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Sections of Analytic Variety

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Abstract. We present a study of sections of a germ of analytic variety V through relations of equivalences defined by geometric subgroups, introduced by J. Damon, that in some way preserve V and we show that under certain conditions these equivalences relate. We also obtain a classification under $\mathcal{R}_V$-equivalence of sections of S_k-singularities.

1. Introduction

Let $(V, 0)$ be the germ of an analytic subvariety of K^n ($K = \mathbb{R}$ or $\mathbb{C}$). The study of equivalence of the germ $f : (K^n, 0) \to (K^p, 0)$ by requiring that the germs of diffeomorphisms on K^n preserve V has been done through groups $\mathcal{R}_V$, $\mathcal{K}_{RV}$, $\mathcal{K}_V$ and $\mathcal{A}_V$ which are geometric subgroups of the classical groups of singularity theory $\mathcal{R}$, $\mathcal{K}$, $\mathcal{A}$ (see [**29**]). This study began in [**1**] with V the discriminant of a versal unfolding and has been followed by [**27**], [**31**], [**4**], [**5**], [**6**], [**11**], etc. Results for unfolding, versality, determinacy theorems and topological triviality for these subgroups were presented in [**10**], [**11**] and [**12**]. A specific study for $\mathcal{K}_V$ was given in [**8**] and [**9**], for $\mathcal{R}_V$ in [**5**]. Results about $\mathcal{R}_V$-topological triviality of families of function germs were given in [**28**] and [**29**].

In this paper we present the equivalence relations defined by these subgroups and show that under certain conditions these equivalences are related. For groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ we give algebraic and geometric conditions necessary and sufficient for finite determinacy and we obtain a classification of sections of S_k-singularities (defined in [**22**]) given by submersions under $\mathcal{R}_V$-equivalence.

2. Equivalence of Sections

Let $\mathcal{O}_n$ be the ring of analytic function germs $f : (K^n, 0) \to K$, with $K = \mathbb{R}$ or $\mathbb{C}$ and let m_n be its maximal ideal.

A germ of a subset $(V, 0) \subset (K^n, 0)$ is the germ of analytic variety if there exist analytic function germs $f_1, ..., f_r$ such that $V = \{x \in K^n : f_1(x) = ... = f_r(x) = 0\}$. We denote by $I(V)$ the ideal in $\mathcal{O}_n$ consisting of germs vanishing on V, that is, $I(V) = \{\phi \in \mathcal{O}_n : \phi(x) = 0 \,\forall\, x \in V\}$.

Definition 2.1. A section of V is a set C obtained by the intersection of V with the germ of a variety $(X, 0) \subset (K^n, 0)$.

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All germs which we consider will be analytic. Classically the three principal notions of equivalence for map germs are $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{K}$-equivalence. We consider the space of analytic germs $f : (K^n, 0) \to (K^p, 0)$, which we denote by $\mathcal{O}_{n,p}$. Let $\mathcal{D}_s$ denote the group of germs of diffeomorphisms $\varphi : (K^s, 0) \to (K^s, 0)$. The group $\mathcal{R} = \mathcal{D}_n$ acts on $\mathcal{O}_{n,p}$ by $\varphi.f = f \circ \varphi^{-1}$. The group $\mathcal{A} = \mathcal{D}_n \times \mathcal{D}_p$ acts on $\mathcal{O}_{n,p}$ by $(\varphi, \phi).f = \phi \circ f \circ \varphi^{-1}$. The group $\mathcal{K}$, contact equivalence, consists of $H \in \mathcal{D}_{n+p}$ such that there is a $\varphi \in \mathcal{D}_n$ so that $H \circ i = i \circ \varphi$ and $\pi \circ H = \varphi \circ \pi$ where $i(x) = (x, 0)$ is the inclusion $i : K^n \to K^{n+p}$ and $\pi(x, y) = x$ is the projection $\pi : K^{n+p} \to K^p$, therefore $H(x, y) = (\varphi(x), \theta(x, y))$ with $\theta(x, 0) = 0$. Then $\mathcal{K}$ acts on $\mathcal{O}_{n,p}$ by $(\varphi(x), H.f(x)) = H(x, f(x))$ that is $\mathrm{graph}(H.f) = H(\mathrm{graph}(f))$. Germs are $\mathcal{R}$, $\mathcal{A}$ or $\mathcal{K}$-equivalent if they lie in common orbits of the group actions.

We can study sections of V in two ways: X is given by a parametrization $g : (K^m, 0) \to (K^n, 0)$ or as an inverse image, $X = l^{-1}(0)$, $l : (K^n, 0) \to (K^p, 0)$. We describe below these different approaches.

In the first case, i.e., taking the section of V as an image of a map germ $g : (K^m, 0) \to (K^n, 0)$, $m < n$, the important equivalence relations are:

Definition 2.2. Two germs g and $g' : (K^m, 0) \to (K^n, 0)$ are:

(i) $\mathcal{A}_V$-equivalent if there exist germs $\varphi \in \mathcal{D}_m$ and $\phi \in \mathcal{D}_n$ with $g \circ \varphi = \phi \circ g'$ and $\phi(V) = V$.

(ii) $\mathcal{K}_V$-equivalent if there exists $H = (\varphi, \theta) \in \mathcal{K}$ such that $H(x, g(x)) = (\varphi(x), g'(\varphi(x)))$ and $H(K^m \times V) \subseteq K^m \times V$. That is,

$$\mathcal{K}_V = \{H = (\varphi, \theta) \in \mathcal{K} \ : \ \theta(K^m \times V) \subseteq V\}.$$

This subgroup of $\mathcal{K}$ can be seen as the semi-direct product of $\mathcal{R}$ and $\mathcal{C}_V$ where

$$\mathcal{C}_V = \{H = (id, \theta) \in \mathcal{K} : \theta(K^n \times V) \subseteq V\}$$

and id is the germ at 0 of the identity mapping on K^m.

The $\mathcal{K}$-equivalence is the one which shows the equivalence of the germs of varieties $g^{-1}(0)$, then the $\mathcal{K}_V$-equivalence shows the equivalence of the germs of varieties $g^{-1}(V)$. The group $\mathcal{K}_V$ is a geometric subgroup of $\mathcal{K}$ which satisfies the basic theorems of theory of singularities, see [**8**]. Relations between $\mathcal{A}$ and $\mathcal{K}_V$ are given in [**9**].

When $X = l^{-1}(0)$ for a germ $l : (K^n, 0) \to (K^p, 0)$, we consider the following relations of equivalence:

Definition 2.3. Two germs l, $l' : (K^n, 0) \to (K^p, 0)$ are:

(i) $\mathcal{R}_V$-equivalent if there exists $\phi \in \mathcal{D}_n$ with $\phi(V) = V$ and $l \circ \phi = l'$. That is,

$$\mathcal{R}_V = \{\phi \in \mathcal{R} : \phi(V) = V\}$$

(ii) $\mathcal{K}_{RV}$-equivalent if there exists $H = (\varphi, \theta) \in \mathcal{K}$ with $\varphi(V) = V$ and $H(x, l(x)) = (\phi(x), l'(\phi(x)))$. That is,

$$\mathcal{K}_{RV} = \{H = (\varphi, \theta) \in \mathcal{K} \ : \ \varphi(V) = V\}$$

When $p = 1$, the group $\mathcal{R}_V$ has been studied by some authors, such as [**5**], [**13**] and [**26**].

The group $\mathcal{K}_{RV}$ can be seen as semi-direct of product $\mathcal{R}_V$ and $\mathcal{C}$. Some results on this group were obtained by R. Atique in [**2**].

