

On the derivative of a normal function associated with a Deligne cohomology class

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ABSTRACT. We prove that for a family of varieties $\pi : X \rightarrow S$ and a *Deligne cohomology* class $\alpha \in H_{\mathcal{D}}^i(X, \mathbb{Z}(k))$, the derivative of the associated normal function, when it exists, can be computed in a purely algebraic way from the class α . We also prove a similar result in syntomic cohomology. An interesting consequence is that when α has a motivic origin, one may derive the corresponding family of elements in syntomic cohomology and obtain exactly the same derivative.

1. Introduction

Let $\pi : X \rightarrow S$ be a smooth family of varieties over \mathbb{C} and let α be a Deligne cohomology class $\alpha \in H_{\mathcal{D}}^i(X, \mathbb{Z}(k))$. One can restrict this class to the fibers and obtain a family of Deligne cohomology classes $\alpha_s \in H_{\mathcal{D}}^i(X_s, \mathbb{Z}(k))$, where $X_s = \pi^{-1}(s)$. Recall that for any variety Y over \mathbb{C} , Deligne cohomology sits in a short exact sequence (see for example [EV88, Cor 2.10])

$$(1.1) \quad \begin{aligned} 0 \rightarrow H^{i-1}(Y, \mathbb{C}) / (F^k H^{i-1}(Y, \mathbb{C}) + H^{i-1}(Y, \mathbb{Z})) &\xrightarrow{\iota} H_{\mathcal{D}}^i(Y, \mathbb{Z}(k)) \\ &\xrightarrow{\tau} H^i(Y, \mathbb{Z}(k)) \cap F^k H^i(Y, \mathbb{C}) \rightarrow 0 \end{aligned}$$

where F^k denotes the Hodge filtration on cohomology (this is slightly off if there is some torsion but it will be of no importance to us). Considering the above short exact sequence for the fibers of π , $Y = X_s$, assume that we have

$$(1.2) \quad \tau(\alpha_s) = 0 \text{ for any } s \in S.$$

Thus we may view $\alpha_s \in H^{i-1}(X_s, \mathbb{C}) / (F^k H^{i-1}(X_s, \mathbb{C}) + H^{i-1}(X_s, \mathbb{Z}))$. One knows that the family of classes α_s described in this way is in fact a *normal function* [Gre94, Lecture 6]. To recall what this means, recall first that the cohomology groups of the fibers $H^{i-1}(X_s, \mathbb{C})$ move in a *Variation of (mixed) Hodge structures*. In particular, they form the fibers of a locally free \mathcal{O}_S -module $\mathcal{H}^{i-1}(X/S)$ with an integrable connection (the *Gauss-Manin* connection), and the Hodge filtration varies in submodules $\mathcal{F}^k \mathcal{H}^{i-1}(X/S)$.

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Now, over a small complex ball $U \subset S_{\text{an}}$, we may lift the family α_s to $\tilde{\alpha} \in \Gamma(U, \mathcal{H}^{i-1}(X/S))$. Applying the Gauss-Manin connection to $\tilde{\alpha}$ we obtain a section $\nabla\tilde{\alpha} \in \Gamma(U, \Omega_S \otimes \mathcal{H}^{i-1}(X/S))$. Since the Gauss-Manin connection kills sections of the local system of cohomology groups this section is determined modulo $\nabla(\Gamma(U, \mathcal{F}^k \mathcal{H}^{i-1}(X/S)))$, which is contained in $\Gamma(U, \Omega_S^1 \otimes \mathcal{F}^{k-1} \mathcal{H}^{i-1}(X/S))$ by Griffiths transversality. The condition that α_s defines a normal functions means that in fact $\nabla\tilde{\alpha} \in \Gamma(U, \Omega_S \otimes \mathcal{F}^{k-1} \mathcal{H}^{i-1}(X/S))$. We thus obtain a well defined element, $\nabla\alpha \in H^1(U, \mathcal{F}^k \mathcal{H}^{i-1}(X/S) \rightarrow \Omega_S \otimes \mathcal{F}^{k-1} \mathcal{H}^{i-1}(X/S) \rightarrow \Omega_S^2 \otimes \mathcal{F}^{k-2} \mathcal{H}^{i-1}(X/S) \rightarrow \dots)$. This is the derivative of the family of Deligne cohomology classes we wish to understand.

Using the short exact sequence (1.1) again, this time with $Y = X$, let β be the image of α in $F^k H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Z})$. Our main result states that β determines $\nabla\alpha$. To explain how, we first make the following assumption.

ASSUMPTION 1.1. There exists a compactification $\bar{\pi} : \bar{X} \rightarrow S$ where the complement is a divisor D with relative normal crossings over S and whose components are smooth over S .

For the syntomic aspects of the theory to follow, it will be important to describe the construction in a more algebraic scenario, so we assume now either the current setup or one where \mathbb{C} is replaced by an arbitrary field of characteristic 0 and U is a Zariski open subset of S .

