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Looking for indecomposable right bounded complexes

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Dedicated to Professor Rüdiger Göbel on the occasion of his 70th birthday.

Abstract. We investigate indecomposable right bounded complexes of projective modules orthogonal to rather special partial tilting complexes.

Introduction

Some "proper" non classical partial tilting modules T have the following property: even though their projective resolution $\dot{T}$ is not a tilting complex, for every non zero module M, there is a morphism from $\dot{T}$ to a shift of the projective resolution of M which is not homotopic to zero.

In this note we investigate indecomposable complexes $\dot{C}$, not left bounded, such that every morphism from $\dot{T}$ to any shift of $\dot{C}$ is homotopic to zero. In Section 1 we fix the notation and the conventions used in the sequel. In Section 2 we describe three completely different situations, where there is an indecomposable not left bounded complex "orthogonal" with all its shifts to a partial tilting complex and with indecomposable non–zero components.

1. Preliminaries

Let R be a ring. We denote by $R-\mathrm{Mod}$ the category of all left R–modules. If $M \in R-\mathrm{Mod}$, then we write $\mathrm{Add}\, M$ for the class of all modules isomorphic to direct summands of direct sums of copies of M. Next, for every cardinal λ, we write $M^{(\lambda)}$ for the direct sum of λ copies of M. Finally, we write $M^{\perp_\infty}$ for the following class

$$M^{\perp_\infty} = \left\{X \in R-\mathrm{Mod} \,|\, \mathrm{Ext}^i_R(M, X) = 0 \text{ for all } i \geq 1\right\}.$$

The symbol $\mathrm{pdim}(M)$ denotes the projective dimension of M.

We shall say that an R–module T is a *partial n–tilting module* if $\mathrm{pdim}(T) \leq n$ and $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ. Given a partial n–tilting module T, we shall say that T is an *n–tilting module* if there is a long exact sequence of the form

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \ldots \longrightarrow T_n \longrightarrow 0,$$

where $T_i \in \mathrm{Add}\, T$ for every $i = 0, 1, \ldots, n$.

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From now on, we shall say, for brevity, that a partial n–tilting module T is a *large partial n–tilting module* if

$$\operatorname{Ker}\operatorname{Hom}(T,-)\cap T^{\perp_\infty}=0.$$

Maintaining the terminology introduced above, we recall some properties of partial n–tilting modules. First of all, for every $n \geq 1$, every n–tilting module T is a large partial n–tilting module [**B**, page 371]. Secondly, a finitely presented module T is a 1-tilting module if and only if T is a large partial 1–tilting module [**C**, Theorem 1] (also see [**CbF**, Theorem 3.2.1 and Section 3.1]). Finally, for every $n \geq 2$, there exist non faithful decomposable large partial n–tilting modules of projective dimension n and Loewy length 2 [**D1**, Example 4].

Given an algebra with n simple modules and a multiplicity free partial tilting module T, then T is an *almost complete tilting* module ([**HU**] and [**R2**, page 413]) if T has exactly $n-1$ indecomposable direct summands.

Given a ring R, we write $K(R)$ for the category of complexes over R with morphisms modulo homotopy. Let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R–modules. Assume the following conditions hold:

(1) $\operatorname{Hom}_{K(R)}(\dot{T},\dot{T}[i])=0$ for every $i \neq 0$,
(2) For every non–zero right bounded complex $\dot{X} \in K(R)$ of projective R–modules there exists some $i \in \mathbb{Z}$, such that $\operatorname{Hom}_{K(R)}(\dot{T},\dot{X}[i]) \neq 0$.

Then $\dot{T}$ is a *tilting complex*, in the sense of Rickard [**Rk**] as observed by Miyachi [**Mi**, condition (iii)', page 184]. In other words, the global, but functorial, condition (2) can replace a global non functorial condition on triangulated categories, which says the following:

- $\operatorname{add}\dot{T}$, the additive category of direct summands of finite direct sums of copies of $\dot{T}$, generates (as a triangulated category) the category of all bounded complexes of finitely generated projective R–modules.

Consequently, given a noetherian ring R, every tilting complex $\dot{T}$ satisfies the following condition ([**Sc-ZI**, condition (2) in the Definition of page 190]).

