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Niho Bent Functions and Subiaco Hyperovals

Tor Helleseth, Alexander Kholosha, and Sihem Mesnager

Abstract. In this paper, the relation between binomial Niho bent functions discovered by Dobbertin *et al.* and o-polynomials that give rise to the Subiaco class of hyperovals is found. This allows to expand the original class of bent functions in the case when $m \equiv 2 \pmod 4$. It is also proven that one of the earlier discovered sporadic Niho bent functions, up to EA-equivalence, belongs to the known infinite class.

1. Introduction and Preliminaries

Boolean functions of n variables are binary functions over the Galois field $\mathbb{F}_{2^n}$ (or over the vector space $\mathbb{F}_2^n$ of all binary vectors of length n). In this paper, we shall always endow this vector space with the structure of a field, thanks to the choice of a basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$. Boolean functions are used in the pseudo-random generators of stream ciphers and play a central role in their security.

Bent functions were introduced by Rothaus [**11**] in 1976. These are Boolean functions of an even number of variables n, that are maximally nonlinear in the sense that their Walsh transform takes precisely the values $\pm 2^{n/2}$. This corresponds to the fact that their Hamming distance to all affine functions is optimal. Bent functions have also attracted a lot of research interest because of their relations to coding theory and applications in cryptography. Despite their simple and natural definition, bent functions turned out to admit a very complicated structure in general. On the other hand, many special explicit constructions are known. Distinguished are primary constructions giving bent functions from scratch and secondary ones building new bent functions from one or several given bent functions. These constructions often look simpler when written in their bivariate representation but, of course, they also have an equivalent univariate form (see Subsection 1.1).

It is well known that some of the explicit constructions belong to the two general families of bent functions which are the original Maiorana-McFarland [**8**] and the Partial Spreads classes. It was in the early seventies when Dillon in his thesis [**5**] introduced the two above mentioned classes plus the third one denoted by H, where bentness is proven under some conditions which were not obvious to achieve (in this class, Dillon was able to exhibit only those functions belonging, up to the affine equivalence, to the Maiorana-McFarland class). He defined the functions in class H in their bivariate representation but they can also be seen in the univariate form as found recently by Carlet and Mesnager [**3**]. They extended

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the class H to a slightly larger class denoted by $\mathcal{H}$ (see Subsection 1.2), also defined in bivariate representation, and observed that this class contains all bent functions of the, so called, Niho type which were introduced in [**6**] by Dobbertin *et al.* (see Subsection 1.3).

1.1. Trace representation, Boolean functions in univariate and bivariate forms. For any positive integer k and any r dividing k, the trace function $\mathrm{Tr}_r^k()$ is the mapping from $\mathbb{F}_{2^k}$ to $\mathbb{F}_{2^r}$ defined by

$$\mathrm{Tr}_r^k(x) := \sum_{i=0}^{\frac{k}{r}-1} x^{2^{ir}} = x + x^{2^r} + x^{2^{2r}} + \cdots + x^{2^{k-r}} \ .$$

In particular, the *absolute trace* over $\mathbb{F}_{2^k}$ is the function $\mathrm{Tr}_1^k(x) = \sum_{i=0}^{k-1} x^{2^i}$. Recall that the trace function satisfies the transitivity property $\mathrm{Tr}_1^k = \mathrm{Tr}_1^r \circ \mathrm{Tr}_r^k$. From now on assume n is even and $n = 2m$. For any set E, denote $E \setminus \{0\}$ by E^*.

The *univariate representation* of a Boolean function is defined as follows: we identify $\mathbb{F}_2^n$ (the n-dimensional vector space over $\mathbb{F}_2$) with $\mathbb{F}_{2^n}$ and consider the arguments of f as elements in $\mathbb{F}_{2^n}$. An inner product in $\mathbb{F}_{2^n}$ is $x \cdot y = \mathrm{Tr}_1^n(xy)$. There exists a unique univariate polynomial $\sum_{i=0}^{2^n-1} a_i x^i$ over $\mathbb{F}_{2^n}$ that represents f (this is true for any vectorial function from $\mathbb{F}_{2^n}$ to itself). The algebraic degree of f is equal to the maximum 2-weight of an exponent having nonzero coefficient, where the 2-weight $w_2(i)$ of an integer i is the number of ones in its binary expansion. Hence, in the case of a bent function, all exponents i whose 2-weight is larger than m have a zero coefficient a_i. Moreover, f being Boolean, its univariate representation can be written in the form of $f(x) = \sum_{j \in \Gamma_n} \mathrm{Tr}_1^{o(j)}(a_j x^j)$, where Γ_n is the set of integers obtained by choosing one element in each cyclotomic coset of 2 modulo $2^n - 1$, $o(j)$ is the size of the cyclotomic coset containing j and $a_j \in \mathbb{F}_{2^{o(j)}}$. This representation is unique up to the choice of cyclotomic coset representatives. Function f can also be written in a non-unique way as $\mathrm{Tr}_1^n(P(x))$ where $P(x)$ is a polynomial over $\mathbb{F}_{2^n}$.

