CONTEMPORARY MATHEMATICS

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and Integrable Systems
July 3–8, 2011
School of Mathematics and Statistics
University of Glasgow
United Kingdom

Chris Athorne
Diane Maclagan
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Introduction

The articles in this volume form the proceedings of the conference 'Tropical Geometry and Integrable Systems' that was held in July 2011 at the University of Glasgow. Over the five days of the conference over twenty talks were given on a wide range of topics, from ultra-discrete soliton equations and cellular automata to curve counting problems in enumerative geometry. While the original motivations for the subjects of these talks were different, common themes were the idea of the max-plus algebra and ideas originating from both tropical geometry and integrable systems.

The historical origins of integrable systems theory lie in three main areas: the 19th century treatment of certain dynamical systems due to Weierstrass and Kovalevskaya; the Cauchy theory of integrability of ordinary differential equations in the complex plane developed in the late 19th and early 20th century by Painléve, Garnier and others; and the geometry of surfaces studied by Bäcklund, Darboux and Goursat. In the late 20th century, particularly under the seminal influence of Gardner, Greene, Kruskal and Miura in the West and of the Russian and Japanese schools in the East the area of integrable systems acquired a momentum that has since kept it centre-stage in the interplay of pure and applied mathematics, drawing on and influencing a plethora of pure subjects such as algebraic geometry, representation theory and combinatorics, while at the same time finding applications in such areas as nonlinear optics and plasma physics.

The first examples of integrable systems were continuous: the dependent fields were continuous and satisfied differential equations. However it was soon realized that the same ideas could be applied to certain discrete systems, where derivative are replaced with difference operators. One may go further and discretize the dependent variables as well, arriving at so-called ultra-discrete integrable systems, while still retaining the notion of integrability. In the most extreme case the dependent variable can take only two values 0 and 1 and since the independent variable is also discrete one arrives at discrete dynamics - a 'box and ball' system - where the movement of balls (where the fields take the value 1) are described by simple sets of rules. Remarkably the essential features of integrability still remain for such systems. In fact such discrete integrable systems can be regarded as more fundamental as many continuous integrable systems may be obtained by taking the continuum limit of discrete systems.

Algebraic geometry begins with the fundamental problem of solving systems of polynomial equation in several variables. Even as early as the 19th century the connection between algebraic curves and solutions to differential equations was well understood (the paradigm being the relations between elliptic curves and elliptic integrals). The development of soliton theory saw a resurgence of such links: solutions correspond to linear flows on the Jacobian of an underlying spectral curve. The relationship, though, between integrable systems and algebraic geometry has, until the last few years, had a tendency to look rather classical: the theory of theta functions, commuting differential operators and the representation theory of Lie algebras. Other developments, particularly that of discrete integrable systems appeared to be outside the natural realm of algebraic geometry.

More recent developments have opened new connections between these two fields. On the integrable systems side the theory of ultra-discrete, or box and ball systems, has developed. At the same time, tropical geometry has emerged as a new theory at the interface between algebraic geometry and combinatorics. In tropical geometry an algebraic variety is replaced by a polyhedral complex, which allows an influx of new techniques into the area. This 'tropicalization' procedure involves replacing the field, such as the complex numbers, by the tropical semiring, which is also known as the max-plus algebra. Surprisingly, there are connections between these two advances that goes deeper than just the appearance of the same max-plus algebra in both areas (see, for example, the survey article by Inoue and Iwao). It is probably safe to say, though, that the full understanding of the connections between these areas remains to be found and fully exploited.

One of the aims of this conference was to bring together two communities working in the field of tropical geometry and its applications, albeit from apparently opposite ends of the spectrum, and to foster a mutual understanding and the development of a common language which would then help in future developments of the area. To help bridge this gap between these two communities two of the speakers (Diane Maclagan and Rei Inoue) gave a series of survey lectures. The aim of the conference is also reflected in the articles in this volume, which cover areas from automata, through cluster algebras to enumerative geometry. In addition to these research articles, the two survey talks mentioned above have been written up as survey articles and these will serve to introduce ideas from one end of this spectrum to researchers working at the other. During the conference a public lecture was given by Prof. Junkichi Satsuma (Aoyama Gakuin University, Tokyo) on 'Nonlinear waves: solitons, boxes and balls', and this was well attended by members of the public as well as the conference participants. We are particularly grateful to Prof. Satsuma for agreeing to give this talk.

No conference can take place without the hard work of a lot of people. We would like to thank the members of the scientific organising committee (Rei Inoue, Tetsuji Tokihiro and Alexander Veselov) for their help, suggestions and guidance, together with the members of the local organising committee (Matthew England, Misha Feigin, Christian Korff, Claire Gilson and Jon Nimmo), as well as all the speakers and contributors to this volume. Financial support was provided by the EPSRC (grant number: EP/I037636/1 - Tropical geometry and integrable systems), the London Mathematical Society (grant number 11031), the Edinburgh Mathematical Society and the Glasgow Mathematical Journal Trust.

Chris Athorne, Diane Maclagan, Ian Strachan

List of Participants

Stuart Andrew
Loughborough University

Chris Athorne
University of Glasgow

Martha Bernal
University of Warwick

Florian Block
University of Warwick

Jan Draisma
Eindhoven University of Technology

Mattthew England
University of Glasgow

Misha Feigin
University of Glasgow

Allan Fordy
University of Leeds

Stéphane Gaubert
INRIA and École Polytechnique

Claire Gilson
University of Glasgow

Takatoshi Ikegami
Waseda University

Rei Inoue Yamazaki
Chiba University

Shinsuke Iwao
Rikkyo University

Paul Johnson
Imperial College, London

Kenji Kajiwara
Kyushu University

Christian Korff
University of Glasgow

Diane Maclagan
University of Warwick

Kazuki Maeda
Kyoto University

Toshiaki Maeno
Kyoto University

Marta Mazzocco
Loughborough University

Tomoki Nakanishi
Nagoya University

Jonathan Nimmo
University of Glasgow

Satyendra Nath Pandey
MNNNIT

Junkichi Satsuma
Aoyama Gakuin University

Sergei Sergeev
INRIA and École Polytechnique

Michael Shapiro
Michigan State University

Ian Strachan
University of Glasgow

Taichiro Takagi
National Defense Academy

Daisuke Takahashi
Waseda University

Tetsuji Tokihiro
The University of Tokyo

Thi Dinh Tran
La Trobe University

Alexander Veselov
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One of the aims of this conference was to bring together researchers in the field of tropical geometry and its applications, from apparently disparate ends of the spectrum, to foster a mutual understanding and establish a common language which will encourage further developments of the area. This aim is reflected in these articles, which cover areas from automata, through cluster algebras, to enumerative geometry. In addition, two survey articles are included which introduce ideas from researchers on one end of this spectrum to researchers on the other.

This book is intended for graduate students and researchers interested in tropical geometry and integrable systems and the developing links between these two areas.

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