- If P is an indecomposable projective module and $\dot{P}$ is the stalk complex $0 \longrightarrow P \longrightarrow 0$, with P in degree 0, then $\dot{P}$ belongs to $\operatorname{add}\dot{T}$.

Finally, let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R–modules. Then, following the definition of [**Sc-ZI**] for complexes over noetherian rings, we shall say that $\dot{T}$ is a *partial tilting complex* if $\operatorname{Hom}_{K(R)}(\dot{T},\dot{T}[i])=0$ for every $i \neq 0$.

Throughout the paper, given a module M, the symbol $\dot{M}$ denotes a right bounded complex of projective modules of the form $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, where P_0 is in degree 0 and $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is a fixed projective resolution of M.

Moreover, we often say, for short, that a complex $\dot{W}$ is *orthogonal with all its shifts* (or just *orthogonal*) to a complex $\dot{V}$ if every morphism from $\dot{V}$ to any shift of $\dot{W}$ is homotopic to zero.

Next, K always denotes an algebraically closed field, and we always identify modules with their isomorphism classes (resp. complexes with their homotopy classes). Moreover, if Λ is a K–algebra given by a quiver and relations, according to [**R1**], then we often replace indecomposable finite dimensional modules by some obvious pictures, describing their composition factors. Over a representation–finite algebra given by a quiver, we often denote by x the simple module $S(x)$ corresponding to the vertex x.

When dealing with a complex $\dot{C}$, we often write d, instead of d_i, for the usual morphism $C_i \longrightarrow C_{i-1}$.

We end with the conventions used to describe morphisms between indecomposable projective modules P and Q (defined over the K–algebra Λ) with the following useful combinatorial property: the K–dimension of the vector space $V = \{f \in \operatorname{Hom}_\Lambda(P,Q) | f(P) \neq Q\}$ is at most one. First of all, the symbol $P \longrightarrow Q$ will denote a fixed generator v of V, and the symbol $P \stackrel{a}{\longrightarrow} Q$ will denote the morphism av for all $a \in K$. Secondly, we shall use Greek or Latin letters $\alpha, \beta, \ldots, a, b, \ldots$ to denote arbitrary morphisms.

For unexplained terminology, we refer to [**AF**] and [**AuReS**].

2. Examples

We begin with two indecomposable left unbounded complexes of the form

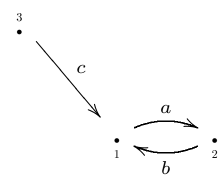
$$\cdots \to P \to P \to P \to Q \to X \to 0,$$

where the modules P and Q are indecomposable modules, while the module X is either indecomposable or equal to 0.

Example 1 (Left cancellation of one component and left addition of infinitely many components). *There are a non faithful large partial 3–tilting module T, defined over a finite dimensional algebra A, and an indecomposable right bounded complex $\dot{C} \in K(A)$ with the following properties:*

(i) $\dot{C}$ is not left bounded, every non zero component of $\dot{C}$ is indecomposable and $\operatorname{Hom}_{K(A)}(\dot{T}, \dot{C}[i]) = 0$ for every $i \in \mathbb{Z}$.
(ii) The direct sum of the homology modules of $\dot{C}$ is a semisimple non homogeneous module of infinite dimension over K.
(iii) T is an almost complete tilting module.

Construction. Let A be the K–algebra given by the quiver



with relations $ac = 0$ and $ba = 0$, and let T denote the injective module $\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus 3$.

Then $\begin{smallmatrix}3\\1\end{smallmatrix}$ and 1 are the indecomposable modules belonging to $\operatorname{Ker}\operatorname{Hom}_A(T, -)$, and we clearly have

$$\operatorname{Ext}^3_A\left(3\ ,\ \begin{smallmatrix}3\\1\end{smallmatrix}\right) \simeq \operatorname{Ext}^2_A\left(1\ ,\ \begin{smallmatrix}3\\1\end{smallmatrix}\right) \simeq \operatorname{Ext}^1_A\left(2\ ,\ \begin{smallmatrix}3\\1\end{smallmatrix}\right) \neq 0$$

and $\operatorname{Ext}^1_A\left(\,3\,,\,1\,\right) \neq 0$. Consequently T is a non faithful large partial 3–tilting module. Let $\dot{C}$ denote the indecomposable complex

$$\cdots \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\end{smallmatrix} \longrightarrow 0\ .$$

Then $\dot{C}$ satisfies (ii) and (iii), and 2 is not a composition factor of $H(\dot{C})$. Hence we deduce from [**D2**, Lemma 1] that

(1) $\quad \operatorname{Hom}_{K(A)}\left(\begin{smallmatrix}\dot{2}\\1\\2\end{smallmatrix}, \dot{C}[i]\right) = 0$ for every i.

