Contemporary Mathematics
Volume **568**, 2012
http://dx.doi.org/10.1090/conm/568/11277

The Expected Retraction Method in Banach Spaces

Manal Gabour and Simeon Reich

Dedicated to the memory of Dan Butnariu

Abstract. We solve the Stochastic Convex Feasibility Problem in Banach spaces by applying the Expected Retraction Method. Both weak and strong convergence theorems are obtained.

1. Introduction

Let X be a separable Banach space and let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space. A measurable point-to-set mapping $Q : \Omega \mapsto 2^X$ is given such that, for each $\omega \in \Omega$, the set Q_ω $(= Q(\omega))$ is a nonempty, closed, and convex subset of X. The Stochastic Convex Feasibility Problem (SCFP) [**4, 5, 6**] is to find a point $z \in X$ such that

$$\mu[\{\omega \in \Omega : z \in Q_\omega\}] = 1. \tag{1.1}$$

We distinguish between two cases: in the consistent case, there exists a point $z \in X$ that satisfies (1.1); in the inconsistent case, there is no such point. In the first case, the point $z \in X$ is called a solution of the SCFP and we name it a μ-almost common point of the family of sets $\{Q_\omega : \omega \in \Omega\}$. We denote the set of all such points by $\bigcap_\mu Q_\omega$. Note that the set $\{\omega \in \Omega : z \in Q_\omega\}$ is measurable for every $z \in X$. Therefore (1.1) makes sense. When Ω is finite, $\mathcal{A}$ is equal to 2^Ω and μ is an atomic measure (that is, $\mu(\omega) > 0$ for every $\omega \in \Omega$), then the SCFP reduces to the well-known Convex Feasibility Problem (CFP).

In this paper, we solve the SCFP in Banach spaces by applying the Expected Retraction Method (ERM) [**12**, p. 49], which is studied in Section 5 below. It is defined by using the expected retraction operator (Section 3). The relaxed expected retraction operator (defined in Section 4) is used in a variant of the ERM, namely the Relaxed Expected Retraction Method (RERM). We establish both weak and strong convergence theorems (see Theorems 5.1 and 5.2, as well as Theorems 5.3 and 5.4). In the next section, we collect several relevant definitions.

2010 *Mathematics Subject Classification.* 46B20, 47H09, 47H10, 47H30, 47J25, 90C30.

Key words and phrases. Banach space, Bochner integral, expected retraction method, fixed point, relaxation, stochastic convex feasibility problem, sunny nonexpansive retraction.

The second author was partially supported by the Israel Science Foundation (Grant 647/07), the Fund for the Promotion of Research at the Technion and by the Technion President's Research Fund.

©2012 M. Gabour, S. Reich

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

2. Definitions

In this section we recall, for the reader's convenience, several pertinent definitions.

Definition 2.1. Nonexpansive Mapping
Let D be a subset of a Banach space $(X, |\cdot|)$. A mapping $T : D \mapsto X$ is called nonexpansive if $|Tx - Ty| \leq |x - y|$ for all $x, y \in D$.

For basic information regarding this class of mappings see, for example, [**7**].

Definition 2.2. Strongly Nonexpansive Mapping
Let D be a subset of a Banach space X. A mapping $T : D \mapsto X$ is said to be strongly non-expansive if it is nonexpansive; and whenever the sequence $\{x_n - y_n\}$ is bounded and $[|x_n - y_n| - |Tx_n - Ty_n|] \to 0$ as $n \to \infty$, it follows that $[(x_n - y_n) - (Tx_n - Ty_n)] \to 0$ as $n \to \infty$.

This class of mappings was introduced in [**3**].

Definition 2.3. Nearest Point Projection
A nearest point projection of the point $x \in X$ onto a subset C of a Banach space $(X, |\cdot|)$ is a point $P_C(x) \in C$ such that $|x - P_C(x)| = d(x, C) := \inf_{y \in C} |x - y|$. If X is a Hilbert space and C is a nonempty, closed and convex subset of X, then $P_C(x)$ exists and is unique for each $x \in X$. If C is a closed subspace of X, then P_C coincides with the orthogonal projection of X onto C.