Definition 2.4. Let $f : (K^n, 0) \to (K^p, 0)$ be an analytic map germ, we denote by I_f the ideal generated by components $f_1, ..., f_p$ of f. A germ $l : (K^p, 0) \to (K^s, 0)$

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is a reduced defining equation for f if the ideal of the functions which vanish on $(f(K^n), 0)$ is generated by I_l, i.e., $I(f(K^n)) = \langle l_1, ..., l_s \rangle$ as an $\mathcal{O}_p$-module.

In [**18**], it's shown that f and $g : (K^n, 0) \to (K^p, 0)$ are $\mathcal{K}$-equivalent if only if there exists $h \in \mathcal{D}_n$ taking I_f to I_g, that is $h^*(I_f) = I_g$, where $h^* : \mathcal{O}_n \to \mathcal{O}_n$ is the ring isomorphism induced by h. It follows that if f and g are $\mathcal{K}$-equivalent then there is diffeomorphism $h : (K^n, 0) \to (K^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$. We remark that the converse is not true, for example, consider $f(x) = x$ and $g(x) = x^2$. However, we can show the following.

LEMMA 2.5. *Let f, $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with I_f and I_g radical ideals. If there exists $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$, then f and g are $\mathcal{K}$-equivalent.*

PROOF. It's enough to show that $h^*(I_g) = I_f$. Let $l \in I_g$ then $l(z) = 0 \, \forall \, z \in g^{-1}(0)$. Taking $x \in f^{-1}(0)$, by hypothesis it follows that $h(x) \in g^{-1}(0)$. Hence,

$$h^*(l)(x) = (l \circ h)(x) = l(h(x)) = 0 \Rightarrow l \circ h \in I(V(I_f)) = \sqrt{I_f} = I_f$$

where the penultimate equality follows from the Hilbert's Nullstellensatz Theorem for germs of analytic functions (see [**21**]), therefore $h^*(I_g) \subseteq I_f$. As h^* is an isomorphism, we can show similarly that $(h^*)^{-1}(I_f) \subseteq I_g$, thus $I_f \subseteq h^*(I_g)$. Therefore $h^*(I_g) = I_f$. □

The following result connects $\mathcal{A}_V$ classification of section to $\mathcal{K}_{RV}$ classification of reduced defining equation of sections.

THEOREM 2.6. *Given g, g' : $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, let l, l' : $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be defining equation for g, g' respectively.*
(a) *If g, g' are $\mathcal{A}_V$- equivalent then l, l' are $\mathcal{K}_{RV}$-equivalent.*
(b) *If g, g' are $\mathcal{A}$-finitely determined with $n > 2$ and l, l' are $\mathcal{K}_{RV}$-equivalent then g, g' are $\mathcal{A}_V$-equivalent.*

PROOF. (a) If g, g' are $\mathcal{A}_V$-equivalent then there exists $\phi \in \mathcal{D}_m$ and $\varphi \in \mathcal{D}_n$ such that $g \circ \phi = \varphi \circ g'$ and $\varphi(V) = V$. Hence, $\varphi(\mathrm{Im}(g')) = \mathrm{Im}(g)$,

$$(l \circ \varphi)^{-1}(0) = \varphi^{-1}(l^{-1}(0)) \supseteq \varphi^{-1}(\mathrm{Im}(g)) = \mathrm{Im}(g'), \text{then}$$

$$I((l \circ \varphi)^{-1}(0)) \subseteq I_{l'} \Rightarrow (l \circ \varphi)^{-1}(0) \supseteq l'^{-1}(0)$$

we have

$$(l' \circ \varphi^{-1})^{-1}(0) = \varphi(l'^{-1}(0)) \supseteq \varphi(\mathrm{Im}(g')) = \mathrm{Im}(g), \text{ then}$$

$$I((l' \circ \varphi^{-1})^{-1}(0)) \subseteq I_l \Rightarrow \varphi(l'^{-1}(0)) \supseteq l^{-1}(0) \Rightarrow l'^{-1}(0) \supseteq (l \circ \varphi)^{-1}(0)$$

Therefore

$$(l \circ \varphi)^{-1}(0) = l'^{-1}(0)$$

as I_l and $I_{l'}$ are radical ideals it follows from Lemma 2.5 that l, l' are $\mathcal{K}_{RV}$-equivalent, since $\varphi(V) = V$.

(b) If l, l' are $\mathcal{K}_{RV}$ equivalent then there exists $h \in \mathcal{D}_n$, $h(V) = V$ such that $(l \circ h)^{-1}(0) = l'^{-1}(0)$. Thus, $(h^{-1} \circ g)(\mathbb{C}^m) = g'(\mathbb{C}^m)$, and by uniqueness of the normalisation (see [**16**]) it follows that there exists $\alpha \in \mathcal{D}_m$ such that $h^{-1} \circ g = g' \circ \alpha$, that is, g, g' are $\mathcal{A}_V$-equivalent. □

With the hypothesis added that g, g' are germs of immersion we can connect the $\mathcal{A}_V$-equivalence to the $\mathcal{K}_V$-equivalence.

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THEOREM 2.7. *Let g, $g' : (K^m, 0) \to (K^n, 0)$ be germs of immersion. Then g and g' are $\mathcal{A}_V$-equivalent if only if they are $\mathcal{K}_V$-equivalent.*

PROOF. We can suppose that $g(x) = (x, \varphi(x))$ and $g'(x) = (\phi_1(x), \phi_2(x))$ (changing the coordinate only in the source). Supposing g, g' are $\mathcal{K}_V$-equivalent then there exists $H \in \mathcal{D}_{m+n}$ given by $H(x, y) = (h(x), \theta(x, y))$ with $\theta(x, 0) = 0$, $\theta(x, V) \subseteq V$ and

$$H(x, x, \varphi(x)) = (h(x), \theta(x, x, \varphi(x))) = (h(x), \phi_1(h(x)), \phi_2(h(x)))$$

Let $\tilde{\theta} : (K^n, 0) \to (K^n, 0)$ with $\tilde{\theta}(y) = \theta(y_1, ..., y_m, y)$. Then

$$\tilde{\theta} \circ g(x) = \tilde{\theta}(x, \phi(x)) = \theta(x, x, \phi(x)) = (\phi_1(h(x)), \phi_2(h(x))) = g' \circ h(x)$$

and $\tilde{\theta}(V) \subseteq V$, $\tilde{\theta} \in \mathcal{D}_n$, since $H \in \mathcal{D}_n$. Therefore g, g' are $\mathcal{A}_V$-equivalent.

Since $\mathcal{A}_V$ is a subgroup of $\mathcal{K}_V$, it is immediate that if g and g' are $\mathcal{A}_V$-equivalent then they are $\mathcal{K}_V$-equivalent. □

The following example shows that if g or g' is not an immersion then Proposition 2.7 can be false.

EXAMPLE 2.8. Define $g, g' : (K, 0) \to (K^2, 0)$ by $g(t) = (t^2, 0)$, $g'(t) = (t^2, t^3)$ and $V = \{(0, y), y \in K\}$. Taking $H(t, x, y) = (t, x, y - tx)$, $H(t, V) \subseteq K \times V$ and $H(t, t^2, t^3) = (t, t^2, 0)$. Therefore g, g' are $\mathcal{K}_V$-equivalent, but g, g' are not $\mathcal{A}_V$-equivalent, since g, g' are not $\mathcal{A}$-equivalent, since g' is 3-$\mathcal{A}$-determined and g is not $\mathcal{A}$-finitely determined.