We now recall [Kat72] that the complexes $\Omega_{\bar{X}}^\bullet(\log D)$ carry the so called *Koszul filtration* (loc. cit. (1.2.1.2))

$$K^j \Omega_{\bar{X}}^\bullet(\log D) = \text{Image} \left(\bar{\pi}^* \Omega_S^j \otimes \Omega_{\bar{X}}^{\bullet-j}(\log D) \rightarrow \Omega_{\bar{X}}^\bullet(\log D) \right)$$

whose associated graded is (loc. cit. (1.4.0.2))

$$\text{gr}_K^j = \bar{\pi}^* \Omega_S^j \otimes \Omega_{\bar{X}/S}^{\bullet-j}(\log D)$$

(we will abbreviate $\bar{\pi}^* \Omega_S^j$ to Ω_S^j). In particular, applying the Hodge filtration we have a short exact sequence (loc. cit. (1.4.1.4))

$$(1.3) \quad 0 \rightarrow \Omega_S^1 \otimes F^{k-1} \Omega_{\bar{X}/S}^\bullet(\log D)[-1] \rightarrow F^k(K^0/K^2) \rightarrow F^k \Omega_{\bar{X}/S}^\bullet(\log D) \rightarrow 0$$

where we have abbreviated, as we will do from now onward $K^0/K^2 \Omega_{\bar{X}}^\bullet(\log D)$ to simply K^0/K^2 . Applying $\mathbb{R}\bar{\pi}_*$, then taking global sections on U and relying on the cohomological triviality of U , we have a long exact sequence

$$\begin{aligned} \dots &\rightarrow \Gamma(U, \mathbb{R}^{i-1} \bar{\pi}_* F^k \Omega_{\bar{X}/S}^\bullet(\log D)) \\ &\rightarrow \Gamma(U, \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* F^{k-1} \Omega_{\bar{X}/S}^\bullet(\log D)) \rightarrow \Gamma(U, \mathbb{R}^i \bar{\pi}_* F^k(K^0/K^2)) \\ &\rightarrow \Gamma(U, \mathbb{R}^i \bar{\pi}_* F^k \Omega_{\bar{X}/S}^\bullet(\log D)) \rightarrow \dots \end{aligned}$$

Restricting β to $\bar{\pi}^{-1}(U)$ we can consider it as an element of $\Gamma(U, \mathbb{R}^i \bar{\pi}_* F^k \Omega_{\bar{X}}^\bullet(\log D))$. Projecting to K^0/K^2 we get an element, still called β , in $\Gamma(U, \mathbb{R}^i \bar{\pi}_* F^k K^0/K^2)$.

The assumption (1.2) implies that β maps to 0 in $\Gamma(U, \mathbb{R}^i \bar{\pi}_* F^k \Omega_{\bar{X}/S}^\bullet(\log D))$. It follows that β is the image of a section

$$\gamma \in \Gamma(U, \Omega_S \otimes \mathbb{R}^{i-1} \bar{\pi}_* F^{k-1} \Omega_{\bar{X}/S}^\bullet(\log D)) = \Gamma(U, \Omega_S \otimes \mathcal{F}^{k-1} \mathcal{H}^{i-1}(X/S))$$

One easily checks that the definition of γ involves the same indeterminacy as that of $\nabla\alpha$. Going back exclusively to the complex setup we have our main result.

THEOREM 1.2. *Suppose assumption 1.1 holds. Then, up to the indeterminacy indicated above we have $\gamma = -\nabla\alpha$.*

It seems a result of this kind has been around in many particular cases for quite some time. We mention a number of examples:

Consider the case where X is affine and α is the regulator of an element of $K_2^M(X)$ (Milnor K-theory of the coordinate ring). In this case, the result is due to Collino [Col97, § 7]. It has also been obtained independently by de Jeu (private communication).

In the case of cycles this result is due to Voisin [Voi88]. Note however, that the result is also true for regulators in K -theory. In this form, the result “explains” the work of Rodriguez-Villegas [Vil99] on the derivatives of Mahler measures (see the discussions in [BD99, 1.7, 2.5-2.6] hinting at the result we discuss here). Indeed, Villegas’s work was the starting point for the observations of this paper.

A special case of this result also appears in the Thesis of Mellit [Mel08].

Finally, this result is closely related with work of M. Saito, S. Saito, Asakura, Green and Griffiths on arithmetic Hodge modules and higher Abel-Jacobi maps. It can probably be proved using tools from that theory.

As far as we know though, this result has never been stated in this generality and simplicity.

An important remark is that unlike the construction of $\nabla\alpha$, the construction of γ is entirely algebraic. Thus, one finds that $\nabla\alpha$ is the restriction to U of an algebraic one form with values in $\mathcal{F}^{k-1}\mathcal{H}^{i-1}(X/S)$ on some Zariski open containing U . This forms the link with the second part of the paper, in which we describe the entirely analogous construction in *syntomic cohomology*. This is the p -adic analogue of Deligne cohomology (see Section 4 for a sketchy introduction), so it is not surprising that a similar result holds in that context as well (see Theorem 4.2).

