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The conormal derivative problem for higher order elliptic systems with irregular coefficients

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Abstract. We prove L_p estimates of solutions to a conormal derivative problem for divergence form complex-valued higher-order elliptic systems on a half space and on a Reifenberg flat domain. The leading coefficients are assumed to be merely measurable in one direction and have small mean oscillations in the orthogonal directions on each small ball. Our results are new even in the second-order case. The corresponding results for the Dirichlet problem were obtained recently in Dong H., Kim D.: *Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains*, (2011).

1. Introduction

This paper is concerned with L_p theory for higher-order elliptic systems in divergence form with *conormal* derivative boundary conditions. Our focus is to seek minimal regularity assumptions on the leading coefficients of elliptic systems defined on regular and irregular domains. The paper is a continuation of [**14, 15**], where the authors considered higher-order systems in the whole space and on domains with *Dirichlet* boundary conditions.

There is a vast literature on L_p theory for second-order and higher-order elliptic and parabolic equations/systems with *constant* or *uniformly continuous* coefficients. We refer the reader to the classical work [**1, 2, 29, 22, 16**]. Concerning possibly discontinuous coefficients, a notable class is the set of bounded functions with vanishing mean oscillations (VMO). This class of coefficients was firstly introduced in [**7, 8**] in the case of second-order non-divergence form elliptic equations, and further considered by a number of authors in various contexts, including higher-order equations and systems; see, for instance, [**9, 17, 26, 27**].

Recently, in [**14, 15**] the authors studied the *Dirichlet* problem for higher-order elliptic and parabolic systems with possibly measurable coefficients. In [**14**], we established the L_p-solvability of both divergence and non-divergence form systems

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with coefficients (called VMO_x coefficients in [**20**]) having locally small mean oscillations with respect to the spatial variables, and measurable in the time variable in the parabolic case. While in [**15**], divergence form elliptic and parabolic systems of arbitrary order are considered in the whole space, on a half space, and on Reifenberg flat domains, with *variably partially BMO coefficients*. This class of coefficients was introduced in [**21**] in the context of second-order non-divergence form elliptic equations in the whole space, and naturally appears in the homogenization of layered materials; see, for instance, [**10**]. It was later considered by the authors of the present article in [**13, 15**] and by Byun and Wang in [**5**]. Loosely speaking, on each cylinder (or ball in the elliptic case), the coefficients are allowed to be merely measurable in one spatial direction called the *measurable direction*, which may vary for different cylinders. It is also assumed that the coefficients have small mean oscillations in the orthogonal directions, and near the boundary the measurable direction is sufficiently close to the "normal" direction of the boundary. Note that the boundary of a Reifenberg flat domain is locally trapped in thin discs, which allows the boundary to have a fractal structure; cf. (2.5). Thus the normal direction of the boundary may not be well defined for Reifenberg flat domains, so instead we take the normal direction of the top surface of these thin discs.

The proofs in [**14, 15**] are in the spirit of [**20**] by N. V. Krylov, in which the author gave a unified approach of L_p estimates for both divergence and non-divergence second-order elliptic and parabolic equations in the whole space with VMO_x coefficients. One of the crucial steps in [**20**] is to establish certain interior *mean oscillation estimates*[1] of solutions to equations with "simple" coefficients, which are measurable functions of the time variable only. Then the estimates for equations with VMO_x coefficients follow from the mean oscillation estimates combined with a perturbation argument. In this connection, we point out that in [**14, 15**] a great deal of efforts were made to derive boundary and interior mean oscillation estimates for solutions to higher-order systems. For systems in Reifenberg flat domains, we also used an idea in [**6**].

In this paper, we study a *conormal* derivative problem for elliptic operators in divergence form of order $2m$:

$$\mathcal{L}\boldsymbol{u} := \sum_{|\alpha|\le m, |\beta|\le m} D^\alpha (a_{\alpha\beta} D^\beta \boldsymbol{u}), \tag{1.1}$$

where α and β are d-dimensional multi-indices, $a_{\alpha\beta} = [a^{ij}_{\alpha\beta}(x)]^n_{i,j=1}$ are $n\times n$ complex matrix-valued functions, and $\boldsymbol{u}$ is a complex vector-valued function. For $\alpha = (\alpha_1, \dots, \alpha_d)$, we use the notation $D^\alpha \boldsymbol{u} = D_1^{\alpha_1} \dots D_d^{\alpha_d} \boldsymbol{u}$. All the coefficients are assumed to be bounded and measurable, and $\mathcal{L}$ is uniformly elliptic; cf. (2.1). Consider the following elliptic system

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha|\le m} D^\alpha \boldsymbol{f}_\alpha \tag{1.2}$$

on a domain Ω in $\mathbb{R}^d$, where $\boldsymbol{f}_\alpha \in L_p(\Omega)$, $p \in (1,\infty)$, and $\lambda \ge 0$ is a constant. A function $\boldsymbol{u} \in W^m_p$ is said to be a weak solution to (1.2) on Ω with the conormal

[1] Also see relevant early work [**18, 11**].

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derivative boundary condition associated with $\boldsymbol{f}_\alpha$ (on $\partial\Omega$) if

(1.3)
$$\int_\Omega \sum_{|\alpha|\le m, |\beta|\le m} (-1)^{m+|\alpha|} D^\alpha \phi \cdot a_{\alpha\beta} D^\beta \boldsymbol{u} + \lambda \phi \cdot \boldsymbol{u}\, dx = \sum_{|\alpha|\le m} \int_\Omega (-1)^{|\alpha|} D^\alpha \phi \cdot \boldsymbol{f}_\alpha\, dx$$

for any test function $\phi = (\phi^1, \phi^2, \dots, \phi^n) \in W^m_q(\Omega)$, where $q = p/(p-1)$. We emphasize that the phrase "associated with $\boldsymbol{f}_\alpha$" is appended after "the conormal derivative boundary condition" because for different representations of the right-hand side of (1.2), even if they are pointwise equal, the weak formulation (1.3) could still be different. In the sequel, we omit this phrase when there is no confusion. We note that the equation above can also be understood as

$$\int_\Omega \sum_{|\alpha|\le m, |\beta|\le m} (-1)^{m+|\alpha|} D^\alpha \phi \cdot a_{\alpha\beta} D^\beta \boldsymbol{u} + \lambda \phi \cdot \boldsymbol{u}\, dx = \boldsymbol{F}(\phi) \quad \forall \phi \in W^m_q(\Omega),$$

where $\boldsymbol{F}$ is a given vector-valued bounded linear functional on $W^m_q(\Omega)$. The main objective of the paper is to show the unique $W^m_p(\Omega)$-solvability of (1.2) on a half space or on a possibly unbounded Reifenberg domain with the same regularity conditions on the leading coefficients, that is, *variably partially BMO coefficients*, as those in [**15**]. See Section 2 for the precise statements of the assumptions and main results.

Notably, our results are new even for second-order scalar equations. In the literature, an L_p estimate for the conormal derivative problem can be found in [**3**], where the authors consider second-order divergence elliptic equations without lower-order terms and with coefficients small BMO with respect to all variables on bounded Reifenberg domains. The proof in [**3**] contains a compactness argument, which does not apply to equations with coefficients measurable in some direction discussed in the current paper. For other results about the conormal derivative problem, we refer the reader to [**23**] and [**25**].

We prove the main theorems by following the strategy in [**15**]. First, for systems with homogeneous right-hand side and coefficients measurable in one direction, we estimate the Hölder norm of certain linear combinations of $D^m \boldsymbol{u}$ in the interior of the domain, as well as near the boundary if the boundary is flat and perpendicular to the measurable direction. Then by using the Hölder estimates, we proceed to establish mean oscillation estimates of solutions to elliptic systems. As is expected, the obstruction is in the boundary mean oscillation estimates, to which we give a more detailed account. Note that when obtaining mean oscillation estimates of solutions, even in the half space case we do not require the measurable direction to be exactly perpendicular to the boundary, but allow it to be sufficiently close to the normal direction. For the Dirichlet problem in [**15**], we used a delicate cut-off argument together with a generalized Hardy's inequality. However, this method no longer works for the conormal derivative problem as solutions do not vanish on the boundary. The key observation in this paper is Lemma 4.2 which shows that if one modifies the right-hand side a little bit, then the function $\boldsymbol{u}$ itself still satisfies the system with the conormal derivative boundary condition on a subdomain with a flat boundary perpendicular to the measurable direction. This argument is also readily adapted to elliptic systems on Reifenberg flat domains with variably partially BMO coefficients.

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The corresponding parabolic problem, however, seems to be still out of reach by the argument mentioned above. In fact, in the modified equation in Lemma 4.2 there would be an extra term involving $\boldsymbol{u}_t$ on the right-hand side. At the time of this writing, it is not clear to us how to estimate this term.