LEMMA 2.9. *If $I(V) = \langle \phi_i \rangle_{i=1,...,r}$ as an $\mathcal{O}_n$-module then $I(K^m \times V) = \langle \phi_i \rangle_{i=1,...,r}$ as an $\mathcal{O}_{m+n}$-module.*

PROOF. $P \in I(K^m \times V)$ if and only if $P(x, y) = 0$, $\forall\, (x, y) \in K^m \times V$.

By Hadamard's Lemma (see [**18**]), $P(x, y) - P(0, y) = \sum_{i=1}^m q_i(x, y)x_i$ where

$$q_i(x, y) = \int_0^1 \frac{\partial P}{\partial x_i}(tx_1, ..., tx_m, y_1, ..., y_n)dt$$

since $P|_{K^m \times \{y\}} \equiv 0$, we have that $q_i|_{K^m \times \{y\}} \equiv 0\ \forall\, y \in V$.

Therefore, $q_i(K^m \times V) \equiv 0$, i.e., $q_i \in I(K^m \times V)$. From the hypothesis, $P(0, y) = \sum_{i=1}^r h_i(y)\phi_i(y)$, with $h_i \in \mathcal{O}_n$, then

$$P(x, y) = \sum_{i=1}^r h_i(y)\phi_i(y) + \sum_{i=1}^m q_i(x, y)x_i,$$

therefore $I(K^m \times V) \subseteq I(V)_{\mathcal{O}_n} + I(K^m \times V)\langle x_1, ..., x_m \rangle$, hence

$$I(K^m \times V) \subseteq I(V)_{\mathcal{O}_{m+n}} + I(K^m \times V)\langle x_1, ..., x_m, y_1, ..., y_n \rangle .$$

By Nakayama Lemma (see [**18**]), we have

$$I(K^m \times V) \subseteq I(V)_{\mathcal{O}_{m+n}} = \langle \phi_i \rangle_{i=1,...,r} \text{ as an } \mathcal{O}_{m+n}\text{-module.}$$

As the other inclusion is immediate, the result follows. □

THEOREM 2.10. *Two germs g, $g' : (K^m, 0) \to (K^n, 0)$ are $\mathcal{K}_V$-equivalent if only if there exists $h \in \mathcal{D}_m$ such that $g' \circ h$, g are $\mathcal{C}_V$-equivalent.*

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PROOF. If g, g' are $\mathcal{K}_V$-equivalent then there exist germs of diffeomorphism H and h with $H(x,y) = (h(x), \theta(x,y))$, $\theta(x,0) = 0$, $\theta(x,V) \subseteq V$ $\forall x \in K^m$ and $H \circ (id, g) = (h, g' \circ h)$. Let $H' = (h^{-1}, \pi_2)$, where $\pi_2(x,y) = y$ thus, $H' \circ H(id, g) = (id, \theta(id, g)) = (id, g' \circ h)$, that is, g and $g' \circ h$ are $\mathcal{C}_V$-equivalent.

If g, $g' \circ h$ are $\mathcal{C}_V$-equivalent then there exist germs of diffeomorphism H, with $H(x,y) = (x, \theta(x,y))$, $\theta(x,V) \subseteq V$ $\forall x \in K^m$ and $H(id, g) = (id, g' \circ h)$. Let $H'(x,y) = (h(x), y)$, thus $H' \circ H(x,y) = (h(x), \theta(x,y))$. Therefore $H' \circ H(id, g) = (h, \theta(id, g)) = (h, g' \circ h)$, that is, g and g' are $\mathcal{K}_V$-equivalent. □

THEOREM 2.11. *Let g, $g' : (K^m, 0) \to (K^n, 0)$ and $I(V)$ be the ideal generated by $\phi_1, ..., \phi_r$ as an $\mathcal{O}_n$-module, then:*
(a) *If g and g' are $\mathcal{C}_V$-equivalent then $\langle \phi_i \circ g \rangle_{i=1,...,r} = \langle \phi_i \circ g' \rangle_{i=1,...,r}$.*
(b) *If g and g' are $\mathcal{K}_V$-equivalent then $\langle \phi_i \circ g \rangle_{i=1,...,r}$ and $\langle \phi_i \circ g' \rangle_{i=1,...,r}$ are isomorphic. Therefore the germs $F = (\phi_1 \circ g, ..., \phi_r \circ g)$ and $G = (\phi_1 \circ g', ..., \phi_r \circ g')$ are $\mathcal{K}$-equivalent.*

PROOF. (a) It is enough to show that $\langle \phi_i \circ g' \rangle \subseteq \langle \phi_i \circ g \rangle$, since the other inclusion is shown analogously. We show that each $\phi_i \circ g'$ can be written $\phi_i \circ g' = \sum_{j=1}^r a_{ij}(\phi_j \circ g)$ with $a_{ij} \in \mathcal{O}_{m+n}$, as g and g' are $\mathcal{C}_V$-equivalent, there exists a germ of diffeomorphism H such that $H(x,y) = (x, \theta(x,y))$, $\theta(x,0) = 0$, $\theta(x,V) \subseteq V$ and $H(x, g(x)) = (x, \theta(x, g(x))) = (x, g'(x))$ as $\theta(x,V) \subseteq V \Rightarrow \phi_i(\theta(x,V)) \equiv 0$ $\Rightarrow \phi_i \circ \theta \in I(K^m \times V)$.

By Lemma 2.9, we can write each $\phi_i \circ \theta$ as

$$\phi_i \circ \theta(x,y) = \sum_{j=1}^r a_{ij}(x,y)\phi_j(y) \text{ with } a_{ij} \in \mathcal{O}_{m+n}$$

therefore

$$\phi_i \circ \theta(x, g(x)) = \sum_{j=1}^r a_{ij}(x, g(x))\phi_j(g(x)),$$

then $\phi_i \circ \theta(x, g(x)) = \phi_i \circ g'(x)$. It follows that $\phi_i \circ g' = \sum_{j=1}^r a_{ij}(\phi_j \circ g)$.

(b) If g and g' are $\mathcal{K}_V$-equivalent then there exists a diffeomorphism h of K^m such that $g' \circ h$ and g are $\mathcal{C}_V$-equivalent, by part (a), $\langle \phi_i \circ g \rangle = \langle \phi_i \circ g' \circ h \rangle$. Thus the isomorphism $h^* : \mathcal{O}_m \to \mathcal{O}_m$ takes $\langle \phi_i \circ g' \rangle$ to $\langle \phi_i \circ g \rangle$. Therefore these ideals are isomorphic. □

The following example shows that the condition $\langle \phi_i \circ g' \rangle$ isomorphic $\langle \phi_i \circ g \rangle$ does not imply that g and g' are $\mathcal{K}_V$-equivalent.

EXAMPLE 2.12. Let $V = \{(x,0), x \in K\} \subseteq K^2$, $I(V) = \langle y \rangle_{\mathcal{O}_2} = \langle \phi \rangle_{\mathcal{O}_2}$. We take g, $g' : (K,0) \to (K^2, 0)$ with $g(t) = (t, t^2)$, $g'(t) = (0, t^2)$, so $(\phi \circ g)(t) = t^2 = (\phi \circ g')(t)$, therefore $\langle \phi \circ g \rangle = \langle \phi \circ g' \rangle$. But g and g' are not $\mathcal{K}_V$-equivalent, since they are not $\mathcal{K}$-equivalent.

However, the following result is true.