Both Deligne cohomology and syntomic cohomology carry a theory of regulators from *motivic cohomology*. It is an essentially immediate consequence (see Corollary 4.3) that when the two constructions can be compared, they are the same. In other words, when the classes in Deligne and syntomic cohomology are coming from an element in motivic cohomology over some number field which may be embedded in both the complex number and the p -adics, then the derivatives of the two regulators are coming from the same form over the number field.

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2. A homological algebra lemma

The simple lemma presented here has a rather long history (see below). We consider the following situation: Suppose we are given a commutative square of complexes

$$\begin{array}{ccc} X^\bullet & \xrightarrow{\alpha} & Y^\bullet \\ \epsilon \downarrow & & \downarrow \phi \\ Z^\bullet & \xrightarrow{\gamma} & W^\bullet \end{array}$$

Then we can extend this to a diagram of distinguished triangles

$$\begin{array}{ccccc}
 X^\bullet & \xrightarrow{\alpha} & Y^\bullet & \xrightarrow{\beta} & C(X^\bullet \rightarrow Y^\bullet) \\
 \downarrow \epsilon & & \downarrow \phi & & \downarrow \\
 Z^\bullet & \xrightarrow{\gamma} & W^\bullet & \longrightarrow & C(Z^\bullet \rightarrow W^\bullet) \\
 \downarrow \eta & & \downarrow & & \downarrow \\
 C(X^\bullet \rightarrow Z^\bullet) & \longrightarrow & C(Y^\bullet \rightarrow W^\bullet) & \longrightarrow & V^\bullet
 \end{array}$$

where V^\bullet stands for either $C(C(X^\bullet \rightarrow Y^\bullet) \rightarrow C(Z^\bullet \rightarrow W^\bullet))$ or $C(C(X^\bullet \rightarrow Z^\bullet) \rightarrow C(Y^\bullet \rightarrow W^\bullet))$. These last two cones are easily seen to be isomorphic (see below). We then get an infinite square of long exact cohomology sequences, in both the vertical and horizontal directions. We denote by δ_1 and δ_2 the boundary maps in these two directions.

The technical foundation for the entire paper is now the following rather simple result.

LEMMA 2.1. *Suppose we have $y \in H^i(Y^\bullet)$ and $z \in H^i(Z^\bullet)$, such that $\phi(y) = \gamma(z)$. Then there exists $v \in H^{i-1}(V^\bullet)$ such that $\delta_1(v) = \beta(y)$ and $\delta_2(v) = -\eta(z)$.*

PROOF. We use standard sign convention for cones [Har66, p. 26], which is that the cone of, e.g., α has in degree n $X^{n+1} \oplus Y^n$ with differential $d(x, y) = (-dx, \alpha(x) + dy)$. It is then easy to compute that an element of V^{i-1} is given by a fourtuple (x, y, z, w) in degrees $i+1, i, i$ and $i-1$ respectively, whose differential is given, when considering the first definition for V^\bullet , by $(dx, -\alpha(x) - dy, \epsilon(x) - dz, dw + \phi(y) + \gamma(z))$, and, when considering the second definition by $(dx, \alpha(x) - dy, -\epsilon(x) - dz, dw + \phi(y) + \gamma(z))$. Indeed, the two definition are isomorphic by taking x to $-x$ and keeping the other components fixed. Our assumption implies that we can find w to make the element $v = (0, y, -z, w)$ closed. The element v maps to $(0, y) = \beta(y)$ in $C(X^\bullet \rightarrow Y^\bullet)$ and to $(0, -z) = -\eta(z)$ in $C(X^\bullet \rightarrow Z^\bullet)$. \square

COROLLARY 2.2. *In the same situation as above, consider $y \in H^i(Y^\bullet)$ such that $\phi(y)$ maps to 0 in $H^i(C(Z^\bullet \rightarrow W^\bullet))$. Then one can form the following two constructions:*

- (1) *Choose a pullback $z \in H^i(Z^\bullet)$ to $\phi(y)$ and compute $\eta(z) \in H^i(C(X^\bullet \rightarrow Z^\bullet))$. This construction is well defined up to the image of the composition $H^{i-1}(C(Z^\bullet \rightarrow W^\bullet)) \rightarrow H^i(Z^\bullet) \xrightarrow{\eta} H^i(C(X^\bullet \rightarrow Z^\bullet))$.*
- (2) *Since $\beta(y)$ maps to 0 in $H^i(C(Z^\bullet \rightarrow W^\bullet))$ it is the image of some $v \in H^{i-1}(V^\bullet)$ and we can compute $\delta_2(v) \in H^i(C(X^\bullet \rightarrow Z^\bullet))$. This is well defined up to the image of the composition $H^{i-1}(C(Z^\bullet \rightarrow W^\bullet)) \rightarrow H^{i-1}(Z^\bullet) \xrightarrow{\delta_2} H^i(C(X^\bullet \rightarrow Z^\bullet))$, which is the same as the indeterminacy in the previous construction.*

Then, these two constructions give the same element up to a -1 sign and the indeterminacy.