The *bivariate representation* of a Boolean function is defined as follows: we identify $\mathbb{F}_{2^n}$ with $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ and consider the argument of f as an ordered pair (x, y) of elements in $\mathbb{F}_{2^m}$. There exists a unique bivariate polynomial $\sum_{0 \le i,j \le 2^m-1} a_{i,j} x^i y^j$ over $\mathbb{F}_{2^m}$ that represents f. The algebraic degree of f is equal to

$$\max_{(i,j) \,|\, a_{i,j} \neq 0} (w_2(i) + w_2(j)) \ .$$

And f being Boolean, its bivariate representation can be written in the form of $f(x, y) = \mathrm{Tr}_1^m(P(x, y))$, where $P(x, y)$ is some polynomial of two variables over $\mathbb{F}_{2^m}$.

Let f be an n-variable Boolean function. Its *"sign" function* is the integer-valued function $\chi_f := (-1)^f$. The *Walsh transform* of f is the discrete Fourier transform of χ_f whose value at point $w \in \mathbb{F}_{2^n}$ is defined by

$$\hat{\chi}_f(w) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \mathrm{Tr}_1^n(wx)} \ .$$

Definition 1.1. For even n, a Boolean function f in n variables is said to be bent if for any $w \in \mathbb{F}_{2^n}$ we have $\hat{\chi}_f(w) = \pm 2^{\frac{n}{2}}$.

1.2. Class $\mathcal{H}$ of Bent Functions. In his thesis [**5**], Dillon introduced the class of bent functions denoted by H. The functions in this class are defined in their bivariate form as

$$f(x,y) = \mathrm{Tr}_1^m\big(y + xG(yx^{2^m-2})\big) \ ,$$

where $x, y \in \mathbb{F}_{2^m}$ and G is a permutation of $\mathbb{F}_{2^m}$ such that $G(x)+x$ does not vanish and for any $\beta \in \mathbb{F}_{2^m}^*$, the function $G(x)+\beta x$ is 2-to-1 (i.e., the pre-image of any element of $\mathbb{F}_{2^m}$ is either a pair or the empty set). As observed by Carlet and Mesnager [**3**, Proposition 1], this class can be slightly extended into a class $\mathcal{H}$ defined as the set of (bent) functions g satisfying

$$g(x,y) = \begin{cases} \mathrm{Tr}_1^m\left(xH\left(\frac{y}{x}\right)\right), & \text{if} \quad x \neq 0 \\ \mathrm{Tr}_1^m(\mu y), & \text{if} \quad x = 0 \ , \end{cases} \tag{1.1}$$

where $\mu \in \mathbb{F}_{2^m}$ and H is a mapping from $\mathbb{F}_{2^m}$ to itself satisfying the following necessary and sufficient conditions

$$G : z \mapsto H(z) + \mu z \text{ is a permutation on } \mathbb{F}_{2^m} \tag{1.2}$$

$$z \mapsto G(z) + \beta z \text{ is 2-to-1 on } \mathbb{F}_{2^m} \text{ for any } \beta \in \mathbb{F}_{2^m}^* \ . \tag{1.3}$$

As proved in [**3**, Lemma 13], condition (1.3) implies condition (1.2) and, thus, is necessary and sufficient for g being bent. It also follows that polynomials $G(z)$ satisfying (1.3) are so-called o-polynomials (oval polynomials) over $\mathbb{F}_{2^m}$ (the additional properties of $G(0) = 0$ and $G(1) = 1$ can be achieved by taking $\frac{G(z)+G(0)}{G(1)+G(0)}$ instead of $G(z)$). o-polynomials arise from hyperovals and define them. Note that class $\mathcal{H}$ contains all bent functions with the property that their restriction to the multiplicative cosets of $\mathbb{F}_{2^m}$ is linear.

1.3. Niho bent functions. Recall that a positive integer d (always understood modulo 2^n-1) is said to be a *Niho exponent* and $t \mapsto t^d$ is a *Niho power function* if the restriction of t^d to $\mathbb{F}_{2^m}$ (and, therefore, to its cosets $u\mathbb{F}_{2^m}$) is linear or, in other words, $d \equiv 2^j \pmod{2^m-1}$ for some $j < n$. As we consider $\mathrm{Tr}_1^n(at^d)$ with $a \in \mathbb{F}_{2^n}$, without loss of generality, we can assume that d is in the normalized form, i.e., with $j = 0$. Then we have a unique representation $d = (2^m-1)s + 1$ with $2 \le s \le 2^m$. The simplest example of an infinite class of Niho bent functions is the quadratic function $\mathrm{Tr}_1^m(at^{2^m+1})$ with $a \in \mathbb{F}_{2^m}^*$. Other known classes are:

- Three examples from [**6**] are binomials of the form $f(t) = \mathrm{Tr}_1^n(\alpha_1 t^{d_1} + \alpha_2 t^{d_2})$, where $2d_1 = 2^m + 1 \in \mathbb{Z}/(2^n-1)\mathbb{Z}$ and $\alpha_1, \alpha_2 \in \mathbb{F}_{2^n}^*$ are such that $(\alpha_1 + \alpha_1^{2^m})^2 = \alpha_2^{2^m+1}$. Equivalently, denoting $a = (\alpha_1 + \alpha_1^{2^m})^2$ and $b = \alpha_2$ we have $a = b^{2^m+1} \in \mathbb{F}_{2^m}^*$ and $f(t) = \mathrm{Tr}_1^m(at^{2^m+1}) + \mathrm{Tr}_1^n(bt^{d_2})$. Note that if $b = 0$ and $a \neq 0$ then f is also bent but becomes quadratic equal to the function mentioned above. The possible values of d_2 are:
  $d_2 = (2^m-1)3 + 1$ (with the condition that, if $m \equiv 2 \pmod 4$ then b is the fifth power of an element in $\mathbb{F}_{2^n}$; otherwise, b can be any nonzero element),
  $4d_2 = (2^m-1) + 4$ (with the condition that m is odd),
  $6d_2 = (2^m-1) + 6$ (with the condition that m is even).
  As observed in [**6, 2**], these functions have algebraic degree m, 3 and m respectively.