THEOREM 2.13 ([**14**]). *Let $V = \phi^{-1}(0)$ where $\phi : (K^n, 0) \to (K, 0)$ be a submersion, let g, $g' : (K^m, 0) \to (K^n, 0)$ be embedding and $h : (K^n, 0) \to (K^n, 0)$ be a diffeomorphism such that $h^*(\langle \phi \circ g' \rangle) = \langle \phi \circ g \rangle$, then g and g' are $\mathcal{K}_V$-equivalent.*

THEOREM 2.14. *Let $(V, 0) \subset (K^n, 0)$. Two germs l, $l' : (K^n, 0) \to (K^r, 0)$ are $\mathcal{K}_{RV}$-equivalent if and only if there exists germ of diffeomorphism $\phi : (K^n, 0) \to (K^n, 0)$ with $\phi(V) = V$ such that $l' \circ \phi$ and l are $\mathcal{C}$-equivalent.*

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PROOF. If l and l' are $\mathcal{K}_{RV}$-equivalent, then there exist germs of diffeomorphism H and ϕ with $H(x,y) = (\phi(x), \theta(x,y))$, $\theta(x,0) = 0$, $\phi(V) = V$ and $H \circ (id, l) = (\phi, l' \circ \phi)$.

Let $H' = (\phi^{-1}, \pi_2)$, where $\pi_2(x,y) = y$. Then $H' \circ H(id, l) = (id, \theta(id, l)) = (id, l' \circ \phi)$, that is, l and $l' \circ \phi$ are $\mathcal{C}$-equivalent and $\phi(V) = V$.

Conversely, if l and $l' \circ \phi$ are $\mathcal{C}$-equivalent and $\phi(V) = V$, then there exists germ of diffeomorphism H with $H(x,y) = (x, \theta(x,y)), H(id, l) = (id, l' \circ \phi)$.

Let $H'(x,y) = (\phi(x), y)$. We have that $H' \circ H(x,y) = (\phi(x), \theta(x,y))$. Therefore $H' \circ H(id, l) = (\phi, \theta(id, l)) = (\phi, l' \circ \phi)$, then l, l' are $\mathcal{K}_{RV}$-equivalent. □

In this work, given an analytic variety germ $(V, 0) \subset (K^n, 0)$, we will always define a section of V through an equation $h = 0$, with $h : (K^n, 0) \to (K^p, 0)$. Hence, our main interest is action of the groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ on the set $\mathcal{O}(n,p)$ of the germs of analytic maps $h : (K^n, 0) \to (K^p, 0)$.

As an application of the equivalence studied, we will see in the following proposition that the $\mathcal{A}$ classification of bigerms reduces to $\mathcal{A}_V$ or $\mathcal{K}_{RV}$ classification of sections. This result extends for any dimension of the result of R. Atique [**2**].

A bigerm $G : (K^m, S) \to (K^n, 0)$, $S = \{x_1, x_2\}$ and $x_i \in K^m$, $i = 1, 2$ is the equivalence classes of the maps that coincide in the neighborhood of S. Without loss of generality we can consider $x_1 = x_2 = 0$ and the germ G as

$$G : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g : (K^m, 0) \to (K^n, 0) \end{cases}$$

(The equivalence of bigerm is defined by different diffeomorphisms in source and a unique diffeomorphism in the target, see [**2**]).

THEOREM 2.15. *Consider the bigerms*

$$G : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g : (K^m, 0) \to (K^n, 0) \end{cases} \qquad G' : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g' : (K^m, 0) \to (K^n, 0) \end{cases}$$

with $n > 2$ and f, g, g' $\mathcal{A}$-finitely determined. Let l and l' be the defining equation for $g(K^m)$ and $g'(K^m)$ respectively and let V be the image of f. The following statements are equivalent
(i) *G, G' are $\mathcal{A}$-equivalent,*
(ii) *g, g' are $\mathcal{A}_V$-equivalent,*
(iii) *l, l' are $\mathcal{K}_{RV}$-equivalent.*

PROOF. If G, G' are $\mathcal{A}$-equivalent then there exist diffeomorphisms ϕ_f, ϕ_g : $(K^m, 0) \to (K^m, 0)$ and $\varphi : (K^n, 0) \to (K^n, 0)$ such that $f \circ \phi_f = \varphi \circ f$ and $g \circ \phi_g = \varphi \circ g'$.

Hence $\varphi(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f)$. It follows that $\varphi(V) \subseteq V$, then g, g' are $\mathcal{A}_V$-equivalent.

Assuming that g, g' are $\mathcal{A}_V$- equivalent, there exist diffeomorphisms ϕ_g : $(K^m, 0) \to (K^m, 0)$ and $\varphi : (K^n, 0) \to (K^n, 0)$ such that $g \circ \phi_g = \varphi \circ g'$ and $\varphi(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f)$. Therefore $(\varphi \circ f)(K^n)) \subseteq f(K^n)$, by uniqueness of normalisation (see [**16**], [**17**]), there exists ϕ such that $f \circ \phi = \varphi \circ f$, hence G, G' are $\mathcal{A}$-equivalent. It follows from Proposition 2.6 that (ii) is equivalent to (iii). □

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3. Finite Determinacy

Results on finite determinacy of groups ($\mathcal{G}_V = \mathcal{R}_V$ or $\mathcal{K}_{RV}$) have been obtained by some authors. The group $\mathcal{R}_V$ was studied in [**5**], [**13**] and [**26**]. There are not many results about group $\mathcal{K}_{RV}$ but there are some results in [**2**].

Our interest is in groups of diffeomorphisms that preserve an analytic variety V. To get these diffeomorphisms the basic technique is the integration of vector fields tangent to V. The main references for studying these vector fields are [**5**], [**8**] and [**30**].

Definition 3.1. ([**5**]) Let $(V,0) \subset (K^n,0)$ be the germ of an analytic variety and let ξ be a germ of analytic vector field on $(K^n,0)$. Then ξ is said to be *logarithmic* for $(V,0)$ if, when considered as a derivation $\xi : \mathcal{O}_n \to \mathcal{O}_n$, $g \mapsto \xi g$, we have $\xi g \in I(V)$ for all $g \in I(V)$, where $\xi g = dg(\xi)$. The $\mathcal{O}_n$- module of such vector fields is denoted by Θ_V or $Derlog(V)$, that is,

$$\Theta_V = \{\xi \in \mathcal{O}_n^n \ : \ \xi g \in I(V), \forall\, g \in I(V)\}$$

The main properties of Θ_V are given by the following proposition.

Theorem 3.2 ([**5**]) (i)). *Θ_V is the set of the vector field germs at $0 \in K^n$ which are tangent to the set of regular points of V.*
(ii) *Θ_V is finitely generated as an $\mathcal{O}_n$-module.*
(iii) *If $\xi \in \Theta_V$ and $\xi(0) = 0$, then the flow ϕ_t generated by ξ preserves $(V,0)$. Thus $\phi_t \in \mathcal{R}_V$ for all t.*

Definition 3.3. A germ $f : (K^n,0) \to (K^p,0)$ is k-$\mathcal{G}_V$-finitely determined if all germs g, such that $j^k g(0) = j^k f(0)$ is $\mathcal{G}_V$-equivalent to f. If f is k-$\mathcal{G}_V$-finitely determined for some k, we say that f is $\mathcal{G}_V$-finitely determined.

Definition 3.4. Let $f : (K^n,0) \to (K^p,0)$. We denote by Θ_V^0 the subset of Θ_V constituted by vector fields that vanish at origin. The tangent spaces and the extended tangent spaces of the groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ are respectively:

$$\begin{array}{ll} T\mathcal{R}_V(f) = df(\Theta_V^0), & T\mathcal{R}_{V,e}(f) = df(\Theta_V) \\ T\mathcal{K}_{RV}(f) = df(\Theta_V^0) + f^*(m_p)\mathcal{O}_n^p, & T\mathcal{K}_{RV,e}(f) = df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p \end{array}$$

The next proposition gives a necessary condition to $\mathcal{G}_V$-finite determinacy.