Definition 2.4. Retraction
Let X be a Banach space and let C be a subset of it. A mapping R of X onto C is called a retraction if $R^2 = R$.

Definition 2.5. Nonexpansive Retraction
Let X be a Banach space and let C be a closed subset of X. A mapping R of X onto C is a nonexpansive retraction if $R^2 = R$ and if $|Rx - Ry| \leq |x - y|$ for all $x, y \in X$. In this case, the set C is called a nonexpansive retract of X.

For information concerning nonexpansive retractions and nonexpansive retracts in Banach spaces see, for instance, [**7**] and [**9**].

Definition 2.6. Sunny Retraction
Let C be a closed subset of a Banach space X. A retraction $R : X \mapsto C$ is called a sunny retraction if $Rx = v$ implies that $R(v + r(x - v)) = v$ for every $x \in X$ and $r \geq 0$.

The nearest point projection of a Hilbert space H onto a nonempty, closed and convex subset C of H is a sunny nonexpansive retraction of H onto C [**7**]. It is also known [**8**] that every closed and convex subset of a two-dimensional Banach space X is a sunny nonexpansive retract of X.

Definition 2.7. Symmetric Set
Let X be a Banach space. A subset C of X is said to be symmetric (with respect to the origin) if $C = -C$.

Definition 2.8. Gâteaux Differentiability
Let X, Y be Banach spaces. Let $D \subset X$ be an open subset, $F : D \mapsto Y$ and $x \in D$.

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

If there exists an operator $F'(x)$ in $L(X,Y)$ (the space of bounded linear operators) such that

$$\lim_{t\to 0}\frac{F(x+ty)-F(x)}{t}=F'(x)y$$

for each $y\in X$, then we say that F is Gâteaux differentiable at x.

If the norm of the Banach space X is Gâteaux differentiable off the origin, then X is said to be smooth.

Definition 2.9. Fréchet Differentiability
We say that $F:D\mapsto Y$ is Fréchet differentiable at $x\in D$ if it is Gâteaux differentiable at x and

$$\lim_{t\to 0}\sup_{|y|=1}|\frac{F(x+ty)-F(x)}{t}-F'(x)y|=0.$$

Definition 2.10. Property (S)
A Banach space X is said to have property (S) if there exists a constant $b>0$ such that if $x,y\in X$ and $|x+ry|\geq|x|$ for all $r\geq 0$, then $|x+y|\geq|x-by|$.

This property was introduced in [**3**, p. 468], where it is shown, *inter alia*, that any L^p space, $1<p<\infty$, has property (S). A Hilbert space has property (S) with constant $b=1$.

3. The expected retraction operator

When C is a nonempty nonexpansive retract of a Banach space X and $R_C:X\mapsto C$ is a nonexpansive retraction of X onto C, we denote by $R_C(x)$ or $R(C,x)$ the image of the point $x\in X$ under R.

Definition 3.1. Assume that for each $\omega\in\Omega$, the set Q_ω is a nonexpansive retract of X and that $R_{Q_\omega}:X\mapsto Q_\omega$ is a nonexpansive retraction of X onto Q_ω. For any fixed $x\in X$, the operator $R_\bullet(x):\Omega\mapsto X$, defined by

$$R_\bullet(x)(\omega):=R(Q_\omega,x),$$

is called the random retraction of x onto the family of sets $\{Q_\omega:\ \omega\in\Omega\}$.

We refer to $R_\bullet(x)(\omega)=R(Q_\omega,x)$ by $R_\omega(x)$ for short.

Definition 3.2. Whenever the random retraction operator $R_\bullet(x)$ is Bochner μ-integrable, its Bochner integral

$$R_\mu(x)=\int_\Omega R_\omega(x)d\mu(\omega) \tag{3.1}$$

is called the μ-expected retraction of x onto the family of sets $\{Q_\omega:\ \omega\in\Omega\}$.

Note that the mapping $x\mapsto R_\mu(x)$ is a natural generalization of the simultaneous projection operator, which is used in the Simultaneous Projection Method (see, for example, [**11**]). If Ω is finite, then $R_\mu(x)$ obviously exists for every $x\in X$. If Ω is infinite, then $R_\mu(x)$ may fail to exist. See [**5**] for examples of sufficient conditions that guarantee its existence.