- An extension by Leander and Kholosha [**7**] of the second class from [**6**] has the form of

$$\mathrm{Tr}_1^n\Big(at^{2^m+1}+\sum_{i=1}^{2^{r-1}-1}t^{(2^m-1)\frac{i}{2^r}+1}\Big) \tag{1.4}$$

with $r>1$ satisfying $\gcd(r,m)=1$ and $a\in\mathbb{F}_{2^n}$ is such that $a+a^{2^m}=1$.
- Functions in a bivariate form obtained from the known o-polynomials (see [**3**, Section 6]).

As was noted in [**6**], all cases except for $d_2=(2^m-1)3+1$ with $m\equiv 2 \pmod 4$ give $\gcd(d_2,2^n-1)=1$ and in the remaining case, $\gcd(d_2,2^n-1)=5$. Therefore, having the condition on b, it can be assumed, without loss of generality, that $b=1$ (this is achieved by substituting t with $b^{-1/d_2}t$). However, in Subsection 3.2, we show that even in the case when $m\equiv 2 \pmod 4$ the value of b can be taken arbitrary under the condition that $a=b^{2^m+1}$.

Since the restriction to $u\mathbb{F}_{2^m}$ of these bent functions is linear, they all belong to $\mathcal{H}$. The question left open in [**6**] was finding the dual and checking if that was of the Niho type (possibly up to affine equivalence). In [**3, 2**] considered were bent functions from the second class (having degree 3) and multinomial (1.4). It was shown that corresponding o-polynomials are Frobenius mappings and dual functions were found that turned out not to be in the Niho class. Moreover, these cases give bent functions in the completed Maiorana-McFarland class. In this paper, we find o-polynomials that arise from the first class of binomial Niho bent functions. However, it still remains to determine the dual. The third class is completely open.

2. Subiaco Hyperovals

Here we define o-polynomials that give rise to the Subiaco family of hyperovals.

THEOREM 2.1 (Theorems 3-5 [**4**]). *Take polynomials $f(x)$ and $g(x)$ and for any $s\in\mathbb{F}_{2^m}$ define*

$$f_s(x)=\frac{f(x)+esg(x)+s^{1/2}x^{1/2}}{1+es+s^{1/2}}\ ,$$

where $e\in\mathbb{F}_{2^m}$ with $\mathrm{Tr}_1^m(e)=1$ is defined further. Then in the following cases, $g(x)$ and $f_s(x)$ are o-polynomials:

(i) *if m is odd then take $e=1$ and*

$$f(x)=\frac{x^2+x}{(x^2+x+1)^2}+x^{\frac12}\quad\text{and}\quad g(x)=\frac{x^4+x^3}{(x^2+x+1)^2}+x^{\frac12}\ ;$$

(ii) *if $m\equiv 2 \pmod 4$ then take $e=w\in\mathbb{F}_{2^m}$ with $w^2+w+1=0$ and*

$$f(x)=\frac{x^2(x^2+wx+w)}{(x^2+wx+1)^2}+w^2x^{\frac12}\quad\text{and}\quad g(x)=\frac{wx(x^2+x+w^2)}{(x^2+wx+1)^2}+w^2x^{\frac12}\ ;$$

(iii) *for any m, take $e=\frac{w^2+w^5+w^{1/2}}{w(1+w+w^2)}$ where $w\in\mathbb{F}_{2^m}$ with $w^2+w+1\neq 0$ and $\mathrm{Tr}_1^m(1/w)=1$, and*

$$f(x)=\frac{w^2(x^4+x)+w^2(1+w+w^2)(x^3+x^2)}{(x^2+wx+1)^2}+x^{\frac12}\quad\text{and}$$

$$g(x)=\frac{w^4x^4+w^3(1+w^2+w^4)x^3+w^3(1+w^2)x}{(w^2+w^5+w^{1/2})(x^2+wx+1)^2}+\frac{w^{1/2}}{w^2+w^5+w^{1/2}}x^{\frac12}\ .$$

It is useful to have the following explicit expressions for $f_s(x)$ in each of the cases considered. Denote $1+es+s^{\frac{1}{2}}=A$, then $f_s(x)$ is equal to

$$\frac{s(x^4+x^3)+x^2+x}{A(x^2+x+1)^2}+x^{\frac{1}{2}}\ , \quad m \text{ odd} \tag{2.1}$$

$$A^{-1}\left(\frac{x^4+w(sw+1)(x^3+x^2)+swx}{(x^2+wx+1)^2}+(w^2+s+s^{\frac{1}{2}})x^{\frac{1}{2}}\right)\ , \quad m/2 \text{ odd} \tag{2.2}$$

$$\begin{aligned}\Bigg(w^2\frac{(1+sw+w^2)x^4+(1+w+w^2)^2(sx^3+x^2)+(s+w+sw^2)x}{(1+w+w^2)(x^2+wx+1)^2}\\ +\left(s^{\frac{1}{2}}+\frac{s+1}{w^{1/2}(1+w+w^2)}\right)x^{\frac{1}{2}}\Bigg)(e+es+s^{\frac{1}{2}})^{-1}\ , \quad m \text{ arbitrary}\ ,\end{aligned} \tag{2.3}$$

where in (2.3), we changed $s+1$ for s in the original definition of $f_s(x)$. Note that for m odd, taking $w=1$ in (2.3) results in (2.1).