Theorem 3.5. *If $f : (K^n,0) \to (K^p,0)$ is $\mathcal{G}_V$-finitely determined, then*

$$T\mathcal{G}_V(f) \supset m_n^l \mathcal{O}_n^p,$$

for some positive integer l.

For groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$, the essential part of proof, as in groups $\mathcal{K}$, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{R}$, is to note that $J^k\mathcal{G}_V$, the set of k-jets of elements $\mathcal{G}_V$, is a Lie group acting smoothly on $J^k(n,p)$. R. Pellikaan in [**25**] shows that $J^k\mathcal{R}_V$ is a Lie group and as $\mathcal{K}_{RV} = \mathcal{R}_V.\mathcal{C}$ (semi-direct product) it follows that $J^k\mathcal{K}_{RV}$ is Lie group.

The theorems on infinitesimal criterion for $\mathcal{G}_V$-finite determinacy are known, see for instance [**2**], [**6**], [**5**].

Theorem 3.6. *If $T\mathcal{G}_V(f) \supseteq m_n^k \mathcal{O}_n^p$ then f is k-$\mathcal{G}_V$-determined, $\mathcal{G}_V = \mathcal{R}_V$, $\mathcal{K}_{RV}$.*

The following corollary is useful for the classification of germs, since it reduces the estimate of tangent space in the finite dimension case.

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COROLLARY 3.7. *If $T\mathcal{G}_V(f) + m_n^{k+1}\mathcal{O}_n^p \supseteq m_n^k\mathcal{O}_n^p$ then f is k-$\mathcal{G}_V$-determined.*

PROOF. This follows Nakayama Lemma, since $T\mathcal{G}_V(f)$ is an $\mathcal{O}_n$-module, for $\mathcal{G}_V = \mathcal{R}_V$, $\mathcal{K}_{RV}$. □

DEFINITION 3.8. (a) We denote by Θ_V^1 the subset of Θ_V constituted by vector fields whose 1-jets are null. If $f : (K^n, 0) \to (K^p, 0)$ we denote

$$\begin{aligned} T\mathcal{R}_V^1(f) &= df(\Theta_V^1) \\ T\mathcal{K}_{RV}^1(f) &= df(\Theta_V^1) + f^*(m_p)m_n\mathcal{O}_n^p. \end{aligned}$$

(b) We denote by $H^k(n,p)$ the vector space of all the maps $K^n \to K^p$, where the components are homogeneous polynomials of degree k.
(c) The standard $k-$jet space $m_n\mathcal{O}_n/m_n^{k+1}\mathcal{O}_n$ is denoted $J^k(n,1)$.

An efficient method for the classification of singularities finitely determined is the *complete transversal method* described by Bruce, Kirk and du Plessis in [**3**].

THEOREM 3.9 ([**3**]). *Let $f : (K^n, 0) \to (K^p, 0)$ be an analytic germ and let T be a vector subspace of the set $H^{k+1}(n,p)$, such that $J^{k+1}(T\mathcal{G}_V^1 f) + T \supset H^{k+1}(n,p)$. Then every $(k+1)$-jet g with $j^k g = j^k f$ lies in the same $\mathcal{G}_V^1$-orbit of some $(k+1)$-jet of the form $j^{k+1}f + t$, for some $t \in T$.*

COROLLARY 3.10. *If $T\mathcal{G}_V^1 f \supseteq m_n^{k+1}\mathcal{O}_n^p$ then f is k-$\mathcal{G}_V$-determined.*

LEMMA 3.11. *Let $f : (K^n, 0) \to (K^p, 0)$ be an analytic germ and let $\{\xi_1, ..., \xi_r\}$ be generators of Θ_V. Then f is $\mathcal{K}_{RV}$-finitely determined if and only if $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$ for some k, where $Jf(\Theta_V)$ is the ideal in $\mathcal{O}_n$ generated by all the $(p \times p)$-minors of the matrix $(\nabla f_i(\xi_j))_{i=1,\dots,p}^{j=1,\dots,r}$ (if $p > r$, then $Jf(\Theta_V) = 0$).*

PROOF. The proof is an adaptation of the result correspondent for the group $\mathcal{K}$ ([**15**], Lemma 2.12).

If f is $\mathcal{K}_{RV}$-finitely determined, then by Proposition 3.5, there exists l such that $df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p \supset m_n^l\mathcal{O}_n^p$. Let $u \in m_n^{l\,p} = m_n^k$, $u = u_1.\dots.u_p$ where $u_i \in m_n^l$. Since

$$(0, \dots, 0, u_i, 0, \dots, 0) \in df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p$$

there exists $\eta_i \in \Theta_V$ such that

$$df(\eta_i) = (0, \dots, 0, u_i, 0, \dots, 0) \mathrm{mod} f^*(m_p)\mathcal{O}_n^p$$

with $\eta_i = a_1^i\xi_1 + \ldots + a_r^i\xi_r$ and $a_j^i \in \mathcal{O}_n$.

Hence,

$$df\begin{pmatrix} a_1^1\xi_1^1 + \ldots + a_r^1\xi_r^1 & \dots & a_1^p\xi_1^1 + \ldots + a_r^p\xi_r^1 \\ a_1^1\xi_1^2 + \ldots + a_r^1\xi_r^2 & \dots & a_1^p\xi_1^2 + \ldots + a_r^p\xi_r^2 \\ \vdots & \vdots & \vdots \\ a_1^1\xi_1^n + \ldots + a_r^1\xi_r^n & \dots & a_1^p\xi_1^n + \ldots + a_r^p\xi_r^n \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

$\mathrm{mod} f^*(m_p)\mathcal{O}_n^p$. Thus,

$$df\begin{pmatrix} \xi_1^1 & \xi_2^1 & \dots & \xi_r^1 \\ \xi_1^2 & \xi_2^2 & \dots & \xi_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^n & \xi_2^n & \dots & \xi_r^n \end{pmatrix} \cdot \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^p \\ a_2^1 & a_2^2 & \dots & a_2^p \\ \vdots & \vdots & \vdots & \vdots \\ a_r^1 & a_r^2 & \dots & a_r^p \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

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$\mathrm{mod} f^*(m_p)\mathcal{O}_n^p$, otherwise,

$$\begin{pmatrix} \nabla f_1(\xi_1) & \dots & \nabla f_1(\xi_r) \\ \nabla f_2(\xi_1) & \dots & \nabla f_2(\xi_r) \\ \vdots & \vdots & \vdots \\ \nabla f_p(\xi_1) & \dots & \nabla f_p(\xi_r) \end{pmatrix} \cdot \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^p \\ a_2^1 & a_2^2 & \dots & a_2^p \\ \vdots & \vdots & \vdots & \vdots \\ a_r^1 & a_r^2 & \dots & a_r^p \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

$\mathrm{mod} f^*(m_p)\mathcal{O}_n^p$. Taking the determinant of each side of the equation, it follows that $u \in Jf(\Theta_V) + f^*(m_p)$.