The remaining part of the paper is organized as follows. We state the main theorems in the next section. Section 3 contains some auxiliary results including L_2-estimates, interior and boundary Hölder estimates, and approximations of Reifenberg domains. In Section 4 we establish the interior and boundary mean oscillation estimates and then prove the solvability of systems on a half space. Finally we deal with elliptic systems on a Reifenberg flat domain in Section 5.

We finish the introduction by fixing some notation. By $\mathbb{R}^d$ we mean a d-dimensional Euclidean space, a point in $\mathbb{R}^d$ is denoted by $x = (x_1, \dots, x_d) = (x_1, x')$, and $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$. Throughout the paper, Ω indicates an open set in $\mathbb{R}^d$. For vectors $\xi, \eta \in \mathbb{C}^n$, we denote

$$(\xi, \eta) = \sum_{i=1}^n \xi^i \overline{\eta^i}.$$

For a function f defined on a subset $\mathcal{D}$ in $\mathbb{R}^d$, we set

$$(f)_{\mathcal{D}} = \fint_{\mathcal{D}} f(x)\,dx = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} f(x)\,dx,$$

where $|\mathcal{D}|$ is the d-dimensional Lebesgue measure of $\mathcal{D}$. Denote

$$\begin{aligned}
\mathbb{R}^d_+ &= \{(x_1, x') \in \mathbb{R}^d : x_1 > 0\},\\
B_r(x) &= \{y \in \mathbb{R}^d : |x-y| < r\}, \quad B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x'-y'| < r\},\\
B^+_r(x) &= B_r(x) \cap \mathbb{R}^d_+, \quad \Gamma_r(x) = B_r(x) \cap \partial\mathbb{R}^d_+, \quad \Omega_r(x) = B_r(x) \cap \Omega.
\end{aligned}$$

For a domain Ω in $\mathbb{R}^d$, we define the solution spaces $W^m_p(\Omega)$ as follows:

$$W^m_p(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega), 1 \le |\alpha| \le m\},$$

$$\|u\|_{W^m_p(\Omega)} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L_p(\Omega)}.$$

We denote $C^\infty_{\text{loc}}(\mathcal{D})$ to be the set of all infinitely differentiable functions on $\mathcal{D}$, and $C^\infty_0(\mathcal{D})$ the set of infinitely differentiable functions with compact support $\Subset \mathcal{D}$.

2. Main results

Throughout the paper, we assume that the $n \times n$ complex-valued coefficient matrices $a_{\alpha\beta}$ are measurable and bounded, and the leading coefficients $a_{\alpha\beta}$, $|\alpha| = |\beta| = m$, satisfy an ellipticity condition. More precisely, we assume:

(1) There exists a constant $\delta \in (0,1)$ such that the leading coefficients $a_{\alpha\beta}$, $|\alpha| = |\beta| = m$, satisfy

$$\delta|\xi|^2 \le \sum_{|\alpha|=|\beta|=m} \Re(a_{\alpha\beta}(x)\xi_\beta, \xi_\alpha), \quad |a_{\alpha\beta}| \le \delta^{-1} \tag{2.1}$$

for any $x \in \mathbb{R}^d$ and $\xi = (\xi_\alpha)_{|\alpha|=m}$, $\xi_\alpha \in \mathbb{C}^n$. Here we use $\Re(f)$ to denote the real part of f.

(2) All the lower-order coefficients $a_{\alpha\beta}$, $|\alpha| \ne m$ or $|\beta| \ne m$, are bounded by a constant $K \ge 1$.

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We note that the ellipticity condition (2.1) can be relaxed. For instance, the operator $\mathcal{L} = D_1^4 + D_2^4$ is allowed when $d = m = 2$. See Remark 2.5 of [**15**].

Throughout the paper we write $\{\bar{a}_{\alpha\beta}\}_{|\alpha|=|\beta|=m} \in \mathbb{A}$ whenever the $n\times n$ complex-valued matrices $\bar{a}_{\alpha\beta} = \bar{a}_{\alpha\beta}(y_1)$ are measurable functions satisfying the condition (2.1). For a linear map $\mathcal{T}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$, we write $\mathcal{T} \in \mathbb{O}$ if $\mathcal{T}$ is of the form

$$\mathcal{T}(x) = \rho x + \xi,$$

where ρ is a $d \times d$ orthogonal matrix and $\xi \in \mathbb{R}^d$.

Let $\mathcal{L}$ be the elliptic operator defined in (1.1). Our first result is about the conormal derivative problem on a half space. The following mild regularity assumption is imposed on the leading coefficients, with a parameter $\gamma \in (0, 1/4)$ to be determined later.

Assumption 2.1 (γ). There is a constant $R_0 \in (0, 1]$ such that the following hold with $B := B_r(x_0)$.

(i) For any $x_0 \in \mathbb{R}^d_+$ and any $r \in (0, R_0]$ so that $B \subset \mathbb{R}^d_+$, one can find $\mathcal{T}_B \in \mathbb{O}$ and coefficient matrices $\{\bar{a}_{\alpha\beta}\}_{|\alpha|=|\beta|=m} \in \mathbb{A}$ satisfying

$$\sup_{|\alpha|=|\beta|=m} \int_B |a_{\alpha\beta}(x) - \bar{a}_{\alpha\beta}(y_1)|\, dx \le \gamma |B|, \tag{2.2}$$

where $y = \mathcal{T}_B(x)$.

(ii) For any $x_0 \in \partial\mathbb{R}^d_+$ and any $r \in (0, R_0]$, one can find $\mathcal{T}_B \in \mathbb{O}$ satisfying $\rho_{11} \ge \cos(\gamma/2)$ and coefficient matrices $\{\bar{a}_{\alpha\beta}\}_{|\alpha|=|\beta|=m} \in \mathbb{A}$ satisfying (2.2).

The condition $\rho_{11} \ge \cos(\gamma/2)$ with a sufficiently small γ means that at any boundary point the y_1-direction is sufficiently close to the x_1-direction, i.e., the normal direction of the boundary.

Theorem 2.2 (Systems on a half space). *Let $\Omega = \mathbb{R}^d_+$, $p \in (1, \infty)$, and*

$$\boldsymbol{f}_\alpha = (f^1_\alpha, \dots, f^n_\alpha)^{tr} \in L_p(\Omega), \quad |\alpha| \le m.$$

Then there exists a constant $\gamma = \gamma(d, n, m, p, \delta)$ such that, under Assumption 2.1 (γ), the following hold true.

(i) For any $\boldsymbol{u} \in W^m_p(\Omega)$ satisfying

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha|\le m} D^\alpha \boldsymbol{f}_\alpha \tag{2.3}$$

in Ω and the conormal derivative condition on $\partial\Omega$, we have

$$\sum_{|\alpha|\le m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha \boldsymbol{u}\|_{L_p(\Omega)} \le N \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)},$$

provided that $\lambda \ge \lambda_0$, where N and $\lambda_0 \ge 0$ depend only on d, n, m, p, δ, K and R_0.

(ii) For any $\lambda > \lambda_0$, there exists a unique solution $\boldsymbol{u} \in W^m_p(\Omega)$ to (2.3) with the conormal derivative boundary condition.

(iii) If all the lower-order coefficients of $\mathcal{L}$ are zero and the leading coefficients are measurable functions of $x_1 \in \mathbb{R}$ only, then one can take $\lambda_0 = 0$.

For elliptic systems on a Reifenberg flat domain which is possibly unbounded, we impose a similar regularity assumption on $a_{\alpha\beta}$ as in Assumption 2.1. Near the boundary, we require that in each small scale the direction in which the coefficients

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are only measurable coincides with the "normal" direction of a certain thin disc, which contains a portion of $\partial\Omega$. More precisely, we assume the following, where the parameter $\gamma \in (0, 1/50)$ will be determined later.

ASSUMPTION 2.3 (γ). There is a constant $R_0 \in (0, 1]$ such that the following hold.

(i) For any $x \in \Omega$ and any $r \in (0, R_0]$ such that $B_r(x) \subset \Omega$, there is an orthogonal coordinate system depending on x and r such that in this new coordinate system, we have

$$\fint_{B_r(x)} \left| a_{\alpha\beta}(y_1, y') - \fint_{B'_r(x')} a_{\alpha\beta}(y_1, z')\, dz' \right| dy \le \gamma. \tag{2.4}$$

(ii) The domain Ω is Reifenberg flat: for any $x \in \partial\Omega$ and $r \in (0, R_0]$, there is an orthogonal coordinate system depending on x and r such that in this new coordinate system, we have (2.4) and

$$\{(y_1, y') : x_1 + \gamma r < y_1\} \cap B_r(x) \subset \Omega_r(x) \subset \{(y_1, y') : x_1 - \gamma r < y_1\} \cap B_r(x). \tag{2.5}$$

In particular, if the boundary $\partial\Omega$ is locally the graph of a Lipschitz continuous function with a small Lipschitz constant, then Ω is Reifenberg flat. Thus all C^1 domains are Reifenberg flat for any $\gamma > 0$.