REMARK 2.3. Clearly the lemma and the corollary apply also to $\mathbb{R}^i G$ replacing H^i , where G is any left exact functor, and to any diagram of distinguished triangles. In particular, we will be applying it to a diagram of mapping fibers (see below) rather than a diagram of cones.

REMARK 2.4. If the diagram of distinguished triangles above is associated with a diagram of short exact sequences, one obtains a result of Jannsen [Jan00, Lemma on p. 268] (and also independently in the first part of [Bes97, Lemma 3.1]). Note that Jannsen already deduces the result for cones, by taking resolutions, but that in fact the proof we give here is simpler.

3. Proof of the main Theorem

Let Y be a smooth variety over \mathbb{C} and suppose we can compactify it as

$$Y = \bar{Y} - D, \quad \bar{Y} \text{ proper}, \quad D \text{ normal crossings divisor.}$$

Recall that the Deligne cohomology of Y , with logarithmic singularities as modified by Beilinson, is defined to be the cohomology on the associated analytic space

$$(3.1) \quad \begin{aligned} H_{\mathcal{D}}^i(Y, \mathbb{Z}(k)) &:= H^i(\bar{Y}_{\text{an}}, \mathbb{Z}_{\mathcal{D}}(k)_Y) \\ \mathbb{Z}_{\mathcal{D}}(k)_Y &:= \text{MF}(\mathbb{Z}(k) \oplus F^k \Omega_{\bar{Y}}^{\bullet}(\log D) \rightarrow \Omega_{\bar{Y}}^{\bullet}(\log D)) \end{aligned}$$

where MF denotes the mapping fiber of a map of complexes (cone shifted by -1). Taking the associated long exact cohomology sequence easily gives (1.1).

We now apply the lemma of the previous section in the following situation. Recall that we consider the smooth map $\pi : X \rightarrow S$ and assume that it may be compactified to a smooth proper map $\bar{\pi} : \bar{X} \rightarrow S$ such that the complement $D = \bar{X} - X$ is a normal crossings divisor relative to S such that each of its components is smooth over S . For a fixed integer k , recalling the Koszul filtration K^{\cdot} from the introduction, we consider the following 3 inclusions of complexes of sheaves in the analytic topology on \bar{X} :

$$\begin{aligned} 0 &\subset \mathbb{Z}(k), \\ \Omega_S^1 \otimes \Omega_{\bar{X}}^{\bullet}(\log D)[-1] &\subset K^0/K^2, \\ \Omega_S^1 \otimes F^{k-1} \Omega_{\bar{X}}^{\bullet}(\log D)[-1] &\subset F^k(K^0/K^2). \end{aligned}$$

Taking the corresponding mapping fibers as in (3.1), we obtain the following diagram (middle row left map being $x \mapsto (0, x)$):

$$(3.2) \quad \begin{array}{ccccc} & & \xrightarrow{\quad} & \tilde{\mathbb{Z}}_{\mathcal{D}}(k)_X & \xrightarrow{\quad} & \mathbb{Z}_{\mathcal{D}}(k)_{X/S} \\ & \downarrow & & \downarrow & & \downarrow \\ \Omega_S^1 \otimes F^{k-1} \Omega_{\bar{X}/S}^{\bullet}(\log D)[-1] & \longrightarrow & \mathbb{Z}(k) \oplus F^k K^0/K^2 & \longrightarrow & \mathbb{Z}(k) \oplus F^k \Omega_{\bar{X}/S}^{\bullet}(\log D) \\ & \downarrow & & \downarrow & & \downarrow \\ \Omega_S^1 \otimes \Omega_{\bar{X}/S}^{\bullet}(\log D)[-1] & \longrightarrow & K^0/K^2 & \longrightarrow & \Omega_{\bar{X}/S}^{\bullet}(\log D) \end{array}$$

where $\mathbb{Z}_{\mathcal{D}}(k)_{X/S}$ is the relative version of the Deligne complex and where $\tilde{\mathbb{Z}}_{\mathcal{D}}(k)_X$ is the quotient of the Deligne complex by the subcomplex $\text{MF}(0 \oplus F^k K^2 \rightarrow K^2)$

The map

$$\begin{aligned}
 \mathcal{H}^{i-1}(X/S) &= \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D) \xrightarrow{\text{boundary on bottom row}} \\
 &\mathbb{R}^{i-1} \bar{\pi}_*(\Omega_S^1 \otimes \Omega_{\bar{X}/S}^\bullet(\log D)) \\
 (3.3) \quad &= \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D) \\
 &= \Omega_S^1 \otimes \mathcal{H}^{i-1}(X/S)
 \end{aligned}$$

is just the Gauss-Manin connection ∇ on $\mathcal{H}^{i-1}(X/S)$ [Kat72, (1.4.0.4)]. Consequently, the map