Assuming now that $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$. It is enough to show that

$$Jf(\Theta_V)\mathcal{O}_n^p \subseteq df(\Theta_V).$$

Let $M = (\nabla f_i(\xi_{j_s}))_{i=1,\dots,p}^{s=1,\dots,p}$ be a $(p \times p)$ sub-matrix of the matrix $(\nabla f_i(\xi_j))_{i=1,\dots,p}^{j=1,\dots,r}$. Let $\{e_1, ..., e_p\}$ be the standard base of K^p, then $(\det M)e_k \in Jf(\Theta_V)\mathcal{O}_n^p$ and

$$(\det M)e_k = \begin{pmatrix} \nabla f_1(\xi_1) & \dots & \nabla f_1(\xi_r) \\ \nabla f_2(\xi_1) & \dots & \nabla f_2(\xi_r) \\ \vdots & \vdots & \vdots \\ \nabla f_p(\xi_1) & \dots & \nabla f_p(\xi_r) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ cof(\nabla f_k(\xi_{j_1})) \\ \vdots \\ \mathrm{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ 0 \end{pmatrix}$$

where $\mathrm{cof}(\nabla f_k(\xi_{j_s}))$ is the cofactor of $\nabla f_i(\xi_{j_s})$ in the matrix M, then

$$(\det M)e_k = df \begin{pmatrix} \xi_1^1 & \xi_2^1 & \dots & \xi_r^1 \\ \xi_1^2 & \xi_2^2 & \dots & \xi_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^n & \xi_2^n & \dots & \xi_r^n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ cof(\nabla f_k(\xi_{j_1})) \\ \vdots \\ \mathrm{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ 0 \end{pmatrix},$$

Hence

$$(\det M)e_k = df \begin{pmatrix} \xi_{j_1}^1 \mathrm{cof}(\nabla f_k(\xi_{j_1})) + \ldots + \xi_{j_p}^1 \mathrm{cof}(\nabla f_k(\xi_{j_p})) \\ \xi_{j_1}^2 \mathrm{cof}(\nabla f_k(\xi_{j_1})) + \ldots + \xi_{j_p}^2 \mathrm{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ \xi_{j_1}^n \mathrm{cof}(\nabla f_k(\xi_{j_1})) + \ldots + \xi_{j_p}^n \mathrm{cof}(\nabla f_k(\xi_{j_p})) \end{pmatrix}$$

$$= df(\mathrm{cof}(\nabla f_k(\xi_{j_1}))\xi_{j_1} + \mathrm{cof}(\nabla f_k(\xi_{j_2}))\xi_{j_2} + \ldots + \mathrm{cof}(\nabla f_k(\xi_{j_p}))\xi_{j_p})$$

therefore, $(\det M)e_k \in df(\Theta_V)$ and $Jf(\Theta_V)\mathcal{O}_n^p \subseteq df(\Theta_V)$. □

The following proposition is the geometric criterion for the $\mathcal{K}_{RV}$-finitely determinacy.

THEOREM 3.12. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and let $(V, 0) \subseteq (\mathbb{C}^n, 0)$. Then f is $\mathcal{K}_{RV}$-finitely determined if and only if $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$.*

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PROOF. By Lemma 3.11, f is $\mathcal{K}_{RV}$-finitely determined if and only if $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$ for some k. Then, $V(Jf(\Theta_V) + f^*(m_p)) \subseteq V(m_n^k)$. Hence,

$$V(Jf(\Theta_V)) \cap V(f^*(m_p)) \subseteq \{0\} \text{ and } V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}.$$

Going in the other direction, assuming that $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$ we get

$$I(V(Jf(\Theta_V) + f^*(m_p))) \supseteq I(\{0\}).$$

Thus, by *Hilbert's Nullstellensatz Theorem* for analytic function germs (see [**21**]),

$$\sqrt{Jf(\Theta_V) + f^*(m_p)} \supseteq m_n,$$

hence $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$ for some k. □

The following theorem is the geometric criterion for the $\mathcal{R}_V$-finite determinacy and its proof is a consequence of the proof of Proposition 3.12.

THEOREM 3.13. *Let $(V, 0) \subseteq (\mathbb{C}^n, 0)$ be an analytic variety germ and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ. Let $V(f) = \{x \in \mathbb{C}^n : \xi f(x) = 0$ for all $\xi \in \Theta_V\}$. Then f is $\mathcal{R}_V$-finitely determined if and only if $V(f) = \{0\}$ or $\emptyset$.*

In the real case, the necessary condition remains true, that is, if f is $\mathcal{G}_V$-finitely determined then: $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$ for $\mathcal{G}_V = \mathcal{K}_{RV}$ and the set $\{x \in \mathbb{R}^n : \xi f(x) = 0$ for all $\xi \in \Theta_V\}$ is $\{0\}$ or $\emptyset$ for $\mathcal{G}_V = \mathcal{R}_V$.

THEOREM 3.14. *Let $(V, 0) \subseteq (K^n, 0)$ be an analytic variety germ and let $f : (K^n, 0) \to (K, 0)$. If f is $\mathcal{R}_V$-finitely determined then $f^{-1}(0)$ is transverse to V away from 0.*

PROOF. By hypothesis there exists a positive integer l such that $df(\Theta_V) \supseteq m_n^l$, thus we have that $V(df(\Theta_V)) \subseteq \{0\}$. However, $V(df(\Theta_V))$ describes the set of points where $f^{-1}(0)$ is not transverse to V, thus $f^{-1}(0)$ is transverse to V away from 0. □

In the following example we show that the converse of Theorem 3.14 is not true.

EXAMPLE 3.15. Let $V \subseteq (\mathbb{C}^2, 0)$ be defined by $\varphi(x, y) = x^3 - y^2 = 0$. We have that Θ_V is generated by $\alpha_1 = (2x, 3y), \alpha_2 = (2y, 3x^2)$. Let $f(x, y) = y^2$, $f^{-1}(0) = x$-axis. It is transverse to V away from 0 but f is not $\mathcal{R}_V$-finitely determined.

Bruce and Roberts, in [**5**], introduced a generalization of Milnor number of a function germ f related to a variety V by $\mu_{BR}(f, V) = \dim \frac{\mathcal{O}_n}{df(\theta_V)}$. We will call it the Bruce-Roberts number of f with respect to V. Like the Milnor number of f, this number shows some geometric properties of f and V. For instance, if one considers the group $\mathcal{R}_V$, then f is finitely determined with respect to the action of $\mathcal{R}_V$ on $\mathcal{O}_n$ if and only if $\mu_{BR}(f, V)$ is finite and, in this case, the codimension of the orbit of f under this action is equal to $\mu_{BR}(f, V)$. We refer to [**5**] for more details.

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4. Section of the Singularities S_k

The goal of this section is to classify germs of submersion $h : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ by action of $\mathcal{R}_V$ where V is the image of S_k-singularities classified by Mond in [**22**], $S_k = \{(x, y^2, y^3 + x^{k+1}y)\}$, $k \geq 1$, in this case the sections are space curves $C_k = S_k \cap h^{-1}(0)$.

We have that $\varphi_k(x, y, z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$ is a reduced defining equation for the images of S_k (obtained using Gröbner basis methods) and the germ of the singular set of S_k is a smooth curve given by $\Sigma_k = \{(x, -x^{k+1}, 0)\}$.

Given a finite function germ $f : (C, 0) \to (\mathbb{C}, 0)$ on a complex analytic reduced space curve $(C, 0)$, in [**24**] is shown that

$$\mu(f|_C) = \mu(C) + \deg(f) - 1,$$

where $\mu(C)$ is the Milnor number of the curve, as defined by Buchweitz and Greuel [**7**], $\deg(f)$ is the degree of f (that is, the number of inverse images of a generic value), and $\mu(f|_C)$ is the Milnor number of f, introduced by Goryunov [**19**] for curves in $(\mathbb{C}^3, 0)$ and by Mond and D. van Straten [**23**] for the general case of curves in $(\mathbb{C}^n, 0)$. As Σ_k is a smooth curve it follows that $\mu(h|_{\Sigma_k}) = \deg(h|_{\Sigma_k}) - 1$.