In each of the cases listed above, the set $(f(x),g(x),a)$ defines a q-clan. On the other hand, by [**4**, Theorem 1], the existence of the q-clan is equivalent to the property that $g(x)$ is an o-polynomial and $f_s(x)$ is an o-polynomial for any $s\in\mathbb{F}_{2^m}$. In [**10**], it was shown that the Subiaco construction provides two inequivalent hyperovals if $m\equiv 2 \pmod 4$ and one hyperoval otherwise.

3. Bent Functions from Subiaco Hyperovals

Take the following function over $\mathbb{F}_{2^n}$

$$f(t)=\mathrm{Tr}_1^m(at^{2^m+1})+\mathrm{Tr}_1^n(bt^{3(2^m-1)+1})\ ,$$

where $a\in\mathbb{F}_{2^m}^*$ and $b\in\mathbb{F}_{2^n}^*$ are such that $b^{2^m+1}=a$. Let (u,v) be a basis of $\mathbb{F}_{2^n}$ as a two-dimensional vector space over $\mathbb{F}_{2^m}$. Then for any $x,y\in\mathbb{F}_{2^m}$, we obtain $f(ux+vy)$ having the form of (1.1) with

$$\begin{aligned}H(z)&=a^{\frac{1}{2}}(u+vz)^{\frac{2^m+1}{2}}+\mathrm{Tr}_m^n\big(b(u+vz)^{3(2^m-1)+1}\big)\\ \mu&=a^{\frac{1}{2}}v^{\frac{2^m+1}{2}}+\mathrm{Tr}_m^n(bv^{3(2^m-1)+1})\ .\end{aligned}$$

Here all notation are from Subsection 1.2. Therefore, with $z\in\mathbb{F}_{2^m}$,

$$G(z)=a^{\frac{1}{2}}v^{\frac{2^m+1}{2}}z+a^{\frac{1}{2}}(u+vz)^{\frac{2^m+1}{2}}+\mathrm{Tr}_m^n\big(b(v^{3(2^m-1)+1}z+(u+vz)^{3(2^m-1)+1})\big)\ .$$

Further, we have that

$$(u+vz)^{\frac{2^m+1}{2}}=u^{\frac{2^m+1}{2}}+\big(\mathrm{Tr}_m^n(u^{2^m}v)\big)^{\frac{1}{2}}z^{\frac{1}{2}}+(vz)^{\frac{2^m+1}{2}}$$

and since $z\in\mathbb{F}_{2^m}$,

$$a^{\frac{1}{2}}v^{\frac{2^m+1}{2}}z+a^{\frac{1}{2}}(u+vz)^{\frac{2^m+1}{2}}=a^{\frac{1}{2}}u^{\frac{2^m+1}{2}}+a^{\frac{1}{2}}\big(\mathrm{Tr}_m^n(u^{2^m}v)\big)^{\frac{1}{2}}z^{\frac{1}{2}}\ . \tag{3.1}$$

Now expand the term $(u+vz)^{3(2^m-1)+1}$. To this end, note that $3(2^m-1)+1=2^{m+1}-1+2^m-1$. Then

$$\begin{aligned}(u+vz)^{3(2^m-1)+1}&=(u+vz)^{2^{m+1}-1}(u+vz)^{2^m-1}\\ &=\sum_{j=0}^{2^{m+1}-1}u^{2^{m+1}-1-j}(vz)^j\sum_{j=0}^{2^m-1}u^{2^m-1-j}(vz)^j\\ &=\sum_{i=0}^{3\cdot 2^m-2}(N_i \bmod 2)u^{3\cdot 2^m-2-i}(vz)^i\ ,\end{aligned}$$

where $N_i = |E_i|$ and

$$E_i = \{(j_1, j_2) \mid j_1 + j_2 = i,\ 0 \le j_1 \le 2^{m+1} - 1,\ 0 \le j_2 \le 2^m - 1\}\ .$$

We compute N_i by enumerating the elements of E_i as follows:

- for $0 \le i \le 2^m - 1$, we have $E_i = \{(i-j, j) \mid 0 \le j \le i\}$ and $N_i = i + 1$;
- for $2^m \le i \le 2^{m+1} - 1$, we have $E_i = \{(i-j, j) \mid 0 \le j \le 2^m - 1\}$ and $N_i = 2^m$;
- for $2^{m+1} \le i \le 3 \cdot 2^m - 2$, we have $E_i = \{(i-j, j) \mid i - 2^{m+1} + 1 \le j \le 2^m - 1\}$ and $N_i = 3 \cdot 2^m - 1 - i$ (indeed, $j_1 + j_2 = i$ implies that $j_2 = i - j_1 \ge i - 2^{m+1} + 1$ since $j_1 \le 2^{m+1} - 1$).