Let $\dot{\alpha}$ be the morphism described by

$$(2)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & P & \longrightarrow & \cdots
\end{array}$$

where $P = \begin{smallmatrix}1\\2\end{smallmatrix}\,,\,\begin{smallmatrix}2\\1\\2\end{smallmatrix}$. Next, let $\dot{\beta}$ be the morphism described by:

$$(3)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle y'} & & \downarrow{\scriptstyle z'} & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0\ .
\end{array}$$

Then ϕ and ϕ' are not automorphisms, otherwise the first squares in (2) and (3) do not commute. Therefore, we have $z' = 0$. It is now easy to show that $\dot{\alpha}$ and $\dot{\beta}$ are homotopic to zero.

Next, let $\dot{\gamma}$ be the morphism described by

$$(4)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0\ . & &
\end{array}$$

Also in this case, proceeding from left to right, we see that $\dot{\gamma}$ is homotopic to zero. On the other hand, a picture of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\ & & {\scriptstyle x}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & \\ \cdots & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0 \end{array} \tag{5}$$

describes a morphism only if $x = 0$. Finally, any morphism of the form

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\ & & & & {\scriptstyle x}\big\downarrow & & & & \\ \cdots & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\end{smallmatrix} & \longrightarrow & 0 & & \end{array} \tag{6}$$

is clearly homotopic to zero. Putting (2), ...,(5) and (6) together, we conclude that

$$\operatorname{Hom}_{K(A)}(\dot{3}, \dot{C}[i]) = 0 \text{ for every } i \in \mathbb{Z}. \tag{7}$$

Hence (i) follows from (1) and (7).

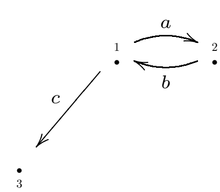
One can prove that, up to shift, $\dot{C}$ is the unique indecomposable right bounded complex satisfying (i). □

In the next example we replace the algebra of Example 1 by its opposite algebra.

EXAMPLE 2 (Left cancellation of two non–zero components and left addition of infinitely many components). *There are a faithful large partial 3–tilting module T, defined over a finite dimensional K–algebra B, and an indecomposable right bounded complex $\dot{C} \in K(B)$ with the following properties:*

- *(i) $\dot{C}$ is not left bounded, every non–zero component of $\dot{C}$ is indecomposable and $\operatorname{Hom}_{K(B)}(\dot{T}, \dot{C}[i]) = 0$. for every $i \in \mathbb{Z}$.*
- *(ii) $\dot{C}$ has infinitely many non–zero homology modules (all indecomposable) and one of them is not simple.*
- *(iii) T is an almost complete tilting module.*

CONSTRUCTION. Let B denote the K–algebra given by the quiver



with relations $ba = 0$ and $cb = 0$, and let T denote the injective module $\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$.

Then 3 is the unique indecomposable module belonging to $\operatorname{Ker}\operatorname{Hom}_B(T, -)$ and

$\mathrm{Ext}^3_B\left(\begin{smallmatrix}1\\3\end{smallmatrix}, 3\right) \simeq \mathrm{Ext}^2_B(2,3) \simeq \mathrm{Ext}^1_B\left(\begin{smallmatrix}1\\2\end{smallmatrix}, 3\right) \neq 0$. Thus T is a faithful large partial 3–tilting module. Now let $\dot{C}$ denote the indecomposable complex

$$\cdots \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} \longrightarrow 0\ .$$

Then (ii) and (iii) holds, and 2 is not a composition factor of $H(\dot{C})$. By [**D2**, Lemma 1], this implies that

(1) $\quad \mathrm{Hom}_{K(B)}\left(\dot{\begin{smallmatrix}2\\1\\2\end{smallmatrix}}, \dot{C}[i]\right) = 0$ for every i.