$$\begin{aligned}
 \mathcal{F}^k \mathcal{H}^{i-1}(X/S) &= \mathbb{R}^{i-1} \bar{\pi}_* F^k \Omega_{\bar{X}/S}^\bullet(\log D) \xrightarrow{\text{boundary on middle row}} \\
 &\mathbb{R}^{i-1} \bar{\pi}_*(\Omega_S^1 \otimes F^{k-1} \Omega_{\bar{X}/S}^\bullet(\log D)) \rightarrow \\
 (3.4) \quad &\mathbb{R}^{i-1} \bar{\pi}_*(\Omega_S^1 \otimes \Omega_{\bar{X}/S}^\bullet(\log D)) \\
 &= \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D) \\
 &= \Omega_S^1 \otimes \mathcal{H}^{i-1}(X/S) .
 \end{aligned}$$

is just the composition

$$\mathcal{F}^k \mathcal{H}^{i-1}(X/S) \rightarrow \mathcal{H}^{i-1}(X/S) \xrightarrow{\nabla} \Omega_S^1 \otimes \mathcal{H}^{i-1}(X/S) .$$

PROOF OF THEOREM 1.2. We show that Corollary 2.2, when applied to diagram (3.2) and the derived functor $\mathbb{R} \bar{\pi}_*$, precisely implies the result we want. Recall that U is a small ball inside S . We start with $\alpha \in H^i(\bar{X}, \mathbb{Z}_{\mathcal{D}}(k)_X)$. Restricting to $\bar{\pi}^{-1}(U)$ and using the cohomological triviality of U we obtain a section $y \in \Gamma(U, \mathbb{R}^i \bar{\pi}_* \mathbb{Z}_{\mathcal{D}}(k)_X)$ which we further project to a section, still called y , in $\Gamma(U, \mathbb{R}^i \bar{\pi}_* \tilde{\mathbb{Z}}_{\mathcal{D}}(k)_X)$. Apply now Corollary 2.2 to diagram (3.2) and the derived functor $\mathbb{R} \bar{\pi}_*$, compose everything with $\Gamma(U, \bullet)$ and considering it all on the element y just defined. The corollary gives us an equality of two sections in

$$\begin{aligned}
 \Gamma(U, \mathbb{R}^{i-1} \bar{\pi}_*(\Omega_S^1 \otimes \Omega_{\bar{X}/S}^\bullet(\log D))) &= \Gamma(U, \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D)) \\
 &= \Gamma(U, \Omega_S^1 \otimes \mathcal{H}^{i-1}(X/S))
 \end{aligned}$$

modulo the image of $\mathbb{R}^{i-1} \bar{\pi}_* F^k \Omega_{\bar{X}/S}^\bullet(\log D)$ under the map which is the composition of the boundary on the middle row and the bottom left vertical map in (3.2), which is just $\Gamma(U, \bullet)$ of the map (3.4). Thus, we obtain an equality of two sections in

$$\Gamma(U, \Omega_S^1 \otimes \mathcal{H}^{i-1}(X/S)) / \nabla \Gamma(U, \mathcal{F}^k \mathcal{H}^{i-1}(X/S))$$

as required.

We now check that the two constructions in Corollary 2.2 give $\nabla \tilde{\alpha}$ and the projection of γ , proving the theorem. Let's start with the second construction. We first push y to $\Gamma(U, \mathbb{R}^i \bar{\pi}_* \mathbb{Z}_{\mathcal{D}}(k)_{X/S})$. This is the family α_s of elements in the Deligne cohomology of the fibers. Our assumption implies that this in turn comes from

$$\Gamma(U, \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D)) = \Gamma(U, \mathcal{H}^{i-1}(X/S)) .$$

This is visibly just the lift $\tilde{\alpha}$. To this we apply the boundary map, which is just the Gauss-Manin connection as we saw. Thus, this construction gives $\nabla \tilde{\alpha}$.

Now using the first construction we apply the map induced by the top arrow in the middle column of (3.2) y to get to $\Gamma(U, \mathbb{R}^i \bar{\pi}_*(\mathbb{Z}(k) \oplus F^k K^0/K^2))$. We can clearly ignore the first component here. We are to lift to an element of

$\Gamma(U, \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* F^{k-1} \Omega_{\bar{X}/S}^\bullet(\log D))$ and map to $\Gamma(U, \Omega_S^1 \otimes \mathbb{R}^{i-1} \bar{\pi}_* \Omega_{\bar{X}/S}^\bullet(\log D))$. By definition, this gives the class γ as required. \square

4. The derivative of a syntomic cohomology class

In this section we sketch the syntomic analogue of the theory developed in the classical case. For the theory of syntomic cohomology and regulators we refer to [Bes00].

Let K be a p -adic field (finite extension of \mathbb{Q}_p) with ring of integers R and residue field κ . We denote by R_0 the ring of Witt vectors of κ , which is a subring of R , and denote its field of fractions by $K_0 \subset K$. All schemes we consider are going to be separated, integral and of finite type over their respective bases.