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

4. The relaxed expected retraction operator

In addition to the expected retraction operator, we also introduce the relaxed expected retraction operator, which is defined as follows. It is used in the RERM.

DEFINITION 4.1. The Relaxed μ-Expected Retraction Operator

$$T_\mu^\lambda : \mathrm{dom}(R_\mu) \mapsto X,$$

is defined by

$$T_\mu^\lambda(x) = \int_\Omega T_\omega^\lambda(x) d\mu(\omega), \tag{4.1}$$

where

$$T_\omega^\lambda(x) = (1-\lambda)x + \lambda R_\omega(x) \tag{4.2}$$

and $0 < \lambda < 2$.

The number λ is called the relaxation parameter.

5. ERM iterations

In this section we consider an iterative method, which generalizes the Simultaneous Projection Method. We call it the Expected Retraction Method (ERM). It is defined by

$$x_{n+1} = R_\mu x_n,\ x_0 \in X,\ n = 0, 1, 2, \dots . \tag{5.1}$$

PROPOSITION 5.1. *Assume that R_ω is a strongly nonexpansive mapping for each $\omega \in \Omega$. If the set $\bigcap_\mu Q_\omega$ (the μ-almost common points of the family $\{Q_\omega : \omega \in \Omega\}$) is not empty, then it equals $F(R_\mu)$ (the fixed point set of R_μ).*

PROOF. It is obvious that $\bigcap_\mu Q_\omega \subset F(R_\mu)$. In order to prove that, conversely, $F(R_\mu) \subset \bigcap_\omega Q_\omega$, we fix a point $y_0 \in \bigcap_\mu Q_\omega$ and consider an arbitrary element $x_0 \in F(R_\mu)$. Then we have

$$|x_0 - y_0| = |R_\mu x_0 - R_\mu y_0| \le \int_\Omega |R_\omega x_0 - R_\omega y_0| d\mu(\omega) \le \int_\Omega |x_0 - y_0| d\mu(\omega) = |x_0 - y_0|.$$

Hence

$$\int_\Omega (|x_0 - y_0| - |R_\omega x_0 - R_\omega y_0|) d\mu(\omega) = 0$$

and

$$|x_0 - y_0| = |R_\omega x_0 - R_\omega y_0|$$

for μ-almost every $\omega \in \Omega$. Recalling that the operators R_ω are strongly nonexpansive and the point y_0 belongs to $\bigcap_\mu Q_\omega$, we conclude that $R_\omega x_0 = x_0$ for μ-almost every $\omega \in \Omega$. In other words, $x_0 \in \bigcap_\mu Q_\omega$, as claimed. □

PROPOSITION 5.2. *If the mappings $\{R_\omega : \omega \in \Omega\}$ are strongly nonexpansive, then so is R_μ.*

PROOF. For every $x, y \in X$, we have

$$|x - y| - |R_\mu x - R_\mu y| = |x - y| - \left| \int_\Omega (R_\omega x - R_\omega y) d\mu(\omega) \right|$$

$$\ge |x - y| - \int_\Omega |R_\omega x - R_\omega y| d\mu(\omega) = \int_\Omega [|x - y| - |R_\omega x - R_\omega y|] d\mu(\omega).$$

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

Suppose that $\{x_n - y_n\}$ is a bounded sequence that satisfies

$$|x_n - y_n| - |(R_\mu x_n - R_\mu y_n)| \to 0.$$

We aim to show that

$$(x_n - y_n) - (R_\mu x_n - R_\mu y_n) \to 0.$$

If not, there exist an $\epsilon > 0$ and a subsequence $\{x_{nk} - y_{nk}\}$ such that

$$| (x_{nk} - y_{nk}) - (R_\mu x_{nk} - R_\mu y_{nk}) | \geq \epsilon \tag{5.2}$$

for every natural number k. But

$$\int_\Omega (|x_{nk} - y_{nk}| - |R_\omega x_{nk} - R_\omega y_{nk}|) d\mu(\omega) \to 0$$

as $k \to \infty$. Therefore there exists a subsequence $\{x_{nkl} - y_{nkl}\}$ that satisfies

$$|x_{nkl} - y_{nkl}| - |R_\omega x_{nkl} - R_\omega y_{nkl}| \to 0$$

as $l \to \infty$, μ-almost everywhere. Since each R_ω is strongly nonexpansive, it follows that

$$(x_{nkl} - y_{nkl}) - (R_\omega x_{nkl} - R_\omega y_{nkl}) \to 0,$$

as $l \to \infty$, μ-almost everywhere.