Let $\dot{\alpha}$ be a morphism described by

(2)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & 0\\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle x}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle y}\downarrow & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots
\end{array}$$

and let $\dot{\beta}$ be a morphism described by

(3)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & 0\\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle x'}\downarrow & & {\scriptstyle \phi'}\downarrow & & {\scriptstyle y'}\downarrow & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & 0\ .
\end{array}$$

Then ϕ and ϕ' are not automorphisms, otherwise the second squares in (2) and (3) would not commute. It is now easy, proceeding from left to right, to show that $\dot{\alpha}$ and $\dot{\beta}$ are homotopic to zero. We also note that all the morphisms of the form

(4)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & 0\\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle x}\downarrow & & {\scriptstyle y}\downarrow & & & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} & \longrightarrow & 0 & &
\end{array}$$

and

$$(5)\qquad \begin{array}{ccccccc} 0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \dots \\ & & \big\downarrow{\scriptstyle x} & & & & \\ \dots & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0 & & \end{array}$$

are homotopic to zero. Finally, a picture of the form

$$(6)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \dots & & \\ & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle x} & & & & \\ \dots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0 & & \end{array}$$

describes a morphism only if $x = 0$. Consequently, we deduce from (2),...,(6) that

$$(7)\qquad \operatorname{Hom}_{K(B)}\left(\dot{\begin{smallmatrix}1\\3\end{smallmatrix}}, \dot{C}[i]\right) = 0 \text{ for every } i.$$

This remark and (1) show that (i) holds. □

The next example shows that the complex $\dot{C}$ of Example 2 is somehow a "limit" of bounded complexes, all orthogonal to $\dot{T}$ together with all their shifts.

EXAMPLE 3 (Left cancellation; left cancellation and central addition). *For any $n \geq 3$ we can find B and T as in the hypotheses of Example 2, and an indecomposable bounded complex $\dot{D} \in K(B)$ with the following properties:*

- *(i) $\dot{D}$ has exactly n components different from zero (all indecomposable) and $\operatorname{Hom}_{K(B)}\left(\dot{T}, \dot{D}[i]\right) = 0$ for every $i \in \mathbb{Z}$.*
- *(ii) $\dot{D}$ has $n-1$ non–zero homology modules (all indecomposable) and one of them is not simple.*

CONSTRUCTION. Fix any $n \geq 3$, and let $\dot{D}$ be the indecomposable complex

$$0 \longrightarrow \begin{smallmatrix}1\\2\,3\end{smallmatrix} \longrightarrow \underbrace{\begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \dots \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix}}_{n-2} \longrightarrow \begin{smallmatrix}1\\2\,3\end{smallmatrix} \longrightarrow 0 \ .$$

Then (ii) holds, and we deduce from [**D2**, Lemma 1] that

$$(1)\qquad \mathrm{Hom}_{K(B)}\left(\begin{smallmatrix}\dot{2}\\1\\2\end{smallmatrix}, \dot{D}[i]\right)=0 \text{ for every } i.$$

Let $\dot\alpha$ be the morphism described by

$$(2)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle z} & & \\
0 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & \cdots
\end{array}$$

where the second line contains an indecomposable complex, and we have either $P=Q=\begin{smallmatrix}2\\1\\2\end{smallmatrix}$, or $P=\begin{smallmatrix}2\\1\\2\end{smallmatrix}$, $Q=\begin{smallmatrix}1\\2\,3\end{smallmatrix}$, or $P=\begin{smallmatrix}1\\2\,3\end{smallmatrix}$, $Q=0$. Then ϕ cannot be injective and we always have $z=0$. Proceeding from left to right, we conclude that $\dot\alpha$ is homotopic to zero. Next, let $\dot\beta$ be the morphism described by

$$(3)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0\\
 & & & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle y} & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & P & \longrightarrow & \cdots
\end{array}$$

where $P=\begin{smallmatrix}1\\2\,3\end{smallmatrix}$ if $n=3$, and $P=\begin{smallmatrix}2\\1\\2\end{smallmatrix}$ if $n>3$. Then $x=0$, ϕ is not an automorphism and $y=0$. In this case, proceeding either from left to right or conversely, we conclude that $\dot\beta$ is homotopic to zero. Finally, let $\dot\gamma$ be the morphism described by

$$(4)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0\\
 & & & & & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \\
 & & & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots
\end{array}\quad .$$

Then x (resp. y) factors on the right (resp. left) through an endomorphism of $\begin{smallmatrix}1\\2\,3\end{smallmatrix}$. Therefore $\dot\gamma$ is homotopic to zero.