The next theorem is about the conormal derivative problem on a Reifenberg flat domain.

THEOREM 2.4 (Systems on a Reifenberg flat domain). *Let Ω be a domain in $\mathbb{R}^d$ and $p \in (1, \infty)$. Then there exists a constant $\gamma = \gamma(d, n, m, p, \delta)$ such that, under Assumption 2.3 (γ), the following hold true.*
(i) Let $\boldsymbol{f}_\alpha = (f^1_\alpha, \dots, f^n_\alpha)^{tr} \in L_p(\Omega)$, $|\alpha| \le m$. For any $\boldsymbol{u} \in W^m_p(\Omega)$ satisfying

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha| \le m} D^\alpha \boldsymbol{f}_\alpha \quad \text{in} \quad \Omega \tag{2.6}$$

with the conormal derivative condition on $\partial\Omega$, we have

$$\sum_{|\alpha| \le m} \lambda^{1 - \frac{|\alpha|}{2m}} \|D^\alpha \boldsymbol{u}\|_{L_p(\Omega)} \le N \sum_{|\alpha| \le m} \lambda^{\frac{|\alpha|}{2m}} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)},$$

provided that $\lambda \ge \lambda_0$, where N and $\lambda_0 \ge 0$ depend only on d, n, m, p, δ, K, and R_0.
(ii) For any $\lambda > \lambda_0$ and $\boldsymbol{f}_\alpha \in L_p(\Omega)$, $|\alpha| \le m$, there exists a unique solution $\boldsymbol{u} \in W^m_p(\Omega)$ to (2.6) with the conormal derivative boundary condition.

For $\lambda = 0$, we have the following solvability result for systems without lower-order terms on bounded domains.

COROLLARY 2.5. *Let Ω be a bounded domain in $\mathbb{R}^d$, and $p \in (1, \infty)$. Assume that $a_{\alpha\beta} \equiv 0$ for any α, β satisfying $|\alpha| + |\beta| < 2m$. Then there exists a constant $\gamma = \gamma(d, n, m, p, \delta)$ such that, under Assumption 2.3 (γ), for any $\boldsymbol{f}_\alpha \in L_p(\Omega)$, $|\alpha| = m$, there exists a solution $\boldsymbol{u} \in W^m_p(\Omega)$ to*

$$(-1)^m \mathcal{L}\boldsymbol{u} = \sum_{|\alpha| = m} D^\alpha \boldsymbol{f}_\alpha \quad \text{in} \quad \Omega \tag{2.7}$$

with the conormal derivative boundary condition, and $\boldsymbol{u}$ satisfies

$$\|D^m \boldsymbol{u}\|_{L_p(\Omega)} \le N \sum_{|\alpha| = m} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)}, \tag{2.8}$$

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where N depends only on d, n, m, p, δ, K, R_0, and $|\Omega|$. Such a solution is unique up to a polynomial of order at most $m-1$.

Finally, we present a result for second-order scalar elliptic equations in the form

$$D_i(a_{ij}D_ju) + D_i(a_iu) + b_iD_iu + cu = \operatorname{div} g + f \quad \text{in } \Omega \tag{2.9}$$

with the conormal derivative boundary condition. The result generalizes Theorem 5 of [**12**], in which bounded Lipschitz domains with small Lipschitz constants are considered. It also extends the main result of [**3**] to equations with lower-order terms and with leading coefficients in a more general class. In the theorem below we assume that all the coefficients are bounded and measurable, and a_{ij} satisfy (2.1) with $m=1$. As usual, we say that $D_ia_i + c \le 0$ in Ω holds in the weak sense if

$$\int_\Omega (-a_iD_i\phi + c\phi)\,dx \le 0$$

for any nonnegative $\phi \in C_0^\infty(\Omega)$. By Assumption (H) we mean that

$$\int_\Omega (-a_iD_i\phi + c\phi)\,dx = 0 \quad \forall \phi \in C^\infty(\overline{\Omega}).$$

Similarly, Assumption (H*) is satisfied if

$$\int_\Omega (b_iD_i\phi + c\phi)\,dx = 0 \quad \forall \phi \in C^\infty(\overline{\Omega}).$$

THEOREM 2.6 (Scalar equations on a bounded domain). *Let $p \in (1,\infty)$ and Ω be a bounded domain. Assume $D_ia_i + c \le 0$ in Ω in the weak sense. Then there exists a constant $\gamma = \gamma(d,p,\delta)$ such that, under Assumption 2.3 (γ), the following hold true.*
(i) If Assumption (H) is satisfied, then for any f, $g = (g_1,\cdots,g_d) \in L_p(\Omega)$, the equation (2.9) has a unique up to a constant solution $u \in W_p^1(\Omega)$ provided that Assumption (H*) is also satisfied. Moreover, we have

$$\|Du\|_{L_p(\Omega)} \le N\|f\|_{L_p(\Omega)} + N\|g\|_{L_p(\Omega)}.$$

(ii) If Assumption (H) is not satisfied, the solution is unique and we have

$$\|u\|_{W_p^1(\Omega)} \le N\|f\|_{L_p(\Omega)} + N\|g\|_{L_p(\Omega)}.$$

The constants N are independent of f, g, and u.

3. Some auxiliary estimates

In this section we consider operators without lower-order terms. Denote

$$\mathcal{L}_0\boldsymbol{u} = \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta}D^\beta\boldsymbol{u}).$$

3.1. L_2-estimates. The following L_2-estimate for elliptic operators in divergence form with measurable coefficients is classical. We give a sketched proof for the sake of completeness.

THEOREM 3.1. *Let $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d_+$. There exists $N = N(d,m,n,\delta)$ such that, for any $\lambda \ge 0$,*

$$\sum_{|\alpha|\le m} \lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha\boldsymbol{u}\|_{L_2(\Omega)} \le N \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}}\|\boldsymbol{f}_\alpha\|_{L_2(\Omega)}, \tag{3.1}$$

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provided that $\boldsymbol{u} \in W_2^m(\Omega)$ *and* $\boldsymbol{f}_\alpha \in L_2(\Omega)$, $|\alpha| \le m$, *satisfy*

$$(-1)^m \mathcal{L}_0 \boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha| \le m} D^\alpha \boldsymbol{f}_\alpha \tag{3.2}$$

in Ω *with the conormal derivative condition on* $\partial\Omega$. *Furthermore, for any* $\lambda > 0$ *and* $\boldsymbol{f}_\alpha \in L_2(\Omega), |\alpha| \le m$, *there exists a unique solution* $\boldsymbol{u} \in W_2^m(\Omega)$ *to the equation* (3.2) *in* Ω *with the conormal derivative boundary condition.*

Proof. By the method of continuity and a standard density argument, it suffices to prove the estimate (3.1) for $\boldsymbol{u} \in C^\infty(\overline{\Omega}) \cap W_2^m(\Omega)$. From the equation, it follows that

$$\int_\Omega \left[(D^\alpha \boldsymbol{u}, a_{\alpha\beta} D^\beta \boldsymbol{u}) + \lambda |\boldsymbol{u}|^2 \right] dx = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \int_\Omega (D^\alpha \boldsymbol{u}, \boldsymbol{f}_\alpha)\, dx.$$

By the uniform ellipticity (2.1), we get

$$\delta \int_\Omega |D^m \boldsymbol{u}|^2\, dx \le \int_\Omega \Re(a_{\alpha\beta} D^\beta \boldsymbol{u}, D^\alpha \boldsymbol{u})\, dx.$$

Hence, for any $\varepsilon > 0$,

$$\begin{aligned} \delta \int_\Omega |D^m \boldsymbol{u}|^2\, dx + \lambda \int_\Omega |\boldsymbol{u}|^2\, dx &\le \sum_{|\alpha| \le m} (-1)^{|\alpha|} \int_\Omega \Re(D^\alpha \boldsymbol{u}, \boldsymbol{f}_\alpha)\, dx \\ &\le \varepsilon \sum_{|\alpha| \le m} \lambda^{\frac{m-|\alpha|}{m}} \int_\Omega |D^\alpha \boldsymbol{u}|^2\, dx + N\varepsilon^{-1} \sum_{|\alpha| \le m} \lambda^{-\frac{m-|\alpha|}{m}} \int_\Omega |\boldsymbol{f}_\alpha|^2\, dx. \end{aligned}$$

To finish the proof, it suffices to use interpolation inequalities and choose ε sufficiently small depending on δ, d, m, and n. □

We say that a function $\boldsymbol{u} \in W_p(\Omega)$ satisfies (1.2) with the conormal derivative condition on $\Gamma \subset \partial\Omega$ if u satisfies (1.3) for any $\phi \in W_q^m(\Omega)$ which is supported on $\Omega \cup \Gamma$.

