notation. If $m > n$, let ϕ_n^m be the covering map from G_m to G_n . Also, let $\Phi_n : X_\Gamma \rightarrow G_n$ be the natural covering map from the inverse limit space. We need to prove that if $1 \neq \gamma \in \Gamma$, then the fixed point set of γ has empty interior. Let $q \in V(G_n)$. Then $\Phi_n^{-1}(q)$ is a basic open set in X_Γ . It is enough to prove that there exists $z \in \Phi_n^{-1}(q)$ such that $\gamma(z) \neq z$. Let $d = \text{dist}_{G_n}(q, p_n)$, where dist is the shortest path distance and $\{p_n\}_{n=1}^\infty$ is the sequence of vertices as above.

By the properties of graph coverings, for any element x in $(\phi_n^m)^{-1}(p_n)$ there exists $r \in (\phi_n^m)^{-1}(q)$ such that $\text{dist}_{G_m}(x, r) = d$. Now let $w(\gamma)$ be the wordlength of γ and consider the vertex $x = p_{n_{d+|w(\gamma)|}}$. Clearly, if $r \in G_{n_{d+|w(\gamma)|}}$ and $\text{dist}(r, x) = d$ then $\gamma(r) \neq r$. Therefore $\Phi_n^{-1}(q)$ contains a point $z \in X_\Gamma$ that is not fixed by γ .

7.3. Hyperfiniteness. Now we finish the proof of Theorem 7.1 by showing that the action of Γ on X_Γ is hyperfinite. Fix $\varepsilon > 0$. Let us recall [BSS] that planar graphs with bounded vertex degree form a hyperfinite family. Hence there exists $K > 0$ such that for each G_n one can remove $\frac{\varepsilon}{10}|V(G_n)|$ edges in such a way that in the remaining graph G'_n the maximal component size is at most K . Since X_Γ is the inverse limit of $\{V(G_n)\}_{n=1}^\infty$ for any $p \in X_\Gamma$ there exists $m(p) \in \mathbb{N}$ such that if $l \geq m(p)$ then the $K + 1$ -neighborhood of p in its Γ -orbit graph and the $K + 1$ -neighborhood of $\Phi_l(p)$ in G_l are isomorphic. Hence we have $m > 0$ such that the Haar-measure of the set A of the points in X_Γ for which the $K + 1$ -neighborhood of x is not isomorphic to the $K + 1$ -neighborhood of $\Phi_m(x) \in G_m$ is less than $\frac{\varepsilon}{10}$.

Let \mathcal{G} denote the graphing of the Γ -action on X_Γ . We remove the edges from \mathcal{G} that are incident to a point in A . Also, we remove the edges that are inverse images of an edge removed from $E(G_n)$. Then the edge-measure of the edges removed from \mathcal{G} is less than ε and in the remaining graphing all the components have size at most K .

References

- [BSS] I. Benjamini, O. Schramm and A. Shapira, *Every minor-closed property of sparse graphs are testable*. Adv. Math. **223** (2010) 2200–2218. MR2601012 (2011m:05292)
- [BLU] M. Blümlinger, *Lévy group action and invariant measures on $\beta\mathbb{N}$* . Trans. Amer. Math. Soc. **348** (1996) no. 12 5087–5111. MR1390970 (97d:54063)
- [BG] N. Bergeron and D. Gaboriau, *Asymptotique des nombres de Betti, invariants l^2 et laminations*. Comment. Math. Helv. **79** (2004) no. 2 362–395. MR2059438 (2005d:58041)
- [EHYP] G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property*. Math. Ann. **332** (2005) 421–441. MR2178069 (2007i:43002)
- [ECOST] G. Elek, *The combinatorial cost*. Enseign. Math **53** (2007) no. 3-4 225–235. MR2455943 (2009m:05083)

- [EFIN] G. Elek, *On limits of finite graphs*. *Combinatorica* **27** (2007) no. 4 503–507. MR2359831 (2008h:05106)
- [ESZA] G. Elek and E. Szabó, *Sofic representations of amenable groups. to appear in Proc. of the AMS* <http://arxiv.org/pdf/1010.3424> MR2823074
- [GN] Y. Glasner and N. Monod, *Amenable actions, free products and a fixed point property*. *Bull. London Math. Soc.* **39** (2007) no. 1 138–150. MR2303529 (2008h:43001)
- [GRIG] R. I. Grigorchuk and V. Nekrashevych, *Amenable actions of nonamenable groups*. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **326** (2005) 85–96. MR2183217 (2006j:43005)
- [GRNS] R.I. Grigorchuk, V.V. Nekrashevich and V.I. Sushchanskii, *Automata, dynamical systems, and groups*. (Russian) *Tr. Mat. Inst. Steklova* 231 (2000), *Din. Sist., Avtom. i Beskon. Gruppy*, 134–214; translation in *Proc. Steklov Inst. Math.* 2000, no. 4 (231), 128–203 MR1841755 (2002m:37016)
- [KAI] V. Kaimanovich, *Amenability, hyperfiniteness, and isoperimetric inequalities*, *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, Volume 325, Issue 9, November 1997, Pages 999-1004 MR1485618 (98j:28014)
- [KECH] A. Kechris and B. Miller, *Topics in Orbit Equivalence Theory*, *Lecture Notes in Mathematics* 1852, Springer (2004). MR2095154 (2005f:37010)
- [MV] S. Moon and A. Valette, *Non-properness of amenable actions on graphs with infinitely many ends*. *Ischia Group Theory*, World Sci. Publ. Hackensack (2007) 227–233. MR2405942 (2009g:20096)
- [MO1] S. Moon, *Amenable actions of amalgamated free products*, *Groups, geometry and dynamics* **4** (2010) 309–332. MR2595094 (2011g:43001)
- [M02] S. Moon, *Permanence properties of amenable, transitive and faithful actions*, *Bull. Belg. Math. Soc. Simon Stevin* **18** (2011), no. 2, 287–296. MR2848804
- [SEW] B. Seward, *Burnside's problem, spanning trees and tilings*. preprint <http://arxiv.org/pdf/1104.1231v1>
- [DOU] E. K. van Douwen, *Measures invariant under actions of F_2* , *Topology Appl.* **34** (1990), no. 1 53–68. MR1035460 (91b:43003)
- [NEU] Johan von Neumann, *Zur allgemeinen Theorie des Maßes*, *Fund. Math.* 13 (1929), 73–116.

ALFRED RÉNYI INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, REALTANODA U. 13-15, 1053 BUDAPEST, HUNGARY
E-mail address: `abert@renyi.hu`

ALFRED RÉNYI INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, REALTANODA U. 13-15, 1053 BUDAPEST, HUNGARY
E-mail address: `elek@renyi.hu`