Let \mathbf{X} be a smooth R -scheme. The syntomic cohomology of \mathbf{X} is by definition the cohomology of a mapping fiber (compare the proof of Proposition 6.3 in [Bes00])

$$(4.1) \quad \mathbb{R}\Gamma_{\text{syn}}(\mathbf{X}, n) := \text{MF}(\mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K_0) \oplus F^n \mathbb{R}\Gamma_{\text{dR}}(\mathbf{X}_K/K) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K_0) \oplus \mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K))$$

where $F^n \mathbb{R}\Gamma_{\text{dR}}(\mathbf{X}_K/K)$ and $\mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K)$ are complexes computing the filtered part of de Rham cohomology and rigid cohomology [Ber96, Ber97] respectively, and the map defining the fiber is given by

$$(4.2) \quad (x, y) \mapsto \left(\left(1 - \frac{\phi}{p^n} \right) x, \text{cb}(x) - \text{sp}(y) \right)$$

Here,

$$(4.3) \quad \text{sp} : \mathbb{R}\Gamma_{\text{dR}}(\mathbf{X}_K/K) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K)$$

is the specialization map [Bes00, BCF04] while

$$(4.4) \quad \text{cb} : \mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K_0) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K)$$

is the base change map [Ber97, Proposition 1.8]. It is important to note that while on cohomology the base change and specialization map are well defined maps, on the level of complexes they are only defined on the derived category level, i.e., after inverting some quasi-isomorphisms, and one has to be careful to include information on which quasi-isomorphisms are inverted (see a detailed account of this issue in [Bes00, Section 3]).

We will need concrete models for the complexes and maps above. For the fact that the models we write down are valid see [Bes00]. To write these models we suppose that we have an open embedding of \mathbf{X} inside a proper R -scheme $\bar{\mathbf{X}}$. The special fiber \mathbf{X}_s is then embedded inside $\bar{\mathbf{X}}_s$ as an open subset.

We need the following notions from the theory of rigid cohomology. For any R -scheme \mathbf{Y} we have a structure of a rigid analytic space on the generic fiber of \mathbf{Y} , denoted \mathbf{Y}_K^{an} . If \mathbf{Y} is proper over R there is a specialization (reduction modulo the maximal ideal) map $\text{sp} : \mathbf{Y}_K^{\text{an}} \rightarrow \mathbf{Y}_\kappa$. If Z is a relatively closed subset of \mathbf{Y}_κ the inverse image of Z under sp is denoted by $]Z[_{\mathbf{Y}}$ and is called the tube of Z in \mathbf{Y} (note that in Berthelot's work everything is associated with the formal scheme which is the p -adic completion of \mathbf{Y}) and has the structure of a rigid analytic space.

Suppose now that X is a κ -scheme, and that we have found an open embedding $j : X \rightarrow \bar{X}$ into a proper \bar{X}/κ and a closed embedding

$$(4.5) \quad \bar{X} \hookrightarrow \mathbf{P}_\kappa, \mathbf{P}/R \text{ proper, smooth in a neighborhood of } X.$$

The auto-functor j^\dagger of sections with overconvergent support on the category of abelian sheaves on $]X[_{\mathbf{P}}$ is defined by

$$j^\dagger(E) = \varinjlim_V (j_V)_* j_V^{-1} E,$$

where the limit runs over all strict neighborhoods V of $]X[_{\mathbf{P}}$ in $]X[_{\mathbf{P}}$ (those admissible opens in the rigid analytic sense for which $]X[_{\mathbf{P}} = V \cup]X[_{\mathbf{P}}$ is an admissible covering) and j_V is the embedding of V in $]X[_{\mathbf{P}}$. The rigid complex is defined as

$$\mathbb{R}\Gamma_{\text{rig}}(X/K) = \mathbb{R}\Gamma(]X[_{\mathbf{P}}, j^\dagger \Omega^\bullet).$$

The specialization map may be realized as follows: embed \mathbf{X} inside a proper R -scheme $\bar{\mathbf{X}}$ as an open set and let $j : \mathbf{X}_\kappa \rightarrow \bar{\mathbf{X}}_\kappa$ be the induced embedding of the special fibers. Then $]X_\kappa[_{\bar{\mathbf{X}}} = \bar{\mathbf{X}}_K$ and $\mathbb{R}\Gamma(\bar{\mathbf{X}}_K, j^\dagger \Omega^\bullet)$ is a model for the rigid complex $\mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}_\kappa/K)$. The specialization map from the complex $\mathbb{R}\Gamma(\mathbf{X}_K^{\text{an}}, \Omega^\bullet)$, which is a model for $\mathbb{R}\Gamma_{\text{rig}}(\mathbf{X}/K)$, comes about because \mathbf{X}_K is a strict neighborhood of $]X_\kappa[_{\bar{\mathbf{X}}}$ inside $\bar{\mathbf{X}}_K$ (see (5.3) and the discussion preceding it in [Bes00]).

To get a semi-linear operator ϕ on $\mathbb{R}\Gamma_{\text{rig}}(X/K_0)$ one may choose the R -scheme \mathbf{P} in (4.5) to have a semi-linear lift φ of (the absolute) Frobenius (for example, if \bar{X} is projective one may take \mathbf{P} to be a projective space). Since the absolute Frobenius acts as the identity on the spaces underlying κ -schemes it is easy to see that φ acts on the tubes $]X[_{\mathbf{P}}$ and $]X[_{\mathbf{P}}$ and furthermore that it acts in a semi-linear way on $j^\dagger \Omega^\bullet$, giving the required action.