Integrating, we arrive at a contradiction to (5.2) by Lebesgue's Dominated Convergence Theorem. □

THEOREM 5.1. *Let X be a separable Banach space, X and X^* be uniformly convex, and let $\{R_\omega : \omega \in \Omega\}$ be sunny nonexpansive retractions. If $\bigcap_\mu Q_\omega \neq \emptyset$, then any sequence $\{x_n\}$ defined by (5.1) converges weakly as $n \to \infty$ to a point in $\bigcap_\mu Q_\omega$.*

PROOF. The family $\{R_\omega : \omega \in \Omega\}$ comprises strongly nonexpansive mappings (see Lemmata 2.1 and 1.2 in [**11**]). By Proposition 5.2, R_μ also is a strongly nonexpansive mapping. Using Proposition 2.4 in [**11**], we see that any sequence defined by (5.1) converges weakly to a fixed point of R_μ. We conclude the proof by appealing to Proposition 5.1. □

This theorem extends a result of [**4**] from Hilbert space to Banach spaces. If R_μ is fixed point free, then clearly $\bigcap_\mu Q_\omega = \emptyset$. Thus we are in the inconsistent case and $|x_n| \to \infty$ as $n \to \infty$ by [**3**, Corollary 1.4].

Our next result is the following strong convergence theorem.

THEOREM 5.2. *Let X be a separable Banach space and let $\{R_\omega : \omega \in \Omega\}$ be a family of sunny nonexpansive retractions. If the sets $\{Q_\omega : \omega \in \Omega\}$ are symmetric (with respect to the origin), and X is smooth and uniformly convex, then any sequence defined by (5.1) converges strongly to a point in $\bigcap_\mu Q_\omega$.*

In order to prove this theorem, we need the following three facts.

LEMMA 5.1. [**11**] *Let C be a sunny nonexpansive retract of a smooth Banach space X. If C is symmetric with respect to the origin, then the unique sunny nonexpansive retraction $R : X \mapsto C$ is odd.*

PROPOSITION 5.3. [**3**] *If $T : D \mapsto D$ is a strongly nonexpansive mapping, then for all $k \geq 1$, $\lim_{n\to\infty} |T^{n+1}x - T^n x| = \lim_{n\to\infty} |T^{n+k}x - T^n x|/k = \lim_{n\to\infty} |T^n x/n|$.*

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

PROPOSITION 5.4. [**1**] *Let D be a closed symmetric subset of a uniformly convex Banach space, $T : D \mapsto D$ a nonexpansive mapping, and x a point in D. If T is odd and $\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$, then $\{T^n x\}$ converges strongly as $n \to \infty$ to a fixed point of T.*

Now we are ready to give the proof of Theorem 5.2.

PROOF. Since each R_ω is strongly nonexpansive (Lemmata 2.1 and 1.2 in [**11**]), R_μ also is a strongly nonexpansive mapping (Proposition 5.2). Since each R_ω is odd (Lemma 5.1), so is R_μ, and the origin belongs to $F(R_\mu)$ (the fixed point set of R_μ). Therefore $\{R_\mu^n x\}$ is bounded and $\lim_{n\to\infty}(R_\mu^{n+1}x - R_\mu^n x) = 0$ (Proposition 5.3). Finally, the strong limit of the sequence defined by (5.1) exists by Proposition 5.4 and belongs to $\bigcap_\mu Q_\omega$ (Proposition 5.1). □

In particular, this theorem adds another case of strong convergence in Hilbert space to those mentioned in [**5**].

Our last two results concern the relaxed expected projection operator and the RERM. The first one is a weak convergence theorem and the second provides strong convergence.