THEOREM 4.1. *Θ_{S_k} is the $\mathcal{O}_3$-module generated by*

$$\begin{array}{ll} \eta_1 = (2x, 2(k+1)y, 3(k+1)z) & \eta_2 = (0, 2z, x^{2(k+1)} + 4x^{k+1}y + 3y^2) \\ \eta_3 = (x^{k+1} + 3y, -2(k+1)x^k y, 0) & \eta_4 = (z, 0, (k+1)(x^{2k+1}y + x^k y^2)) \end{array}$$

PROOF. If $\varphi_k = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y$ then $S_k = \varphi_k^{-1}(0)$. We have that

$$d\varphi_k = (\varphi_{k_x}, \varphi_{k_y}, \varphi_{k_z}) = (2(k+1)x^k y^2 + 2(k+1)x^{2k+1}y, 4x^{k+1}y + 3y^2 + x^{2(k+1)}, -2z),$$
$$d\varphi_k(\eta_1) = 6(k+1)\varphi_k \text{ and } d\varphi_k(\eta_2) = d\varphi_k(\eta_3) = d\varphi_k(\eta_4) = 0,$$

so η_1, η_2, η_3 and η_4 are in Θ_{S_k}.

Now if $\xi \in \Theta_{S_k}$ then $d\varphi_k(\xi) = \lambda\varphi_k$ for some $\lambda \in \mathcal{O}_3$, then

$$d\varphi_k(\xi - \frac{1}{6(k+1)}\lambda\eta_1) = d\varphi_k(\xi) - \frac{\lambda}{6(k+1)}d\varphi_k(\eta_1) = d\varphi_k(\xi) - \lambda\varphi_k = 0,$$

so we need only to check that η_1, η_2, η_3 and η_4 generate all germs of vector ξ such that $d\varphi_k(\xi) = 0$. Writing $\xi = (\xi_1, \xi_2, \xi_3)$, we have $d\varphi_k(\xi) = 0$ if and only if

$$\xi_1\varphi_{k_x} + \xi_2\varphi_{k_y} - \xi_3\varphi_{k_z} = 0,$$

if and only if (ξ_1, ξ_2, ξ_3) is a syzygy of $(\varphi_{k_x}, \varphi_{k_y}, \varphi_{k_z})$ which is generated by

$$(\varphi_{k_y}, -\varphi_{k_x}, 0) = (y + x^{k+1})\eta_1,\ (\varphi_{k_z}, 0, -\varphi_{k_x}) = -\frac{1}{2}\eta_4 \text{ and } (0, -\varphi_{k_z}, \varphi_{k_y}) = \eta_2,$$

so the result follows. □

We follow the standard classification techniques, namely to classify germs inductively at the jet level until a sufficient jet is obtained (and hence produces a finitely determined germ). For this we use the complete transversal method, Mather's Lemma and the results on finite determinacy. To calculate the Milnor number and the Bruce-Roberts number we use SINGULAR (see [**20**]).

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THEOREM 4.2. *Let V be the image of S_k, $k \geq 1$ and $h : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a germ of submersion $\mathcal{R}_V$-finitely determined with $\mu(h_{|_{\Sigma_k}}) \leq k$. Then h is $\mathcal{R}_V$-equivalent to one of the following germs*

h	$\mu_{BR}(h, S_k)$	$\mu(C_k)$	$\mu(h_{\vert_{\Sigma_k}})$
x	1	2	0
$y + a_2x^2,\ \ a_2 \neq 0$	3	5	1
$y + a_3x^3 + a_4x^4,\ \ a_3 \neq 0$	5	8	2
$y + a_4x^4 + a_5x^5 + a_6x^6,\ \ a_4 \neq 0$	7	11	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \cdots + a_{2k}x^{2k}$, $a_{k+1} \neq 0, 1$	$2k+1$	$3k+2$	k
$y + x^{k+1} + a_lx^l + \cdots + a_{l+k-1}x^{l+k-1}$, $l \geq k+2,\ a_l \neq 0$	$l+k$	$2l+k$	k
$y + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{l+k-1}x^{l+k-1}$, $l \geq k+2,\ a_l \neq 0$	$l+k$	$l+2k+1$	k
$z + a_2x^2,\ \ a_2 \neq 0$	4	6	1
$z + a_3x^3 + a_4x^4,\ \ a_3 \neq 0$	7	10	2
$z + a_4x^4 + a_5x^5 + a_6x^6,\ \ a_4 \neq 0$	10	14	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2}$, $a_l \neq 0, l \leq k+1$	$3l-2$	$4l-2$	$l-1$

In what follows we give a sketch of the proof.

With the intention of setting ideas, we start with the case $k = 1$:

Section of S_1.

We start classifying by the 1-jets. By Mather's Lemma and change of scale the orbit in $J^1(3,1)$ of the action of $J^1\mathcal{R}_V$ reduces to: x, y, z and 0.

Case (i): Classification of germs h for which the 1-jet is x.

Consider the germ $h(x,y,z) = x$. Then $T\mathcal{R}_V(h) + \mathcal{M}^2 \supset \mathcal{M}^1$ and x is 1-$\mathcal{R}_V$-determined.

Case (ii): Classification of germs h for which the 1-jet is y.

Consider the germ $h(x,y,z) = y$, by geometric criterion, y is not $\mathcal{R}_V$-finite. Then a complete transversal is given by $h = y + a_lx^l$.

(a): If $l = 2$, $a_2 \neq 0$ and $a_2 \neq 1$ then $h(x,y,z) = y + a_2x^2$ is 2-$\mathcal{R}_V$-determined.

When $a_2 = 1$, $h = y + x^2$ is not $\mathcal{R}_V$-finite, a complete transversal is $y + x^2 + a_lx^l$.

Let $h = y + x^2 + a_lx^l$ with $a_l \neq 0$, then $T\mathcal{R}_V(h) + \mathcal{M}^{l+2} \supseteq \mathcal{M}^{l+1}$ hence $y + x^2 + a_lx^l$ is $(l+1)$-$\mathcal{R}_V$-determined. It shows that in fact it is (l)-$\mathcal{R}_V$-determined for all $a_l \neq 0$.

(b): If $l \neq 2$ and $a_l \neq 0$ then $h = y + a_lx^l$ is l-$\mathcal{R}_V$-determined.

Case (iii): Classification of the germs for which the 1-jet is z.

Consider the germ $h(x,y,z) = z$ which is not $\mathcal{R}_V$-finite.

As $J^2(T\mathcal{R}_V^1(h)) + \mathbb{C}\{x^2, xy, y^2\} \supseteq H^2$, all the 2-jets which 1-jet is z are $\mathcal{R}_V$-equivalent to $z + a_2x^2 + bxy + cy^2$. We have

$$J^2(T\mathcal{R}_V(h)) = \left\langle 2a_2x^2 + 3bxy + 4cy^2 + 3z, 2bxz + 4cyz + 3y^2, 6a_2xy + 3by^2, 2a_2xz + byz \right\rangle,$$

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therefore $y^2 \in j^2(T\mathcal{R}_V(h))$, $\forall\, a_2$, b, c if $a_2 \neq 0$, $xy \in j^2(T\mathcal{R}_V(h))$, $\forall\, b, c$. Hence, by Mather's Lemma, we have:

For $a_2 \neq 0$, $h = z + a_2x^2 + bxy + cy^2 \sim z + a_2x^2$ is $2-\mathcal{R}_V$-determined.
For $a_2 = 0$, $b \neq 0$, $h = z + bxy + cy^2 \sim z + bxy$ and $\mu(h|_{\Sigma_1}) = 2 > 1$.
For $a_2 = b = 0$, $j^2(h) = z$ and $\mu(h|_{\Sigma_1}) \geq 2$.