Therefore $N_i \bmod 2 = 1$ if and only if $i = 2l$ with $0 \le l \le 2^{m-1} - 1$ or $i = 2^{m+1} + 2l$ with $0 \le l \le 2^{m-1} - 1$ and

$$\begin{aligned}
(u+vz)^{3(2^m-1)+1} &= \sum_{l=0}^{2^{m-1}-1} u^{3\cdot 2^m - 2 - 2l}(vz)^{2l} + \sum_{l=0}^{2^{m-1}-1} u^{3\cdot 2^m - 2 - 2^{m+1} - 2l}(vz)^{2^{m+1}+2l} \\
&\overset{(*)}{=} \sum_{l=0}^{2^{m-1}-1} u^{3\cdot 2^m - 2(l+1)}(vz)^{2l} + \sum_{l=0}^{2^{m-1}-1} u^{2^m - 2(l+1)} v^{2^{m+1}-2}(vz)^{2(l+1)} \\
&= \sum_{l=0}^{2^{m-1}-1} u^{3\cdot 2^m - 2(l+1)}(vz)^{2l} + \sum_{l=1}^{2^{m-1}} u^{2^m - 2l} v^{2^{m+1}-2}(vz)^{2l} \\
&= u^{3\cdot 2^m - 2} + (u^{3\cdot 2^m - 2} + u^{2^m} v^{2^{m+1}-2}) \sum_{l=1}^{2^{m-1}-1} \left(u^{-1}vz\right)^{2l} + v^{3\cdot 2^m - 2} z \\
&= u^{3\cdot 2^m - 2} + u^{2^m}(u^{2(2^m-1)} + v^{2(2^m-1)}) \left(1 + \frac{1 + (u^{-1}vz)^{2^m}}{1 + u^{-2}v^2 z^2}\right) + v^{3\cdot 2^m - 2} z \\
&= u^{2^m} v^{2(2^m-1)} + u^{2^m}(u^{2(2^m-1)} + v^{2(2^m-1)})(1 + u^{-1}vz)^{2^m - 2} + v^{3\cdot 2^m - 2} z \\
&= u^{2^m} v^{2(2^m-1)} + u^2 (u^{2(2^m-1)} + v^{2(2^m-1)})(u + vz)^{2^m - 2} + v^{3\cdot 2^m - 2} z\ .
\end{aligned}$$

In the second sum after $(*)$, we used that $z^{2^{m+1}+2l} = (z^{2^m})^2 z^{2l} = z^2 z^{2l} = z^{2(l+1)}$. Finally, denoting

$$c = a^{\frac{1}{2}} u^{\frac{2^m+1}{2}} + \mathrm{Tr}_m^n(b u^{2^m} v^{2(2^m-1)})$$

and using (3.1), we obtain that

(3.2)
$$G(z) = c + a^{\frac{1}{2}} \left(\mathrm{Tr}_m^n(u^{2^m} v)\right)^{\frac{1}{2}} z^{\frac{1}{2}} + \mathrm{Tr}_m^n\left(b u^2 (u^{2(2^m-1)} + v^{2(2^m-1)})(u + vz)^{2^m - 2}\right)\ .$$

Now assume $v = 1$ and take $u \in \mathbb{F}_{2^n} \setminus \{1\}$ with $u^{2^m+1} = 1$ that means $u \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^m}$. Also denote $u + u^{2^m} = w \in \mathbb{F}_{2^m}^*$ and observe that $\mathrm{Tr}_1^m(1/w) = 1$ (since this is equivalent to $u^2 + wu + 1$ being irreducible over $\mathbb{F}_{2^m}$). Moreover, all $w \in \mathbb{F}_{2^m}$ with such a trace property are obtained in this way from u. Then $u^{2^m - 1} = w/u + 1$ and

$$\begin{aligned}
\mathrm{Tr}_m^n(u^{2^m} v) &= w \\
u^2\left(v^{2(2^m-1)} + u^{2(2^m-1)}\right) &= w^2\ .
\end{aligned}$$

Under these conditions, $c = a^{\frac{1}{2}} + \mathrm{Tr}_m^n(bu^{2^m})$ and

(3.3)

$$\begin{aligned}
G(z) &= c + (awz)^{\frac{1}{2}} + \frac{bw^2(u^{2^m}+z)}{(u+z)^2} + \frac{b^{2^m}w^2(u+z)}{(u^{2^m}+z)^2} \\
&= c + (awz)^{\frac{1}{2}} + w^2\frac{b(u+w+z)^3 + b^{2^m}(u+z)^3}{(u+z)^2(u+w+z)^2} \\
&= c + (awz)^{\frac{1}{2}} + w^2\frac{(b+b^{2^m})(u+z)^3 + bw(z^2+wz+u^{2^m+1}+w^2)}{(z^2+wz+u^{2^m+1})^2} \\
&\overset{(3.4)}{=} c + (awz)^{\frac{1}{2}} \\
&\quad + \frac{w^2(b+b^{2^m})(z^3+uz^2+u^2z) + bw^3(z^2+wz) + \mathrm{Tr}_m^n(b^{2^m}(u^5+u))}{(z^2+wz+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b^{2^m}u^5) + (awz)^{\frac{1}{2}} \\
&\quad + \frac{w^2(b+b^{2^m})(z^3+uz^2+u^2z) + bw^3(z^2+wz) + \mathrm{Tr}_m^n(b^{2^m}(u^5+u))(z^2+wz)^2}{(z^2+wz+1)^2} \\
&\overset{(3.5,3.6)}{=} a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b^{2^m}u^5) + (awz)^{\frac{1}{2}} \\
&\quad + \frac{\mathrm{Tr}_m^n(b^{2^m}(u^5+u))z^4 + \mathrm{Tr}_m^n(b)w^2z^3 + \mathrm{Tr}_m^n(b^{2^m}u^5)w^2z^2 + \mathrm{Tr}_m^n(b^{2^m}(u^4+1))z}{(z^2+wz+1)^2} \ .
\end{aligned}$$