On the other hand, for any morphism of the form

$$(5)\qquad \begin{array}{ccccccc} \cdots \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & 0 & \\ & & & \Big\downarrow{\scriptstyle x} & & & \\ & 0 & \longrightarrow & \begin{smallmatrix}1\\2\,3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow \cdots \end{array}$$

we obviously have $x = 0$. Combining the above remarks with (2),. . . ,(5) and (6) in Example 2, we get

$$(6)\qquad \operatorname{Hom}_{K(B)}\left(\begin{smallmatrix}\dot{1}\\3\end{smallmatrix}, \dot{D}[i]\right) = 0 \text{ for every } i.$$

Hence (i) follows from (1) and (6). □

One can prove that, up to shift, the right bounded complex $\dot{C}$ and the bounded complexes $\dot{D}$, constructed in Examples 2 and 3 respectively, are the unique indecomposable right bounded complexes satisfying condition (i) of the same examples.

The next example describes a completely different situation.

Example 4 (Lego–type construction). *There are a K–algebra R, a large partial 5–tilting module T, and complexes $\dot{C}(m)$ with $1 \le m \le \aleph_0$ with the following properties:*

(i) $T = P \oplus S$ with P indecomposable projective–injective, S simple injective, but T is not an almost complete tilting module.

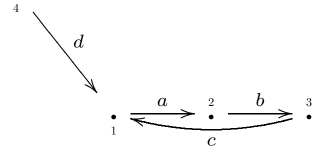
(ii) Every non–zero component of $\dot{C}(m)$ is an indecomposable projective module, and exactly m components of $\dot{C}(m)$ are equal to P.

(iii) $\dot{C}(m)$ is orthogonal to $\dot{T}$, but no proper subcomplex of $\dot{C}(m)$ is orthogonal to $\dot{T}$.

(iv) Every component of $\dot{C}(m)$ is injective if and only if $m \ge 2$.

(v) Up to shift, there exist $2^{\aleph_0}$ indecomposable right bounded complexes orthogonal to $\dot{T}$ and with indecomposable non–zero components.

Construction. Let R be the K–algebra given by the quiver



with relations $cba = 0$ and $ad = 0$. Let T denote the module $\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \oplus 4$. Then T is injective, $p\dim T = 5$ and the indecomposable modules belonging to $\operatorname{Ker}\operatorname{Hom}_R(T,-)$ are 1 , 3 , $\begin{smallmatrix}3\\1\end{smallmatrix}$, $\begin{smallmatrix}4\\1\end{smallmatrix}$ and $\begin{smallmatrix}3\,4\\1\end{smallmatrix}$. Since

$$\operatorname{Ext}^5_R(4,-) \simeq \operatorname{Ext}^4_R(1,-) \simeq \operatorname{Ext}^3_R(\begin{smallmatrix}2\\3\end{smallmatrix},-) \simeq \operatorname{Ext}^2_R(\begin{smallmatrix}1\\2\end{smallmatrix},-) \simeq \operatorname{Ext}^1_R(3,-),$$

it immediately follows that T is large. Hence (i) holds.

Next, let $\dot{X}$ be the indecomposable complex

$$(1) \qquad 0 \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow 0 \ .$$

Finally, let $\dot{Y}$ be the indecomposable complex

$$(2) \qquad 0 \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\1\end{smallmatrix} \longrightarrow 0 \ .$$

We first note that $\dot{4}$ is of the form

$$(3) \qquad 0 \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\1\end{smallmatrix} \longrightarrow 0 \ .$$

Consequently, we obtain $\dot{Y}$ from $\dot{4}$ after left cancellation of two components. On the other hand $\dot{T}$ is a partial tilting complex. Therefore any morphism from $\dot{4}$ to a shift of $\dot{Y}$ of the form

$$(4) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \dots \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ \dots & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is homotopic to zero.