Sections of S_k.
We have that orbits in $J^1(3,1)$ of the act of $J^1\mathcal{R}_V$ are: x, y, z and 0.

Case (i): Classification of the germs h for which the 1-jet is x.
Let $h(x,y,z) = x$, so $T\mathcal{R}_V(h) \supset \mathcal{M}^1$ and x is 1-$\mathcal{R}_V$-determined.

Case (ii): Classification of the germs h for which the 1-jet is y.
The generators of the tangent space are: $\eta_1 h = 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = -2(k+1)x^3y$ and $\eta_4 h = 0$ and a l-complete transversal is $y + a_l x^l$.
Let $h = y + a_l x^l$, then $\eta_1 h = 2la_l x^l + 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = la_l x^{k+l} + 3la_l x^{l-1}y - 2(k+1)x^k y$ and $\eta_4 h = la_l x^{l-1}z$.

(a): If $l = k+1$, $a_{k+1} \neq 0$ and $a_{k+1} \neq 1$ then $h = y + a_{k+1}x^{k+1}$ is $\mathcal{R}_V$-finitely determined.
In this case $T\mathcal{R}_V(h) \supseteq \mathcal{M}^{2k+1}$, if $a_{k+1} \neq 0$ and $a_{k+1} \neq 1$, thus $y + a_{k+1}x^{k+1}$ is $(2k+1)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $F = y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \ldots + a_{2k}x^{2k} + a_{2k+1}x^{2k+1}$, as $x^{2k+1} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \ldots + a_{2k}x^{2k}$.
For $a_{k+1} = 1$, let $h = y + x^{k+1}$, then $\eta_1 h = 2(k+1)x^{k+1} + 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = (k+1)x^{2k+1} + (k+1)x^k y$ and $\eta_4 h = (k+1)x^k z$. Hence by geometric criterion, $y + x^{k+1}$ is not $\mathcal{R}_V$-finite. A l-complete transversal for $y + x^{k+1}$ is $y + x^{k+1} + a_l x^l$, $l > (k+1)$.
Let $h = y + x^{k+1} + a_l x^l$ with $a_l \neq 0$. Then $\eta_1 h = 2(k+1)x^{k+1} + 2(k+1)y + 2a_l l x^l$, $\eta_2 h = 2z$, $\eta_3 h = (k+1)x^{2k+1} + (k+1)x^k y + la_l x^{k+l} + 3la_l x^{l-1}y$ and $\eta_4 h = (k+1)x^k z + la_l x^{l-1}z$. Hence $T\mathcal{R}_V(h) + \mathcal{M}^{k+l+1} \supseteq \mathcal{M}^{k+l}$, thus $y + x^{k+1} + a_l x^l$ is $(k+l)$-$\mathcal{R}_V$-determined. Successively using complete transversal method we obtain $F = y + x^{k+1} + a_l x^l + \ldots + a_{k+l-1}x^{k+l-1} + a_{k+l}x^{k+l}$, as $x^{k+l} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + x^{k+1} + a_l x^l + \ldots + a_{k+l-1}x^{k+l-1}$.

(b): If $l \neq (k+1)$ and $a_l \neq 0$ then $h = y + a_l x^l$ is $\mathcal{R}_V$-finitely determined.
If $l > (k+1)$, $T\mathcal{R}_V(h) + \mathcal{M}^{k+l+1} \supseteq \mathcal{M}^{k+l}$ and $y + a_l x^l$ is $(k+l)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $F = y + a_l x^l + \ldots + a_{k+l-1}x^{k+l-1} + a_{k+l}x^{k+l}$, as $x^{k+l} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + x^{k+1} + a_l x^l + \ldots + a_{k+l-1}x^{k+l-1}$.
If $l < (k+1)$ then $T\mathcal{R}_V(h) + \mathcal{M}^{2l} \supseteq \mathcal{M}^{2l-1}$ and $y + a_l x^l$ is $(2l-1)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $y + a_l x^l + \ldots + a_{2l-2}x^{2l-2} + a_{2l-1}x^{2l-1}$, as $x^{2l-1} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + a_l x^l + \ldots + a_{2l-2}x^{2l-2}$.

Case (iii): Classification of the germs h for which the 1-jet is z.
By geometric criterion, $h = z$ is not $\mathcal{R}_V$-finite. A 2-complete transversal for z is $h = z + a_2x^2 + bxy + cy^2$.

$$\begin{aligned} J^2(T\mathcal{R}_V) = \quad & \big\langle 4a_2x^2 + (4+2k)bxy + 4(k+1)cy^2 + 3(k+1)z, \\ & 2bxz + 4cyz + 3y^2, 6a_2xy + 3by^2, 2a_2xz + byz \big\rangle . \end{aligned}$$

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We have $j^2x\eta_1h = 3(k+1)xz$, $j^2y\eta_1h = 3(k+1)yz$, $j^2\eta_2h = 2bxz + 4cyz + 3y^2$ e $j^2\eta_3h = 6a_2xy + 3by^2$. Then $a_2xy, y^2 \in j^2(T\mathcal{R}_V(h))$ and by Mather's Lemma:

For $a_2 \neq 0$, $h = z + a_2x^2 + bxy + cy^2 \sim z + a_2x^2$.

For $a_2 = 0$, $b \neq 0$, $h = z + bxy + cy^2 \sim z + bxy$.

For $a_2 = b = 0$, $h = z$.

If $a_2 \neq 0$, $h = z + a_2x^2$ is 2-$\mathcal{R}_V$-determined.

If $a_2 = 0$ and $b \neq 0$ taking $h = z + bxy$. We have $\mu(h|_{\Sigma_k}) = k+1 > k$.

If $b = 0$, let $j^{l-1}h = z$, $l \geq 3$, a l-complete transversal for z is $z + a_lx^l + b_{l-1}x^{l-1}y$ if $3 \leq l \leq (2k+2)$ and $z + a_lx^l$ if $l \geq (2k+3)$.

Then for $h = z + a_lx^l + b_{l-1}x^{l-1}y$, $\eta_1h = 2la_lx^l + 2(l+k)b_{l-1}x^{l-1}y + 3(k+1)z$, $\eta_2h = 2b_{l-1}x^{l-1}z + x^{2k+2} + 4x^{k+1}y + 3y^2$, $\eta_3h = la_lx^{l+k} + 3la_lx^{l-1}y + (l-2k-3)b_{l-1}x^{l+k-1}y + (3l-3)b_{l-1}x^{l-2}y^2$ and $\eta_4h = la_lx^{l-1}z + (l-1)b_{l-1}x^{l-2}yz + (k+1)x^{2k+1}y + (k+1)x^ky^2$

thus, for $a_l \neq 0$, $x^{l-1}y \in T\mathcal{R}_V(h)$ and by Mather's Lemma $h \sim z + a_lx^l$.

Let $h = z + a_lx^l$, with $a_l \neq 0$. If $l > k+1$, $\mu(h|_{\Sigma_k} \geq k+1 > k$, then we do not need to classify. If $l \leq k+1$, h is $(2l-1)$-$\mathcal{R}_V$-determined and $x^{2l-1} \in T\mathcal{R}_V(\tilde{h})$ with $\tilde{h} = z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2}$, then $h \sim z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2}$.

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