Here we used the following identities

$$w^2(b+b^{2^m})u^3 + bw^3(1+w^2) = \mathrm{Tr}_m^n(b^{2^m}(u^5+u)) \ ; \tag{3.4}$$

$$u(b+b^{2^m}) + bw + \mathrm{Tr}_m^n(b^{2^m}(u^5+u)) = \mathrm{Tr}_m^n(b^{2^m}u^5) \ ; \tag{3.5}$$

$$w^2(b+b^{2^m})u^2 + bw^4 = \mathrm{Tr}_m^n(b^{2^m}(u^4+1)) \ . \tag{3.6}$$

Further, we consider three separate cases defined by the value of m.

3.1. m **odd.** In this case, take $u \in \mathbb{F}_4 \setminus \{0,1\}$. Note that $u \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^m}$ and $w = u + u^{2^m} = u + u^2 = 1$. Then, by (3.3),

$$\begin{aligned}
G(z) &= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(bu) + (az)^{\frac{1}{2}} + \frac{\mathrm{Tr}_m^n(b)(z^4+z^3) + \mathrm{Tr}_m^n(bu)(z^2+z)}{(z^2+z+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(bu) + (az)^{\frac{1}{2}} + a^{\frac{1}{2}}\frac{(B+B^{-1})(z^4+z^3) + (B^{-1}u^2+Bu)(z^2+z)}{(z^2+z+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(bu) + a^{\frac{1}{2}}f_s(z) \ ,
\end{aligned}$$

where $B = ba^{-\frac{1}{2}}$ with $B^{-1} = b^{2^m}a^{-\frac{1}{2}} = B^{2^m}$ since $a = b^{2^m+1}$. Polynomial $f_s(z)$ with $s = \frac{1+B^2}{u^2+B^2u} \in \mathbb{F}_{2^m}$ is an o-polynomial (2.1) (assuming $u^2 + B^2u \neq 0$). In the case when $u^2 = B^2u$ (or, equivalently, $b^{2^m-1} = u^2$) we obtain

$$G(z) = bu + buz^{\frac{1}{2}} + bu\frac{z^4+z^3}{(z^2+z+1)^2} = bu(1+g(z)) \ ,$$

since $a^{\frac{1}{2}} = (b^{2^m+1})^{\frac{1}{2}} = bu = b + b^{2^m}$ and where o-polynomial $g(z)$ comes from Theorem 2.1 Item (i).

Assuming $b^{2^m-1} \neq u^2$, note that equation $s = \frac{b^{2^m-1}+1}{b^{2^m-1}u^2+u}$ can be solved for the unknown $b \in \mathbb{F}_{2^n}^*$ for any $s \in \mathbb{F}_{2^m}$ since $s \neq u$. We conclude that the set of

bent functions with $b \in \mathbb{F}_{2^n}^*$ corresponds exactly to all o-polynomials described in Theorem 2.1 Item (i). This means that the existence of this set of bent functions is equivalent to the existence of the corresponding q-clan.

3.2. $m \equiv 2 \pmod 4$**.** In this case, take $u \in \mathbb{F}_{16} \setminus \mathbb{F}_4$ with $u^5 = 1$. Note that $u \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^m}$ and $u^{2^m+1} = u^5 = 1$. Then $u + u^{2^m} = u + u^4 = w \in \mathbb{F}_4 \subset \mathbb{F}_{2^m}$. Obviously, $w \neq 0$. It can be checked directly that u with the prescribed properties also satisfies $w \neq 1$ and, thus, $w^2 + w = 1$. There are *four* options for choosing u with these properties and both $w \in \mathbb{F}_4 \setminus \{0, 1\}$ can be obtained. Then, by (3.3),

$$\begin{aligned}
G(z) &= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + (awz)^{\frac{1}{2}} \\
&\quad + \frac{\mathrm{Tr}_m^n(b(u^4+1))z^4 + \mathrm{Tr}_m^n(b)w^2(z^3+z^2) + \mathrm{Tr}_m^n(b(u+1))z}{(z^2+wz+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + (awz)^{\frac{1}{2}} + \mathrm{Tr}_m^n(b(u^4+1))\frac{z^4 + w(sw+1)(z^3+z^2) + swz}{(z^2+wz+1)^2} \\
&\overset{(*)}{=} a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + (1 + ws + s^{\frac{1}{2}})\mathrm{Tr}_m^n(b(u^4+1))f_s(z) \ ,
\end{aligned}$$

where polynomial $f_s(z)$ with $s = \frac{w^2\mathrm{Tr}_m^n(b(u+1))}{\mathrm{Tr}_m^n(b(u^4+1))}$ is an o-polynomial (2.2) (assuming $\mathrm{Tr}_m^n(b(u^4+1)) \neq 0$). In the case when $\mathrm{Tr}_m^n(b(u^4+1)) = 0$ (or, equivalently, $b^{2^m-1} = (u+1)^3 = u^4$) we obtain