By Theorem 3.1 and adapting the proofs of Lemmas 3.2 and 7.2 in [**15**] to the conormal case, we have the following local L_2-estimate.

Lemma 3.2. *Let* $0 < r < R < \infty$. *Assume that* $\boldsymbol{u} \in C_{loc}^\infty(\overline{\mathbb{R}^d_+})$ *satisfies*

$$\mathcal{L}_0 \boldsymbol{u} = 0 \tag{3.3}$$

in B_R^+ *with the conormal derivative boundary condition on* Γ_R. *Then there exists a constant* $N = N(d, m, n, \delta)$ *such that for* $j = 1, \dots, m$,

$$\|D^j \boldsymbol{u}\|_{L_2(B_r^+)} \le N(R-r)^{-j} \|\boldsymbol{u}\|_{L_2(B_R^+)}.$$

Corollary 3.3. *Let* $0 < r < R < \infty$ *and* $a_{\alpha\beta} = a_{\alpha\beta}(x_1)$, $|\alpha| = |\beta| = m$. *Assume that* $\boldsymbol{u} \in C_{loc}^\infty(\overline{\mathbb{R}^d_+})$ *satisfies* (3.3) *in* B_R^+ *with the conormal derivative boundary condition on* Γ_R. *Then for any multi-index* θ *satisfying* $\theta_1 \le m$ *and* $|\theta| \ge m$, *we have*

$$\|D^\theta \boldsymbol{u}\|_{L_2(B_r^+)} \le N \|D^m \boldsymbol{u}\|_{L_2(B_R^+)},$$

where $N = N(d, m, n, \delta, R, r, \theta)$.

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PROOF. It is easily seen that $D^{mk}_{x'}\boldsymbol{u}, k = 1, 2, \ldots,$ also satisfies (3.3) with the conormal derivative boundary condition on Γ_R. Then by applying Lemma 3.2 repeatedly, we obtain

$$\|D^m D^{mk}_{x'}\boldsymbol{u}\|_{L_2(B^+_{R'})} \le N\|D^m\boldsymbol{u}\|_{L_2(B^+_R)},$$

where $R' = (r + R)/2$. From this inequality and the interpolation inequality, we get the desired estimate. □

By using a Sobolev-type inequality, we shall obtain from Corollary 3.3 a Hölder estimate of all the m-th derivatives of $\boldsymbol{u}$ except $D^{\bar\alpha}\boldsymbol{u}$, where $\bar\alpha = me_1 = (m, 0, \ldots, 0)$. To compensate this lack of regularity of $D^{\bar\alpha}\boldsymbol{u}$, we consider the quantity

$$\Theta := \sum_{|\beta|=m} a_{\bar\alpha\beta} D^\beta \boldsymbol{u}.$$

We recall the following useful estimate proved in [**15**, Corollary 4.4].

LEMMA 3.4. *Let $k \ge 1$ be an integer, $r \in (0, \infty)$, $p \in [1, \infty]$, $\mathcal{D} = [0, r]^d$, and $u(x) \in L_p(\mathcal{D})$. Assume that $D^k_1 u = f_0 + D_1 f_1 + \ldots + D^{k-1}_1 f_{k-1}$ in $\mathcal{D}$, where $f_j \in L_p(\mathcal{D}), j = 0, \ldots, k-1$. Then $D_1 u \in L_p(\mathcal{D})$ and*

$$\|D_1 u\|_{L_p(\mathcal{D})} \le N\|u\|_{L_p(\mathcal{D})} + N\sum_{j=0}^{k-1}\|f_j\|_{L_p(\mathcal{D})},$$

where $N = N(d, k, r) > 0$.

COROLLARY 3.5. *Let $0 < r < R < \infty$ and $a_{\alpha\beta} = a_{\alpha\beta}(x_1)$. Assume $\boldsymbol{u} \in C^\infty_{loc}(\overline{\mathbb{R}^d_+})$ satisfies (3.3) in B^+_R with the conormal derivative boundary condition on Γ_R. Then, for any nonnegative integer j,*

$$\|D^j_{x'}\Theta\|_{L_2(B^+_r)} + \|D^j_{x'}D_1\Theta\|_{L_2(B^+_r)} \le N\|D^m\boldsymbol{u}\|_{L_2(B^+_R)},$$

where $N = N(d, m, n, r, R, \delta, j) > 0$.

PROOF. Due to Corollary 3.3 and the fact that $D^j_{x'}\boldsymbol{u}$ satisfies (3.3) with the conormal derivative boundary condition, it suffices to prove the desired inequality when $j = 0$ and R is replaced by another R' such that $r < R' < R$. Obviously, we have

$$\|\Theta\|_{L_2(B^+_r)} \le N\|D^m\boldsymbol{u}\|_{L_2(B^+_r)}.$$

Thus we prove that, for $R' = (r + R)/2$,

$$\|D_1\Theta\|_{L_2(B^+_r)} \le N\|D^m\boldsymbol{u}\|_{L_2(B^+_{R'})}. \tag{3.4}$$

From (3.3), in B^+_R we have

$$D^m_1\Theta = -\sum_{\substack{|\alpha|=|\beta|=m\\ \alpha_1<m}} D^\alpha(a_{\alpha\beta}D^\beta\boldsymbol{u}) = -\sum_{\substack{|\alpha|=|\beta|=m\\ \alpha_1<m}} D^{\alpha_1}_1(a_{\alpha\beta}D^{\alpha'}_{x'}D^\beta\boldsymbol{u}).$$

Then the estimate (3.4) follows from Lemma 3.4 with a covering argument and Corollary 3.3. The corollary is proved. □

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3.2. Hölder estimates. By using the L_2 estimates obtained in Section 3.1, in this section we shall derive several Hölder estimates of derivatives of $\boldsymbol{u}$. As usual, for $\mu \in (0,1)$ and a function u defined on $\mathcal{D} \subset \mathbb{R}^d$, we denote

$$[u]_{C^\mu(\mathcal{D})} = \sup_{\substack{x,y\in\mathcal{D}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\mu},$$

$$\|u\|_{C^\mu(\mathcal{D})} = [u]_{C^\mu(\mathcal{D})} + \|u\|_{L_\infty(\mathcal{D})}.$$

LEMMA 3.6. *Let $a_{\alpha\beta} = a_{\alpha\beta}(x_1)$. Assume that $\boldsymbol{u} \in C^\infty_{loc}(\overline{\mathbb{R}^d_+})$ satisfies* (3.3) *in B_2^+ with the conormal derivative boundary condition on Γ_2. Then for any α satisfying $|\alpha| = m$ and $\alpha_1 < m$, we have*

$$\|\Theta\|_{C^{1/2}(B_1^+)} + \|D^\alpha \boldsymbol{u}\|_{C^{1/2}(B_1^+)} \le N\|D^m \boldsymbol{u}\|_{L_2(B_2^+)},$$

where $N = N(d,m,n,\delta) > 0$.

PROOF. The lemma follows from the proof of Lemma 4.1 in [**15**] by using Corollaries 3.3 and 3.5. □

For $\lambda \ge 0$, let

$$U = \sum_{|\alpha|\le m} \lambda^{\frac{1}{2}-\frac{|\alpha|}{2m}} |D^\alpha \boldsymbol{u}|, \quad U' = \sum_{|\alpha|\le m, \alpha_1<m} \lambda^{\frac{1}{2}-\frac{|\alpha|}{2m}} |D^\alpha \boldsymbol{u}|.$$

Notably, since the matrix $[a^{ij}_{\bar\alpha\bar\alpha}]^n_{i,j=1}$ is positive definite, we have

$$N^{-1}U \le U' + |\Theta| \le NU, \tag{3.5}$$

where $N = N(d,m,n,\delta)$.

LEMMA 3.7. *Let $a_{\alpha\beta} = a_{\alpha\beta}(x_1)$ and $\lambda \ge 0$. Assume that $\boldsymbol{u} \in C^\infty_{loc}(\overline{\mathbb{R}^d_+})$ satisfies*

$$(-1)^m \mathcal{L}_0 \boldsymbol{u} + \lambda \boldsymbol{u} = 0$$

in B_2^+ with the conormal derivative condition on Γ_2. Then we have

$$\|\Theta\|_{C^{1/2}(B_1^+)} + \|U'\|_{C^{1/2}(B_1^+)} \le N\|U\|_{L_2(B_2^+)}, \tag{3.6}$$

$$\|U\|_{L_\infty(B_1^+)} \le N\|U\|_{L_2(B_2^+)}, \tag{3.7}$$

where $N = N(d,m,n,\delta) > 0$.