THEOREM 5.3. *Let X be a separable and uniformly convex Banach space that has property (S) with constant b and a Fréchet differentiable norm (off the origin). Assume that the mappings $\{R_\omega : \omega \in \Omega\}$ are sunny nonexpansive retractions from X onto the closed and convex subsets $\{Q_\omega : \omega \in \Omega\}$ of X and that $\bigcap_\mu Q_\omega \neq \emptyset$. If T_μ^λ is the relaxed expected projection operator with relaxation parameter $0 < \lambda < 1 + b$, then the sequence defined by*

$$x_{n+1} = T_\mu^\lambda x_n,\ x_0 \in X, n = 0, 1, 2, \ldots, \tag{5.3}$$

converges weakly to a point in $\bigcap_\mu Q_\omega$.

PROOF. Each R_ω is firmly nonexpansive [**11**] and hence $(1 + b)R_\omega - bI$ is nonexpansive. It follows that $T_\omega^\lambda = (1 - \lambda)I + \lambda R_\omega$ is an averaged mapping for $0 < \lambda < 1+b$ [**12**, p. 46]. Thus each T_ω^λ is strongly nonexpansive [**3**, Proposition 1.3] and so is T_μ^λ (by the proof of Proposition 5.2). The result now follows from the nonlinear mean ergodic theorem [**2, 10**], the proof of Proposition 2.4 in [**11**], and Proposition 5.1. □

THEOREM 5.4. *Let X be a separable, smooth and uniformly convex Banach space that has property (S) with constant b. Assume that the mappings $\{R_\omega : \omega \in \Omega\}$ are sunny nonexpansive retractions from X onto the closed, convex and symmetric subsets $\{Q_\omega : \omega \in \Omega\}$ of X. Let T_μ^λ be the relaxed expected projection operator with relaxation parameter $0 < \lambda < 1 + b$. Then the sequence defined by (5.3) converges strongly to a point in $\bigcap_\mu Q_\omega$.*

PROOF. We already know that the relaxed expected projection operator T_μ^λ is strongly nonexpansive. Since it also is odd, the result follows from Propositions 5.3 and 5.4. □

References

[1] J.-B. Baillon, R. E. Bruck and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1–9. MR0473932 (57:13590)

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms

[2] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107–116. MR531254 (80j:47066)

[3] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470. MR0470761 (57:10507)

[4] D. Butnariu, *The expected-projection method: its behavior and applications to linear operator equations and convex optimization*, J. Applied Analysis **1** (1995), 95–108. MR1390423 (97b:65071)

[5] D. Butnariu and S. D. Flåm, *Strong convergence of expected projection methods in Hilbert spaces*, Numer. Func. Anal. Optim. **16** (1995), 601–636. MR1341102 (97h:65084)

[6] D. Butnariu, A. N. Iusem and R. S. Burachik, *Iterative methods of solving stochastic convex feasibility problems and applications*, Comput. Optim. Appl. **15** (2000), 269–307. MR1752487 (2001b:90050)

[7] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984. MR744194 (86d:58012)

[8] L. A. Karlovitz, *The construction and application of contractive retractions in two-dimensional normed linear spaces*, Indiana Univ. Math. J. **22** (1972), 473–481. MR0308755 (46:7869)

[9] E. Kopecká and S. Reich, *Nonexpansive retracts in Banach spaces*, Banach Center Publications **77** (2007), 161–174. MR2338582 (2008j:46008)

[10] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276. MR528688 (80d:47090)

[11] S. Reich, *A limit theorem for projections*, Linear and Multilinear Algebra **13** (1983), 281–290. MR700890 (84i:47070)

[12] M. Souliman-Gabour, *Parallel Algorithms for Convex Feasibility Problems*, M. Sc. Thesis, Technion, Haifa, 1997 (in Hebrew).

Department of Mathematics, The Technion - Israel Institute of Mathematics, 32000 Haifa, Israel

E-mail address: manalj@tx.technion.ac.il

Department of Mathematics, The Technion - Israel Institute of Mathematics, 32000 Haifa, Israel

E-mail address: sreich@tx.technion.ac.il

Licensed to AMS.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/publications/ebooks/terms