As a consequence of the definition of syntomic cohomology (4.1) we have the following short exact sequence

$$(4.6) \quad 0 \rightarrow J_p^i(\mathbf{X}, n) \rightarrow H_{\text{syn}}^i(\mathbf{X}, n) \rightarrow \text{Hdg}_p^i(\mathbf{X}, n) \rightarrow 0$$

where $J_p^i(\mathbf{X}, n)$ (the p -adic intermediate jacobian) and $\text{Hdg}_p^i(\mathbf{X}, n)$ (p -adic Hodge classes) are respectively the cokernel, with $j = i - 1$, and the kernel, with $j = i$, of the map

$$(4.7) \quad H_{\text{rig}}^j(\mathbf{X}_\kappa/K_0) \oplus F^n H_{\text{dR}}^j(\mathbf{X}_K/K) \rightarrow H_{\text{rig}}^j(\mathbf{X}_\kappa/K_0) \oplus H_{\text{rig}}^j(\mathbf{X}_\kappa/K)$$

induced by (4.2).

Consider now the relative situation where $\pi : \mathbf{X} \rightarrow \mathbf{S}$ is a smooth map of smooth R -schemes. Let s_0 be a κ -rational point of \mathbf{S}_κ . Then over the “residue disc” $U =]s_0[_{\mathbf{S}}$ of s_0 we can vary the p -adic intermediate jacobians of the fibers \mathbf{X}_s of π as follows: The space $H_{\text{rig}}^{i-1}(\mathbf{X}_{s_\kappa}/K_0) = H_{\text{rig}}^{i-1}(\mathbf{X}_{s_0}/K_0)$ is fixed. The spaces $H_{\text{rig}}^{i-1}(\mathbf{X}_{s_\kappa}/K)$ vary in a free \mathcal{O}_U module $\mathcal{H}_{\text{rig}}^{i-1}(\mathbf{X}_\kappa/\mathbf{S}_\kappa)$ with a flat connection ∇ , forming an *overconvergent isocrystal*. (see [Ber86, Théorème 5] and [Tsu03]). Finally, the $F^n H_{\text{dR}}^{i-1}(\mathbf{X}_{s_K}/K)$ form a locally free \mathcal{O}_U module $\mathcal{F}^n \mathcal{H}^{i-1}(\mathbf{X}_K/\mathbf{S}_K)$ that maps to $\mathcal{H}_{\text{rig}}^{i-1}(\mathbf{X}_\kappa/\mathbf{S}_\kappa)$. Via base change, the space $H_{\text{rig}}^{i-1}(\mathbf{X}_{s_0}/K_0)$ maps to the horizontal sections of $\mathcal{H}_{\text{rig}}^{i-1}(\mathbf{X}_\kappa/\mathbf{S}_\kappa)$. This suggests the following obvious p -adic analogue of the notion of a normal function and its derivative.

DEFINITION 4.1. In the above situation a normal function over U is an element of the cokernel of (the relative analogue of (4.7))

$$(4.8) \quad H_{\text{rig}}^{i-1}(\mathbf{X}_{s_0}/K_0) \oplus \Gamma(U, \mathcal{F}^n \mathcal{H}^{i-1}(\mathbf{X}_K/\mathbf{S}_K)) \rightarrow H_{\text{rig}}^{i-1}(\mathbf{X}_{s_0}/K_0) \oplus \Gamma(U, \mathcal{H}_{\text{rig}}^{i-1}(\mathbf{X}_\kappa/\mathbf{S}_\kappa))$$

such that under the well defined map of this cokernel to

$$H^1(U, \mathcal{F}^n \mathcal{H}^{i-1}(\mathbf{X}_K/\mathbf{S}_K)) \xrightarrow{\nabla} \mathcal{O}_U^1 \otimes \mathcal{H}_{\text{rig}}^{i-1}(\mathbf{X}_\kappa/\mathbf{S}_\kappa) \text{ induced by } (c, \alpha) \mapsto \nabla \alpha$$

it maps to the image of an element of $\Omega_U^1 \otimes \mathcal{F}^{n-1} \mathcal{H}^{i-1}(\mathbf{X}_K/\mathbf{S}_K)$. We call this last element the derivative of the normal function.

THEOREM 4.2. *Let $\pi : \mathbf{X} \rightarrow \mathbf{S}$ be as above and let $\alpha \in H_{\text{syn}}^i(\mathbf{X}, n)$ be in the kernel of*

$$H_{\text{syn}}^i(\mathbf{X}, n) \rightarrow F^n H_{\text{dR}}^i(\mathbf{X}_K/K) \rightarrow H_{\text{dR}}^i(\mathbf{X}_K/K) \rightarrow \Gamma(\mathbf{S}_K, \mathcal{H}_{\text{dR}}^i(\mathbf{X}_K/\mathbf{S}_K)).$$

Then, over a fixed residue disc U the restrictions to fibers $\alpha|_{\mathbf{X}_s} \in J_p^i(\mathbf{X}_s, n)$ give a p -adic normal function, in the sense of Definition 4.1, whose derivative, again in the sense of Definition 4.1, is given by $-\gamma$, where γ is deduced from the image of α in $F^n H_{\text{dR}}^i(\mathbf{X}_K/K)$ as in the introduction over an affine open containing U and then restricted to U .