Dually, let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{Y}$ of the form

$$(5) \qquad \begin{array}{ccccccccc} \dots & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \overset{d}{\longrightarrow} & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} \longrightarrow 0 \ . \\ & & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 0} \\ & & 0 & \longrightarrow & L & \overset{d'}{\longrightarrow} & M & \longrightarrow & N \longrightarrow \dots \end{array}$$

If $L = M = 0$ (resp. $L = 0$, and $M = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$), then we have $\phi = \psi = 0$. If $L = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ and $M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, then there is a morphism h such that $\phi = h \circ d$ and $\psi = d' \circ h$. Hence, $\dot{\alpha}$

is always homotopic to zero. Let now $\dot\beta$ be a morphism of the form

$$(6)\qquad \begin{array}{ccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0
\end{array}$$

Proceeding from right to left, we obtain $c = b = a = 0$. Therefore $\dot\beta$ is homotopic to zero. Next, let $\dot\gamma$ be a morphism of the form

$$(7)\qquad \begin{array}{ccccccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle l}\downarrow & & & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 & &
\end{array}\quad .$$

In this case, proceeding from left to right, the existence of suitable endomorphisms of $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $\begin{smallmatrix}4\\1\end{smallmatrix}$ shows that $\dot\gamma$ is homotopic to zero. Finally, let $\dot\delta$ be a morphism of the form

$$(8)\qquad \begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \xrightarrow{\;d\;} & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \xrightarrow{\;d'\;} & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0
\end{array}\quad .$$

Proceeding from left to right, we have $m = 0$, $\sigma^2 = 0$ and $p = q = 0$. Hence there is a morphism r such that $r \circ d = m$ and $d' \circ r = \sigma$. Therefore $\dot\delta$ is homotopic to zero. This observation completes the proof that

(9) $\quad \mathrm{Hom}_{K(R)}(\dot 4, \dot Y[i]) = 0$ for every $i \in \mathbb{Z}$.

Now, let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{X}$ of the form

$$(10)\qquad \begin{array}{ccccccccccc} \cdots & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\ & & & & \downarrow f & & \downarrow g & & \downarrow 0 & & \\ & & 0 & \longrightarrow & L & \xrightarrow{d'} & M & \longrightarrow & N & \longrightarrow & \cdots \end{array} .$$

If $L = M = 0$ (resp. $L = 0$, $M = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$), then we have $f = g = 0$. If $L = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ and $M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, then there is a morphism s such that $f = s \circ d$ and $g = d' \circ s$. Consequently, $\dot{\alpha}$ is homotopic to zero. Let $\dot{\beta}$ be a morphism from $\dot{4}$ to a shift of $\dot{X}$ of the form

$$(11)\qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \cdots . \\ & & \downarrow r & & \downarrow \sigma & & \downarrow t & & & & \\ \cdots & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 & & \end{array}$$

If $M = N = 0$ and $L = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ (resp. $L = M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $N = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$), then we have $\sigma = t = 0$. If $L = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, $M = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ and $N = 0$, then $t = 0$ and $\sigma^2 = 0$. Proceeding from left to right, we conclude that $\dot{\beta}$ is homotopic to zero. Now let $\dot{\gamma}$ be a morphism of the form

$$(12)\qquad \begin{array}{ccccccccccccc} 0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \cdots \\ & & \downarrow p & & \downarrow q & & \downarrow v & & \downarrow w & & & & \\ \cdots & \longrightarrow & I & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & 0 & & \end{array}$$

with I indecomposable projective–injective. Proceeding from left to right, we see that $\dot\gamma$ is homotopic to zero. Let $\dot\delta$ be a morphism of the form

$$
(13)\qquad
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 0} & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \xrightarrow{d'} & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & I & \longrightarrow & \cdots
\end{array}
$$

with I indecomposable projective–injective. In this case we have $a = 0$, $\tau^2 = 0$ and $b = 0$. Hence there is a morphism t such that $t \circ d = 0 = a$ and $d' \circ t = \tau$. Thus $\dot\delta$ is homotopic to zero. Finally, let $\dot\epsilon$ denote a morphism of the form

$$
(14)\qquad
\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 0} & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & I & \longrightarrow & L & \longrightarrow & \cdots
\end{array}
$$

with I and L projective–injective and I indecomposable. We first note that $\phi^2 = 0$, $\psi^2 = 0$ and $q = 0$. Proceeding from left to right, we conclude that $\dot\epsilon$ is homotopic to zero. This remark completes the proof that

$$
(15)\qquad \mathrm{Hom}_{K(R)}(\dot 4, \dot X[i]) = 0 \text{ for every } i \in \mathbb{Z}.
$$