$$\begin{aligned}
G(z) &= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + (awz)^{\frac{1}{2}} + \frac{\mathrm{Tr}_m^n(b)w^2(z^3+z^2) + \mathrm{Tr}_m^n(b(u+1))z}{(z^2+wz+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + bu^2w^2z^{\frac{1}{2}} + bu^2\frac{wz(z^2+z+w^2)}{(z^2+wz+1)^2} \\
&= a^{\frac{1}{2}} + \mathrm{Tr}_m^n(b) + bu^2g(z) \ ,
\end{aligned}$$

since $a = b^{2^m+1} = b^2u^4$ and $\mathrm{Tr}_m^n(b)w = b(1+u^4)(u+u^4) = bu^2$ and where o-polynomial $g(z)$ comes from Theorem 2.1 Item (ii). On the other hand, if $b^{2^m-1} = u^4$ then it suffices just to take another u with the above defined properties (recall that four options exist). To obtain $(*)$ we used the following identities

$$\begin{aligned}
&(w + s^2 + s)\mathrm{Tr}_m^n(b(u^4+1))^2 \\
&= w\mathrm{Tr}_m^n(b(u^4+1))^2 + w\mathrm{Tr}_m^n(b(u+1))^2 + w^2\mathrm{Tr}_m^n(b(u+1))\mathrm{Tr}_m^n(b(u^4+1)) \\
&= w^2\big(\mathrm{Tr}_m^n(bu)\mathrm{Tr}_m^n(bu^4) + \mathrm{Tr}_m^n(b)\mathrm{Tr}_m^n(b(u^4+u)) + \mathrm{Tr}_m^n(b)^2\big) + w\mathrm{Tr}_m^n(b(u^4+u))^2 \\
&= w^2(bu + b^{2^m}u^4)(bu^4 + b^{2^m}u) + w^2\mathrm{Tr}_m^n(b)^2 = aw \ .
\end{aligned}$$

It is important to observe that there are no restrictions on the value of b here. It means that this technique allows to enlarge the original class of Niho bent functions proved in [**6**].

Assuming $b^{2^m-1} \neq u^4$, note that equation $s = \frac{w^2\mathrm{Tr}_m^n(b(u+1))}{\mathrm{Tr}_m^n(b(u^4+1))}$ can be solved for the unknown $b \in \mathbb{F}_{2^n}^*$ for any $s \in \mathbb{F}_{2^m}$. Indeed, this equation can be rewritten as

$$\begin{aligned}
b(u^4s + s + uw^2 + w^2) &= b^{2^m}(us + s + u^4w^2 + w^2) \quad \text{or} \\
b(u^4s + s + u^4 + u^2) &= b^{2^m}(us + s + u^3 + u) \ .
\end{aligned}$$

Since $s \in \mathbb{F}_{2^m}$, it is easy to see that this equation has nonzero sides and its right-hand side is a 2^mth power of the left-hand side. We conclude that the set of bent functions with $b \in \mathbb{F}_{2^n}^*$ corresponds exactly to all o-polynomials described in

Theorem 2.1 Item (ii). This means that the existence of this set of bent functions is equivalent to the existence of the corresponding q-clan.

3.3. $m \equiv 0 \pmod 4$**.** In this case, $w^2+w+1 \neq 0$ since the opposite is equivalent to $u^4+u^3+u^2+u+1=0$ that gives $u \in \mathbb{F}_{2^4}$ which is a contradiction because $\mathbb{F}_{2^4} \subset \mathbb{F}_{2^m}$. As was noted in Subsection 1.3, without loss of generality, we can assume $b=a=1$. Then, by (3.3),

$$\begin{aligned} G(z) &= 1+\mathrm{Tr}_m^n(u^5)+(wz)^{\frac{1}{2}}+\frac{\mathrm{Tr}_m^n(u^5+u)z^4+\mathrm{Tr}_m^n(u^5)w^2z^2+\mathrm{Tr}_m^n(u^4)z}{(z^2+wz+1)^2} \\ &\overset{(*)}{=} 1+\mathrm{Tr}_m^n(u^5)+(wz)^{\frac{1}{2}}+\frac{(w^5+w^3)z^4+w^3(1+w+w^2)^2z^2+w^4z}{(z^2+wz+1)^2} \\ &= 1+\mathrm{Tr}_m^n(u^5)+(w^2+w^5+w^{\frac{1}{2}})f_0(z) \ , \end{aligned}$$

where $(*)$ follows by $w(1+w+w^2)^2=\mathrm{Tr}_m^n(u^5)$ and $f_0(z)$ is an o-polynomial from (2.3).

Remark 3.1. In 2004, using computer calculations, the following sporadic bent function of Niho type was found. For $m=4$,

$$f(t)=\mathrm{Tr}_1^m(t^{2^m+1})+\mathrm{Tr}_1^n\big(t^{5(2^m-1)+1}+t^{7(2^m-1)+1}\big) \ . \tag{3.7}$$

The question open since then is whether this function is a new one or if it is EA-equivalent to one of the known Niho bent functions. Here we resolve this open problem.