PROOF. First we prove (3.6). The case when $\lambda = 0$ follows from Lemma 3.6. To deal with the case $\lambda > 0$, we follow an idea by S. Agmon, which was originally used in a quite different situation. Let $\eta(y) = \cos(\lambda^{1/(2m)}y) + \sin(\lambda^{1/(2m)}y)$ so that η satisfies

$$D^{2m}\eta = (-1)^m \lambda\eta, \quad \eta(0) = 1, \quad |D^j\eta(0)| = \lambda^{j/(2m)}, \ \ j = 1,2,\dots.$$

Let $z = (x,y)$ be a point in $\mathbb{R}^{d+1}$, where $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, and $\hat{\boldsymbol{u}}(z)$ and $\hat{B}_r^+$ be given by

$$\hat{\boldsymbol{u}}(z) = \hat{\boldsymbol{u}}(x,y) = \boldsymbol{u}(x)\eta(y), \quad \hat{B}_r^+ = \{|z| < r : z \in \mathbb{R}^{d+1}, x_1 > 0\}.$$

Also define

$$\hat\Theta = \sum_{|\beta|=m} a_{\bar\alpha\beta} D^{(\beta,0)} \hat{\boldsymbol{u}}.$$

It is easily seen that $\hat{\boldsymbol{u}}$ satisfies

$$(-1)^m \mathcal{L}_0 \hat{\boldsymbol{u}} + (-1)^m D_y^{2m} \hat{\boldsymbol{u}} = 0$$

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in $\hat{B}_2^+$ with the conormal derivative condition on $\hat{B}_2 \cap \partial\mathbb{R}^{d+1}_+$. By Lemma 3.6 applied to $\hat{\boldsymbol{u}}$ we have

$$\|\hat{\Theta}\|_{C^{1/2}(\hat{B}_1^+)} + \left\|D_z^\beta \hat{\boldsymbol{u}}\right\|_{C^{1/2}(\hat{B}_1^+)} \le N(d,m,n,\delta)\|D_z^m \hat{\boldsymbol{u}}\|_{L_2(\hat{B}_2^+)} \tag{3.8}$$

for any $\beta = (\beta_1, \dots, \beta_{d+1})$ satisfying $|\beta| = m$ and $\beta_1 < m$. Notice that for any $\alpha = (\alpha_1, \dots, \alpha_d)$ satisfying $|\alpha| \le m$ and $\alpha_1 < m$,

$$\lambda^{\frac{1}{2}-\frac{|\alpha|}{2m}} \left\|D^\alpha \boldsymbol{u}\right\|_{C^{1/2}(B_1^+)} \le N\left\|D_z^\beta \hat{\boldsymbol{u}}\right\|_{C^{1/2}(\hat{B}_1^+)}, \quad \beta = (\alpha_1, \dots, \alpha_d, m - |\alpha|),$$

$$\|\Theta\|_{C^{1/2}(B_1^+)} \le \|\hat{\Theta}\|_{C^{1/2}(\hat{B}_1^+)},$$

and $D_z^m \hat{\boldsymbol{u}}$ is a linear combination of

$$\lambda^{\frac{1}{2}-\frac{k}{2m}} \cos(\lambda^{\frac{1}{2m}} y) D_x^k \boldsymbol{u}, \quad \lambda^{\frac{1}{2}-\frac{k}{2m}} \sin(\lambda^{\frac{1}{2m}} y) D_x^k \boldsymbol{u}, \quad k = 0, 1, \dots, m.$$

Thus the right-hand side of (3.8) is less than the right-hand side of (3.6). This completes the proof of (3.6). Finally, we get (3.7) from (3.6) and (3.5). □

Similarly, we have the following interior estimate.

LEMMA 3.8. *Let $a_{\alpha\beta} = a_{\alpha\beta}(x_1)$ and $\lambda \ge 0$. Assume that $\boldsymbol{u} \in C^\infty_{loc}(\mathbb{R}^d)$ satisfies*

$$(-1)^m \mathcal{L}_0 \boldsymbol{u} + \lambda \boldsymbol{u} = 0$$

in B_2. Then we have

$$\|\Theta\|_{C^{1/2}(B_1)} + \|U'\|_{C^{1/2}(B_1)} \le N\|U\|_{L_2(B_2)},$$
$$\|U\|_{L_\infty(B_1)} \le N\|U\|_{L_2(B_2)},$$

where $N = N(d,m,n,\delta) > 0$.

3.3. The maximal function theorem and a generalized Fefferman-Stein theorem. We recall the maximal function theorem and a generalized Fefferman-Stein theorem. Let

$$\mathcal{Q} = \{B_r(x) : x \in \mathbb{R}^d, r \in (0,\infty)\}.$$

For a function g defined in $\mathbb{R}^d$, the maximal function of g is given by

$$\mathcal{M}g(x) = \sup_{B \in \mathcal{Q}, x \in B} \fint_B |g(y)|\, dy.$$

By the Hardy–Littlewood maximal function theorem,

$$\|\mathcal{M}g\|_{L_p(\mathbb{R}^d)} \le N\|g\|_{L_p(\mathbb{R}^d)},$$

if $g \in L_p(\mathbb{R}^d)$, where $1 < p < \infty$ and $N = N(d,p)$.

Theorem 3.9 below is from [**21**] and can be considered as a generalized version of the Fefferman-Stein Theorem. To state the theorem, let

$$\mathbb{C}_l = \{C_l(i_1, \dots, i_d), i_1, \dots, i_d \in \mathbb{Z}, i_1 \ge 0\}, \quad l \in \mathbb{Z}$$

be the collection of partitions given by dyadic cubes in $\mathbb{R}^d_+$

$$[i_1 2^{-l}, (i_1+1)2^{-l}) \times \dots \times [i_d 2^{-l}, (i_d+1)2^{-l}).$$

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THEOREM 3.9. *Let $p \in (1,\infty)$, and $U, V, F \in L_{1,loc}(\mathbb{R}^d_+)$. Assume that we have $|U| \le V$ and, for each $l \in \mathbb{Z}$ and $C \in \mathbb{C}_l$, there exists a measurable function U^C on C such that $|U| \le U^C \le V$ on C and*

$$\int_C |U^C - (U^C)_C|\,dx \le \int_C F(x)\,dx.$$

Then

$$\|U\|^p_{L_p(\mathbb{R}^d_+)} \le N(d,p)\|F\|_{L_p(\mathbb{R}^d_+)}\|V\|^{p-1}_{L_p(\mathbb{R}^d_+)}.$$

3.4. Approximations of Reifenberg domains. Let Ω be a domain in $\mathbb{R}^d$. Throughout this subsection, we assume that, for any $x \in \partial\Omega$ and $r \in (0,1]$, Ω satisfies (2.5) in an appropriate coordinate system. That is, Ω satisfies the following assumption with $\gamma < 1/50$.

ASSUMPTION 3.10 (γ). There is a constant $R_0 \in (0,1]$ such that the following holds. For any $x \in \partial\Omega$ and $r \in (0,R_0]$, there is a coordinate system depending on x and r such that in this new coordinate system, we have

$$\{(y_1,y') : x_1+\gamma r < y_1\} \cap B_r(x) \subset \Omega_r(x) \subset \{(y_1,y') : x_1-\gamma r < y_1\} \cap B_r(x). \tag{3.9}$$

For any $\varepsilon \in (0,1)$, we define

$$\Omega^\varepsilon = \{x \in \Omega \,|\, \operatorname{dist}(x,\partial\Omega) > \varepsilon\}. \tag{3.10}$$

We say that a domain is a Lipschitz domain if locally the boundary is the graph of a Lipschitz function in some coordinate system. More precisely,

ASSUMPTION 3.11 (θ). There is a constant $R_1 \in (0,1]$ such that, for any $x \in \partial\Omega$ and $r \in (0,R_1]$, there exists a Lipschitz function ϕ: $\mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \,:\, x_1 > \phi(x')\}$$

and

$$\sup_{x',y' \in B'_r(x'_0), x' \ne y'} \frac{|\phi(y')-\phi(x')|}{|y'-x'|} \le \theta$$

in some coordinate system.

We note that if Ω satisfies Assumption 3.11 (θ) with a constant R_1, then Ω satisfies Assumption 3.10 with R_1 and θ in place of R_0 and γ, respectively.