PROOF. We describe the proof of this Theorem only under the additional (very special) restrictions (we later sketch how to remove them): $R = R_0$, π extends to a diagram

$$(4.9) \quad \begin{array}{ccc} \mathbf{X} & \xrightarrow{j} & \bar{\mathbf{X}} \\ & \searrow \pi & \swarrow \bar{\pi} \\ & \mathbf{S} & \end{array}$$

where the complement $\mathbf{D} := \bar{\mathbf{X}} - \mathbf{X}$ is a relative normal crossings divisor over \mathbf{S} with components which are smooth over \mathbf{S} , and finally that there is semi-linear lift of Frobenius acting on the entire diagram (4.9). The reasons for these restriction is that with them we can write down a similar diagram to (3.2) on a single space. Namely, Let \bar{X} be the rigid analytic space $\bar{\pi}^{-1}(U)$ and let D be the intersection of \bar{X} with \mathbf{D}_K . Consider the syntomic analogue of diagram (3.2), a diagram of sheaves on \bar{X} , where the maps on the left are to the second component.

$$\begin{array}{ccccc} & & \mathcal{S}_{\mathbf{X}} & \longrightarrow & \mathcal{S}_{\mathbf{X}/\mathbf{S}} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{\mathbf{S}}^1 \otimes F^{k-1} \Omega_{\bar{X}/\mathbf{S}}^{\bullet}(\log D)[-1] & \longrightarrow & j^{\dagger} \Omega_{\bar{X}}^{\bullet} \oplus F^k K^0/K^2 & \longrightarrow & j^{\dagger} \Omega_{\bar{X}}^{\bullet} \oplus F^k \Omega_{\bar{X}/\mathbf{S}}^{\bullet}(\log D) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{\mathbf{S}}^1 \otimes j^{\dagger} \Omega_{\bar{X}/\mathbf{S}}^{\bullet}[-1] & \longrightarrow & j^{\dagger} \Omega_{\bar{X}}^{\bullet} \oplus j^{\dagger} K^0/K^2 & \longrightarrow & j^{\dagger} \Omega_{\bar{X}}^{\bullet} \oplus j^{\dagger} \Omega_{\bar{X}/\mathbf{S}}^{\bullet} \end{array}$$

Now take the cohomology on \bar{X} and deduce the theorem exactly in the same way as the proof of Theorem 1.2.

In the general case each of the complexes making up the syntomic complex “lives” on a different space. Thus, one needs to take injective resolutions on each of the spaces separately, push forward to U and then apply Corollary 2.2 and go through all the details of the construction of the specialization (4.3) and base change (4.4) maps (for this reason we do not write this in detail). \square

COROLLARY 4.3. *Let F be a number field and suppose given embeddings $\tau : F \rightarrow \mathbb{C}$ and $\tau_p : F \rightarrow K$, where K is some p -adic field. Let $\pi : X \rightarrow S$ be a smooth map of F -varieties and let $\theta \in H_{\mathcal{M}}^i(X, \mathbb{Q}(n))$ such that the de Rham regulator of θ , $\beta \in F^n H_{\text{dR}}^i(X/F)$, restricts to 0 in $\Gamma(S, \mathcal{F}^n \mathcal{H}_{\text{dR}}^i(X/S))$. Suppose*

that the map $\pi \otimes_{\tau_p} K : X \otimes_{\tau_p} K \rightarrow S \otimes_{\tau_p} K$ admits a smooth lift $\mathbf{X} \rightarrow \mathbf{S}$ over R and that the class $\tau_p(\theta)$ lifts to $\theta \in H_{\mathcal{M}}^i(\mathbf{X}, \mathbb{Q}(n))$. Let $\alpha \in H_{\mathbb{P}}^i(X \otimes \mathbb{C}, \mathbb{Z}(n))$ (respectively $\alpha_p \in H_{\text{syn}}^i(\mathbf{X}, n)$) be the Beilinson regulator of $\tau(\theta)$ (respectively the syntomic regulators of θ). Finally, let s be an F -rational point of S such that $\tau_p(s)$ has a reduction s_0 , and let U (respectively U_p) be a small ball in $(S \otimes \mathbb{C})_{\text{an}}$ containing $\tau(s)$ (respectively the residue disc of s_0). Then both α and α_p define normal functions and there exists a section γ of $\Omega_S^1 \otimes \mathcal{F}^{n-1} \mathcal{H}_{\text{dR}}^{i-1}(X/S)$ over a Zariski open affine subset of S containing s such that the derivative of the normal function associated with α is $\tau(\gamma)|_U$ and the derivative of the normal function associated with α_p is $\tau_p(\gamma)|_{U_p}$. In fact, γ is deduced from β as in the introduction.

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