Take basis elements $v=1$ and u with $u+u^{2^m}=1$. Since

$$x^{16}+x+1=(1+x+x^3+x^4+x^5+x^6+x^8)(1+x^3+x^5+x^6+x^8) \ ,$$

we get that either

$$1+u+u^3+u^4+u^5+u^6+u^8=0 \quad \text{or} \quad 1+u^3+u^5+u^6+u^8=0 \ . \tag{3.8}$$

By direct calculations, we obtain that $\mu=1$ and

$$\begin{aligned} G_1(z) &= z+(u+z)^{\frac{2^m+1}{2}}+\mathrm{Tr}_m^n\big((u+z)^{76}+(u+z)^{106}\big) \\ &= 1+u+u^4+u^6+u^8+u^{10}+u^{12} \\ &\quad +(u^4+u^8)z^2+(1+u^2+u^8)z^4+z^6+(1+u^2+u^4)z^8+z^{10}+z^{12} \ , \end{aligned}$$

since $z^2+(u+z)^{2^m+1}=u^{2^m+1}+\mathrm{Tr}_m^n(u)z=u^{2^m+1}+z$. As observed in [**3**, Sec. 3.1.2], adding a constant to $G_1(z)$ results into EA-equivalent bent functions, thus, the constant term in $G_1(z)$ can be ignored. Define $\beta=1+u+u^4 \in \mathbb{F}_{2^m}$ and note that $\beta^4=\beta+1$ and β is primitive in $\mathbb{F}_{2^m}$ (this is checked easily). Then, depending on (3.8), $G_1(z)$ without a constant term is respectively equal to either

$$\begin{aligned} &\beta^9z^2+\beta^2z^4+z^6+\beta^{11}z^8+z^{10}+z^{12} \quad \text{or} \\ &\beta^7z^2+\beta^2z^4+z^6+\beta^{12}z^8+z^{10}+z^{12} \ . \end{aligned}$$

Both polynomials belong to the list of 2040 o-polynomials representing the Lunelli-Sce hyperoval (numbers 119 and 120 in the list [**9**]). By [**1**, Theorem 26], the Lunelli-Sce hyperoval is a member of the Subiaco family of hyperovals. Thus, it is natural to expect that function (3.7) is EA-equivalent to the following Niho bent function from Subsection 1.3

$$f(t)=\mathrm{Tr}_1^m(t^{2^m+1})+\mathrm{Tr}_1^n(t^{3(2^m-1)+1}) \tag{3.9}$$

with $m = 4$. However, this does not come automatically since equivalent hyperovals do not necessarily correspond to EA-equivalent bent functions (see [**3**, Sec. 3.1.2]).

Now, take basis elements $v = 1$ and $w = u^2$ (where u is the second element in the basis chosen for analyzing function (3.7)) and recall that different choices of basis lead to EA-equivalent functions. Then $w + w^{2^m} = 1$ and using (3.2), we obtain that function (3.9) corresponds to the following polynomial

$$\begin{aligned} G_2(z) &= w^{8(2^m+1)} + 1 + z^8 + \mathrm{Tr}_m^n\big((w+z)^{14}\big) \\ &= 1 + w + w^2 + w^4 + w^6 + w^{10} + w^{12} \\ &\quad + (1 + w^4 + w^8)z^2 + (1 + w^2 + w^8)z^4 + z^6 + (w^2 + w^4)z^8 + z^{10} + z^{12} \\ &= 1 + u + u^5 + u^9 + u^{12} \\ &\quad + (u + u^8)z^2 + (u + u^4)z^4 + z^6 + (u^4 + u^8)z^8 + z^{10} + z^{12} \ . \end{aligned}$$

Similarly, if $\eta = 1 + w + w^4 = \beta^2 \in \mathbb{F}_{2^m}$ (obviously, η is also primitive in $\mathbb{F}_{2^m}$ and $\eta^4 = \eta + 1$) then, depending on (3.8) (where u is replaced by w), $G_2(z)$ without a constant term is respectively equal to either

$$\eta^7 z^2 + \eta^2 z^4 + z^6 + \eta^{12} z^8 + z^{10} + z^{12} \quad \text{or}$$
$$\eta^9 z^2 + \eta^2 z^4 + z^6 + \eta^{11} z^8 + z^{10} + z^{12}$$

using the fact that the sum of all coefficients in the latter polynomials has to be equal to one. These are the same Lunelli-Sce o-polynomials as obtained before but in the reverse order.

Now observe that

$$G_2(z+u^4+u^8) = c_u + (u^4+u^8)z^2 + (1+u^2+u^8)z^4 + z^6 + (1+u^2+u^4)z^8 + z^{10} + z^{12},$$

where c_u is a constant depending on u. Finally, note that the latter polynomial without the constant term c_u is exactly $G_1(u)$ without the constant term. Since adding a constant term to the argument of an o-polynomial is one of the transformations that preserves EA-equivalence of the corresponding bent functions (see [**3**, Sec. 3.1.2]), we conclude that bent functions (3.7) and (3.9) are EA-equivalent.

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Department of Informatics, University of Bergen, P.O. Box 7800, N-5020 Bergen, Norway
E-mail address: Tor.Helleseth@ii.uib.no

Department of Informatics, University of Bergen, P.O. Box 7800, N-5020 Bergen, Norway
E-mail address: Alexander.Kholosha@ii.uib.no

Department of Mathematics, University of Paris 8 and University of Paris 13, 2 rue de la liberté, 93526 Saint-Denis Cedex, France
E-mail address: smesnager@univ-paris8.fr