Next we show that Ω^ε is a Lipschitz domain and Reifenberg flat with uniform parameters if Ω is Reifenberg flat. A related result was proved in [**4**] which, in our opinion, contains a flaw.

LEMMA 3.12. *Let Ω satisfy Assumption 3.10 (γ). Then for any $\varepsilon \in (0,R_0/4)$, Ω^ε satisfies Assumption 3.10 ($N_0\gamma^{1/2}$) with $R_0/2$ in place of R_0, and satisfies Assumption 3.11 ($N_0\gamma^{1/2}$) with $R_1 = \varepsilon$. Here N_0 is a universal constant.*

PROOF. We first prove that Ω^ε satisfies Assumption 3.11 ($N_0\gamma^{1/2}$) with $R_1 = \varepsilon > 0$. In particular, we show that, for each $x_0 \in \partial\Omega^\varepsilon$, there exists a function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega^\varepsilon \cap B_\varepsilon(x_0) = \{x \in B_\varepsilon(x_0) : x_1 > \phi(x')\}, \quad \frac{|\phi(y')-\phi(x')|}{|x'-y'|} \le N_0\gamma^{1/2} \tag{3.11}$$

for all $x', y' \in B'_\varepsilon(x'_0)$, $x' \ne y'$. Indeed, this implies Assumption 3.11 ($N_0\gamma^{1/2}$) since for a fixed $x_0 \in \partial\Omega^\varepsilon$ we can use the same ϕ for all $r \in (0,\varepsilon)$.

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Let 0 be a point on $\partial\Omega$ such that $|x_0 - 0| = \varepsilon$. That is, we have a coordinate system and $r_0 := 4\varepsilon < R_0$ such that $\partial\Omega \cap B_{r_0}(0)$ is trapped between $\{x_1 = \gamma r_0\}$ and $\{x_1 = -\gamma r_0\}$. See Figure 1. Note that $B_\varepsilon(x_0) \subset B_{r_0}(0)$ since, for $x \in B_\varepsilon(x_0)$,

$$|x| \le |x - x_0| + |x_0| < 2\varepsilon < r_0 = 4\varepsilon.$$

We show that for any $y, z \in \partial\Omega^\varepsilon \cap B_\varepsilon(x_0)$

$$|y_1 - z_1| \le N_0 \gamma^{1/2} |y' - z'|, \tag{3.12}$$

which implies (3.11). For $y, z \in \partial\Omega^\varepsilon \cap B_\varepsilon(x_0)$, we see that

$$\varepsilon - \gamma r_0 < y_1 < \varepsilon + \gamma r_0, \quad \varepsilon - \gamma r_0 < z_1 < \varepsilon + \gamma r_0. \tag{3.13}$$

Without loss of generality we assume that $y_1 \ge z_1$. To prove (3.12), let us consider two cases. First, let $\varepsilon\gamma^{1/2} \le |y' - z'|$. In this case, due to the inequalities in (3.13), we have

$$\frac{|y_1 - z_1|}{|y' - z'|} \le \frac{2\gamma r_0}{\varepsilon\gamma^{1/2}} = 8\gamma^{1/2},$$

which proves (3.12).

Now let $|y' - z'| \le \varepsilon\gamma^{1/2}$. In this case, find $w \in \partial\Omega$ such that $|y - w| = \varepsilon$. Note that $B_\varepsilon(w) \subset B_{r_0}(0)$ since

$$|w| \le |w - y| + |y - x_0| + |x_0| < 3\varepsilon < r_0 = 4\varepsilon.$$

We estimate $|w' - z'|$ as follows. Using the fact that $-\gamma r_0 < w_1 < \gamma r_0$ and the first inequality in (3.13), we have

$$|y_1 - w_1| \ge \varepsilon - 2\gamma r_0 > 0.$$

Thus using the equality

$$|w' - y'|^2 + |w_1 - y_1|^2 = \varepsilon^2,$$

we see that

$$|w' - y'|^2 \le \varepsilon^2 - (\varepsilon - 2\gamma r_0)^2 \le 4\varepsilon\gamma r_0 = 4^2\varepsilon^2\gamma.$$

Hence

$$|w' - z'| \le |w' - y'| + |y' - z'| < 5\varepsilon\gamma^{1/2}.$$

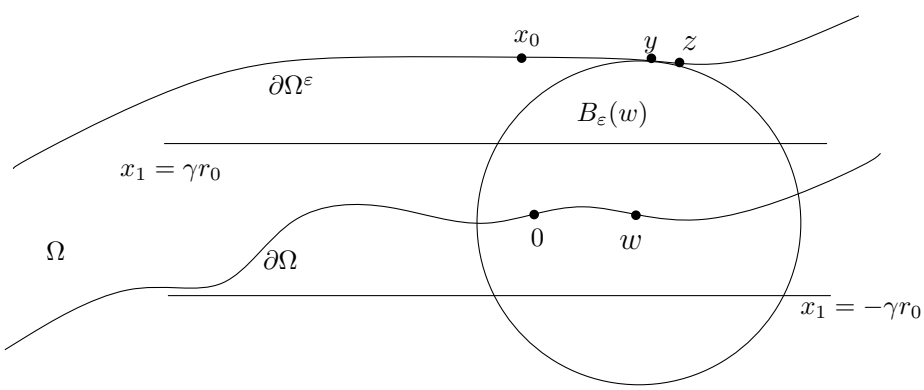

FIGURE 1

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Since $y_1 \geq z_1$, $|w' - z'| \leq 5\varepsilon\gamma^{1/2}$, and z is above the ball $B_\varepsilon(\omega)$ (recall that $\gamma < 1/50$), it follows that

$$\frac{y_1 - z_1}{|y' - z'|} \leq \frac{d}{dx}\left(-\sqrt{\varepsilon^2 - x^2}\right)\Big|_{x=5\varepsilon\gamma^{1/2}} \leq N_0\gamma^{1/2}.$$

Thus (3.12) is proved. Therefore, we have proved that Ω^ε satisfies Assumption 3.11 $(N_0\gamma^{1/2})$ with $R_1 = \varepsilon$. As pointed out earlier, this shows that Ω^ε satisfies (3.9) for all $0 < r < \varepsilon$. Thus in order to completely prove that Ω^ε satisfies Assumption 3.10 $(N_0\gamma^{1/2})$ with $R_0/2$, we need to prove that Ω^ε satisfies (3.9) for $\varepsilon \leq r < R_0/2$.

Let $\varepsilon \leq r < R_0/2$ and $x_0 \in \partial\Omega^\varepsilon$. Find $0 \in \partial\Omega$ such that $|x_0 - 0| = \varepsilon$. Then

$$B_r(x_0) \subset B_R(0),$$

where $R = \varepsilon + r < R_0$. Then the first coordinate x_1 of the point $x \in \partial\Omega^\varepsilon \cap B_r(x_0)$ is trapped by

$$\varepsilon - \gamma R < x_1 < \varepsilon + \gamma R,$$

which is the same as

$$\varepsilon - \gamma(\varepsilon + r) < x_1 < \varepsilon + \gamma(\varepsilon + r).$$

Note that

$$\gamma(\varepsilon + r) \leq 2\gamma r \leq 2\gamma^{1/2} r.$$

Thus each x_1 of $x \in \partial\Omega^\varepsilon \cap B_r(x_0)$ satisfies (3.9) with $2\gamma^{1/2}$ in place of γ. The lemma is proved. □

The next approximation result is well known. See, for instance, [**24**].

LEMMA 3.13. *Let Ω be a domain in $\mathbb{R}^d$ and satisfy Assumption* 3.11 *(θ) with some $\theta > 0$ and $R_1 \in (0, 1]$. Then there exists a sequence of expanding smooth subdomains $\Omega^k, k = 1, 2, \ldots$, such that $\Omega^k \to \Omega$ as $k \to \infty$ and each Ω^k satisfies Assumption* 3.11 *($N_0\theta$) with $R_1/2$ in place of R_1. Here N_0 is a universal constant.*

4. Systems on a half space

4.1. Estimates of mean oscillations. Now we prove the following estimate of mean oscillations. As in Section 3, we assume that all the lower-order coefficients of $\mathcal{L}$ are zero. For $\boldsymbol{f}_\alpha = (f^1_\alpha, \ldots, f^n_\alpha)^{\mathrm{tr}}$, we denote

$$F = \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} |\boldsymbol{f}_\alpha|.$$

PROPOSITION 4.1. *Let $x_0 \in \overline{\mathbb{R}^d_+}$, $\gamma \in (0, 1/4)$, $r \in (0, \infty)$, $\kappa \in [64, \infty)$, $\lambda \geq 0$, $\nu \in (2, \infty)$, $\nu' = 2\nu/(\nu - 2)$, and $\boldsymbol{f}_\alpha = (f^1_\alpha, \ldots, f^n_\alpha)^{tr} \in L_{2,loc}(\overline{\mathbb{R}^d_+})$. Assume that $\kappa r \leq R_0$ and $\boldsymbol{u} \in W^m_{\nu,loc}(\overline{\mathbb{R}^d_+})$ satisfies*

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha| \leq m} D^\alpha \boldsymbol{f}_\alpha \tag{4.1}$$

in $B^+_{\kappa r}(x_0)$ with the conormal derivative condition on $\Gamma_{\kappa r}(x_0)$. Then under Assumption 2.1 *(γ), there exists a function U^B depending on $B^+ := B^+_{\kappa r}(x_0)$ such that $N^{-1}U \leq U^B \leq NU$ and*

$$\left(|U^B - (U^B)_{B^+_r(x_0)}|\right)_{B^+_r(x_0)} \leq N(\kappa^{-1/2} + (\kappa\gamma)^{1/2}\kappa^{d/2})\left(U^2\right)^{1/2}_{B^+_{\kappa r}(x_0)}$$

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$$+N\kappa^{d/2}\left[(F^2)^{1/2}_{B^+_{\kappa r}(x_0)}+\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{B^+_{\kappa r}(x_0)}\right], \tag{4.2}$$

where $N=N(d,m,n,\delta,\nu)>0$.