Putting (9), (15) and [**D2**, Lemma1] together, we conclude that $\dot Y$ and $\dot X$ satisfy (ii) and (iii) with $m = 1$ and $m = 2$ respectively. Now fix some $m \geq 3$, and let $\dot Z$ denote the indecomposable complex

$$
0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0
$$

Let $\dot\alpha$ be a morphism of the form

$$
(16)\qquad
\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & U & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} & \longrightarrow & W & \longrightarrow & I & \longrightarrow & \cdots
\end{array}
$$

where U, W and I (resp U and W) are projective–injective (resp. indecomposable) modules. Then f and g factor through a morphism from $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ to U. On the other hand, we clearly have $l^2 = 0$ and $s = 0$. Hence, proceeding from left to right, we see that $\dot{\alpha}$ is homotopic to zero. Since we know from (15) that $\dot{X}$ is orthogonal to $\dot{T}$, we deduce from (12), (13), (14), (16) and [**D2**, Lemma 1] that the complex $\dot{Z}$ satisfies (ii) and (iii) for the natural number $m \geq 3$. Let now $\dot{U}$ denote the indecomposable complex

$$\cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0.$$

Then any morphism form $\dot{4}$ to a shift of $\dot{U}$ induces a morphism from $\dot{4}$ to a bounded complex of the form

$$0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{n} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0. \qquad \text{for some } n \geq 2.$$

This remark and [**D2**, Lemma 1] guarantee that $\dot{U}$ satisfies (ii) and (iii) for $m = \aleph_0$.

Since (iv) clearly holds, it remains to prove (v). To see this, we first note that there are $2^{\aleph_0}$ sequences $\sigma = (m(i))_{i\geq 1}$ of integers $m(i) \geq 2$.

Next, for any σ as above and any n-uple $s = (m(1), \ldots, m(n))$ where n and the $m(i)$'s are natural numbers ≥ 2, we denote by $\square$ the indecomposable complexes

$$\cdots \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(2)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \xrightarrow{\delta} \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(1)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \quad \text{and}$$

$$0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(n)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(1)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$$

respectively. Keeping the above notation, let $\square \to 0$ denote one of the indecomposable complexes $\dot{C}(\sigma)$ or $\dot{C}(s)$. Next, let $\dot{C}(\sigma, 1)$ or $\dot{C}(s, 1)$ denote the indecomposable complex

$$\square \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}4\\1\end{smallmatrix} \to 0.$$

Finally, let $\blacksquare$ denote an indecomposable bounded complex such that $0 \to \blacksquare$ is a bounded complex, say $\dot{B}$, of the form $\dot{C}(s)$ or $\dot{C}(s, 1)$ for some s. Then we denote by $\dot{B}(\aleph_0)$ the indecomposable complex $\cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \blacksquare$. Let $\dot{C}$ be one of the complexes just defined, and let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{C}$. Then $\dot{\alpha}$ induces a morphism, say $\dot{\beta}$, from $\dot{4}$ to a bounded complex, say $\dot{D}$, as above, obtained by "glueing together" finitely many bounded complexes, say $\dot{C}(n(1)), \ldots, \dot{C}(n(r))$, as indicated by the following picture

$$\begin{array}{cccc} \cdots\cdots & \cdots\cdots & \overset{\delta}{\frown\!\!\to} & \\ \hline \multicolumn{1}{|c|}{\dot{C}(n(r))} & \multicolumn{1}{c|}{\ldots\ldots} & \multicolumn{1}{c|}{\dot{C}(n(2))} & \multicolumn{1}{c|}{\dot{C}(n(1))} \\ \hline \end{array} .$$

We also note that the connecting maps δ are morphisms $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to P$ with simple image.

On the other hand, the canonical inclusion $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ is the unique arrow in $\dot{4}$ ending in $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$. Moreover, if $f : \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $d' : \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ are morphisms, then we have $d' \circ f = 0$. Consequently, finitely many applications of [**D3**, Lemma 5] guarantee that $\dot{D}$ is orthogonal to $\dot{4}$. Hence $\dot{\beta}$ is homotopic to zero, and so $\dot{\alpha}$ has the same property. Therefore $\dot{C}$ is orthogonal to $\dot{4}$. Moreover, the definition of $\dot{C}$ and [**D2**, Lemma 1] imply that $\dot{C}$ is orthogonal to $\dot{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}$. This observation completes the proof of (v). □

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