The proof of the proposition is split into two cases.

Case 1: the first coordinate of $x_0\geq \kappa r/16$. In this case, we have

$$B^+_r(x_0)=B_r(x_0)\subset B_{\kappa r/16}(x_0)\subset \mathbb{R}^d_+.$$

With $B_{\kappa r/16}$ in place of $B^+_{\kappa r}$ in the right-hand side of (4.2), the problem is reduced to an interior mean oscillation estimate. Thus the proof can be done in the same way as in Proposition 7.10 in [**15**] using Theorem 3.1 and Lemma 3.8.

Case 2: $0\leq$ *the first coordinate of* $x_0<\kappa r/16$. Notice that in this case,

$$B^+_r(x_0)\subset B^+_{\kappa r/8}(\hat{x}_0)\subset B^+_{\kappa r/4}(\hat{x}_0)\subset B^+_{\kappa r/2}(\hat{x}_0)\subset B^+_{\kappa r}(x_0), \tag{4.3}$$

where $\hat{x}_0:=(0,x_0')$. Denote $R=\kappa r/2(<R_0)$. Because of Assumption 2.1, after a linear transformation, which is an orthogonal transformation determined by $B=B_R(\hat{x}_0)$ followed by a translation downward, we may assume

$$B^+_R(\hat{y}_0)\subset \Omega_R(\hat{y}_0)\subset \{(y_1,y'):-2\gamma R<y_1\}\cap B_R(\hat{y}_0) \tag{4.4}$$

and

$$\sup_{|\alpha|=|\beta|=m}\int_{B_R(\hat{y}_0)}|a_{\alpha\beta}(x)-\bar{a}_{\alpha\beta}(y_1)|\,dy\leq \gamma|B_R|. \tag{4.5}$$

Here Ω is the image of $\mathbb{R}^d_+$ under the linear transformation and $\hat{y}_0$ (or y_0) is the

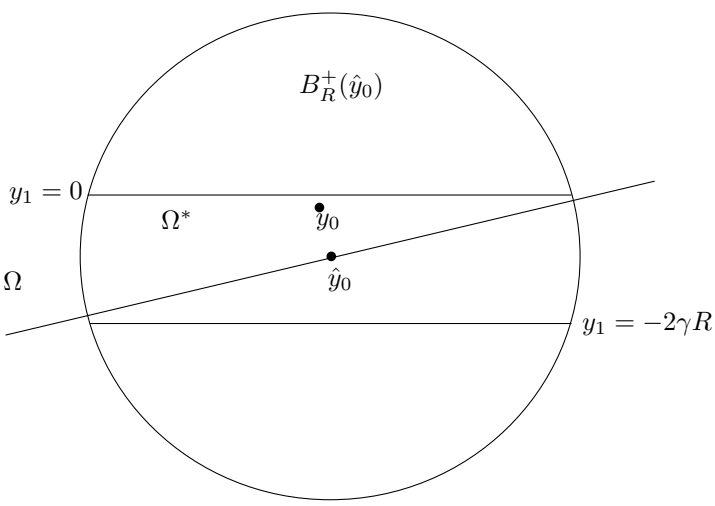

FIGURE 2. $\hat{y}_0$ (or y_0) is the new coordinates of $\hat{x}_0$ (or x_0).

new coordinates of $\hat{x}_0$ (or x_0). See Figure 2. Then (4.3) becomes

$$\Omega_r(y_0)\subset \Omega_{R/4}(\hat{y}_0)\subset \Omega_{R/2}(\hat{y}_0)\subset \Omega_R(\hat{y}_0)\subset \Omega_{\kappa r}(y_0). \tag{4.6}$$

For convenience of notation, in the new coordinate system we still denote the corresponding unknown function, the coefficients, and the data by $\boldsymbol{u}$, $a_{\alpha\beta}$, $\bar{a}_{\alpha\beta}$, and $\boldsymbol{f}_\alpha$, respectively. Note that, without loss of generality, we may assume that the coefficients $\bar{a}_{\alpha\beta}(y_1)$ in (4.5) are infinitely differentiable.

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Below we present a few lemmas, which should be read as parts of the proof of the second case.

Let us introduce the following well-known extension operator. Let $\{c_1, \cdots, c_m\}$ be the solution to the system:

$$\sum_{k=1}^{m} \left(-\frac{1}{k}\right)^j c_k = 1, \quad j = 0, \cdots, m-1. \tag{4.7}$$

For a function w defined on $\mathbb{R}^d_+$, set

$$\mathcal{E}_m w = \begin{cases} w(y_1, y') & \text{if} \quad y_1 > 0 \\ \displaystyle\sum_{k=1}^{m} c_k w(-\frac{1}{k} y_1, y') & \text{otherwise} \end{cases}.$$

Note that $\mathcal{E}_m w \in W^m_{2,\mathrm{loc}}(\mathbb{R}^d)$ if $w \in W^m_{2,\mathrm{loc}}(\overline{\mathbb{R}^d_+})$. Indeed, by (4.7)

$$D_1^j \left(\sum_{k=1}^{m} c_k w(-\frac{1}{k} y_1, y') \right) \Big|_{y_1=0} = \sum_{k=1}^{m} \left(-\frac{1}{k}\right)^j c_k D_1^j w(0, y') = D_1^j w(0, y')$$

for $j = 0, \cdots, m-1$.

Denote $\Omega^* = \mathbb{R}^d_- \cap \Omega \cap B_R(\hat{y}_0)$. Recall that in the new coordinate system we still denote the corresponding unknown function, the coefficients, and the data by $\boldsymbol{u}$, $a_{\alpha\beta}$, $\bar{a}_{\alpha\beta}$, and $\boldsymbol{f}_\alpha$, respectively. Throughout the end of this subsection, the derivatives are taken with respect to the y-coordinates. The following lemma contains the key observation in the proof of Proposition 4.1.

Lemma 4.2. *The function $\boldsymbol{u}$ satisfies*

$$\begin{aligned} (-1)^m \mathcal{L}_0 \boldsymbol{u} + \lambda \boldsymbol{u} = (-1)^m & \sum_{|\alpha|=|\beta|=m} D^\alpha \left((\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^\beta \boldsymbol{u} \right) \\ & + \sum_{|\alpha| \le m} D^\alpha \tilde{\boldsymbol{f}}_\alpha + \sum_{|\alpha|=m} D^\alpha \boldsymbol{g}_\alpha - \lambda \boldsymbol{h} \end{aligned} \tag{4.8}$$

in $B^+_R(\hat{y}_0)$ with the conormal derivative boundary condition on $\Gamma_R(\hat{y}_0)$. In the above, $\mathcal{L}_0$ is the differential operator with the coefficients $\bar{a}_{\alpha\beta}$ from (4.5)*, and*

$$\tilde{\boldsymbol{f}}_\alpha = \boldsymbol{f}_\alpha + c_{\alpha,k} \boldsymbol{f}_\alpha(-ky_1, y')\, 1_{(-ky_1, y') \in \Omega^*},$$

$$\boldsymbol{g}_\alpha = c_{\alpha,k} (-1)^{m+1} \sum_{|\beta|=m} \sum_{k=1}^{m} a_{\alpha\beta}(-ky_1, y') (D^\beta \boldsymbol{u})(-ky_1, y')\, 1_{(-ky_1, y') \in \Omega^*},$$

$$\boldsymbol{h} = \sum_{k=1}^{m} k c_k \boldsymbol{u}(-ky_1, y')\, 1_{(-ky_1, y') \in \Omega^*},$$

where $c_{\alpha,k} = (-1)^{\alpha_1} c_k k^{-\alpha_1+1}$ are constants.

Proof. Take a test function $\phi = (\phi^1, \phi^2, \ldots, \phi^n) \in W^m_2(B^+_R(\hat{y}_0))$ which vanishes near $\mathbb{R}^d_+ \cap \partial B_R(\hat{y}_0)$. Due to (4.4), it is easily seen that $\mathcal{E}_m \phi \in W^m_2(\Omega_R(\hat{y}_0))$ and vanishes near $\Omega \cap \partial B_R(\hat{y}_0)$. Since $\boldsymbol{u}$ satisfies (4.1) with the conormal derivative condition on $\partial\Omega \cap B_R(\hat{y}_0)$, we have

$$\int_{\Omega_R(\hat{y}_0)} \sum_{|\alpha|=|\beta|=m} D^\alpha \mathcal{E}_m \phi \cdot a_{\alpha\beta} D^\beta \boldsymbol{u} + \lambda \mathcal{E}_m \phi \cdot \boldsymbol{u}\, dy$$

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$$= \sum_{|\alpha|\le m} \int_{\Omega_R(\hat{y}_0)} (-1)^{|\alpha|} D^\alpha \mathcal{E}_m \phi \cdot \boldsymbol{f}_\alpha \, dy.$$

From this identity and the definition of the extension operator $\mathcal{E}_m$, a straightforward calculation gives

$$\int_{B_R^+(\hat{y}_0)} \sum_{|\alpha|=|\beta|=m} D^\alpha \phi \cdot \bar{a}_{\alpha\beta} D^\beta \boldsymbol{u} + \lambda \phi \cdot \boldsymbol{u} \, dy$$

$$= \sum_{|\alpha|=|\beta|=m} \int_{B_R^+(\hat{y}_0)} D^\alpha \phi \cdot (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^\beta \boldsymbol{u} \, dy + \sum_{|\alpha|\le m} \int_{B_R^+(\hat{y}_0)} (-1)^{|\alpha|} D^\alpha \phi \cdot \tilde{\boldsymbol{f}}_\alpha \, dy$$

$$= \sum_{|\alpha|=m} \int_{B_R^+(\hat{y}_0)} D^\alpha \phi \cdot (-1)^{|\alpha|} \boldsymbol{g}_\alpha \, dy - \lambda \int_{B_R^+(\hat{y}_0)} \phi \cdot \boldsymbol{h} \, dy.$$

The lemma is proved. □

Set

$$G_\alpha = (-1)^m \sum_{|\beta|=m} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \, D^\beta u + \tilde{\boldsymbol{f}}_\alpha + \boldsymbol{g}_\alpha \quad \text{for} \quad |\alpha| = m,$$

$$G_\alpha = \tilde{\boldsymbol{f}}_\alpha \quad \text{for} \quad 0 \le |\alpha| < m.$$

We see that $G_\alpha \in L_2(B_R^+(\hat{y}_0))$, and by (4.8)

$$(-1)^m \mathcal{L}_0 \boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha|\le m} D^\alpha G_\alpha - \lambda \boldsymbol{h}.$$

Take φ to be an infinitely differentiable function such that

$$0 \le \varphi \le 1, \quad \varphi = 1 \text{ on } B_{R/2}(\hat{y}_0), \quad \varphi = 0 \text{ outside } B_R(\hat{y}_0).$$

Then we find a unique solution $\boldsymbol{w} \in W_2^m(\mathbb{R}^d_+)$ satisfying

$$(-1)^m \mathcal{L}_0 \boldsymbol{w} + \lambda \boldsymbol{w} = \sum_{|\alpha|\le m} D^\alpha (\varphi G_\alpha) - \lambda \varphi \boldsymbol{h} \tag{4.9}$$

with the conormal derivative condition on $\partial \mathbb{R}^d_+$. By Theorem 3.1 we have

$$\sum_{|\alpha|\le m} \lambda^{\frac{1}{2} - \frac{|\alpha|}{2m}} \|D^\alpha \boldsymbol{w}\|_{L_2(\mathbb{R}^d_+)} \le N \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} \|\varphi G_\alpha\|_{L_2(\mathbb{R}^d_+)} + N\lambda \|\varphi \boldsymbol{h}\|_{L_2(\mathbb{R}^d_+)}. \tag{4.10}$$

Now we set $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{w}$ in $B_R(\hat{y}_0) \cap \overline{\mathbb{R}^d_+}$. Then $\boldsymbol{v}$ satisfies

$$(-1)^m \mathcal{L}_0 \boldsymbol{v} + \lambda \boldsymbol{v} = 0 \tag{4.11}$$

in $B_{R/2}^+(\hat{y}_0)$ with the conormal derivative condition on $\Gamma_{R/2}(\hat{y}_0)$. Since the coefficients of $\mathcal{L}_0$ are infinitely differentiable, by the classical theory $\boldsymbol{v}$ is infinitely differentiable in $B_{R/2}(\hat{y}_0) \cap \overline{\mathbb{R}^d_+}$.

Recall that

$$U = \sum_{|\alpha|\le m} \lambda^{\frac{1}{2} - \frac{|\alpha|}{2m}} |D^\alpha \boldsymbol{u}|, \quad F = \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} |\boldsymbol{f}_\alpha|.$$

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LEMMA 4.3. *We have*

$$\sum_{k=0}^{m} \lambda^{\frac{1}{2}-\frac{k}{2m}} (|D^k \boldsymbol{w}|^2)^{1/2}_{B_R^+(\hat{y}_0)} \le N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)} + N(F^2)^{1/2}_{\Omega_R(\hat{y}_0)}, \tag{4.12}$$

where ν and ν' are from Proposition 4.1.

PROOF. By (4.10) and the definition of G_α, we have

$$\begin{aligned}\sum_{|\alpha|\le m} \lambda^{\frac{1}{2}-\frac{|\alpha|}{2m}} \|D^\alpha \boldsymbol{w}\|_{L_2(\mathbb{R}^d_+)} &\le N \sum_{|\alpha|=|\beta|=m} \|\varphi(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^\beta \boldsymbol{u}\|_{L_2(\mathbb{R}^d_+)} \\ &\quad + N \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|\varphi \tilde{\boldsymbol{f}}_\alpha\|_{L_2(\mathbb{R}^d_+)} + N \sum_{|\beta|=m} \|\varphi \boldsymbol{g}_\alpha\|_{L_2(\mathbb{R}^d_+)} + N\lambda \|\varphi \boldsymbol{h}\|_{L_2(\mathbb{R}^d_+)}.\end{aligned}$$

Note that Ω^* lies in the strip $B_R(\hat{y}_0) \cap \{y : -2\gamma R < y_1 < 0\}$. Thus, by the definitions of $\tilde{\boldsymbol{f}}_\alpha$, $\boldsymbol{g}_\alpha$, and $\boldsymbol{h}$, it follows that the left-hand side of (4.12) is less than a constant times

$$\begin{aligned}&\sum_{|\alpha|=|\beta|=m} \left(|(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^\beta \boldsymbol{u}|^2\right)^{1/2}_{B_R^+(\hat{y}_0)} + \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \left(|\boldsymbol{f}_\alpha|^2\right)^{1/2}_{\Omega_R(\hat{y}_0)} \\ &\qquad + \left(I_{\{-2\gamma R<y_1<0\}} |D^m \boldsymbol{u}|^2\right)^{1/2}_{\Omega_R(\hat{y}_0)} + \lambda \left(I_{\{-2\gamma R<y_1<0\}} |\boldsymbol{u}|^2\right)^{1/2}_{\Omega_R(\hat{y}_0)} \\ &\qquad := I_1 + I_2 + I_3 + I_4.\end{aligned}$$

By using (4.5) and Hölder's inequality, we see that

$$I_1 \le N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}.$$

It is clear that I_2 is bounded by $N(F^2)^{1/2}_{\Omega_R(\hat{y}_0)}$. Observe that by Hölder's inequality we have

$$\begin{aligned}\left(I_{\{-2\gamma R<y_1<0\}} |D^m \boldsymbol{u}|^2\right)^{1/2}_{\Omega_R(\hat{y}_0)} &\le \left(I_{\{-2\gamma R<y_1<0\}}\right)^{1/\nu'}_{\Omega_R(\hat{y}_0)} (|D^m \boldsymbol{u}|^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)} \\ &\le N\gamma^{1/\nu'} (|D^m \boldsymbol{u}|^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}.\end{aligned}$$

Thus I_3 is also bounded by $N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}$. In a similar way, I_4 is bounded by $N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}$. Therefore, we conclude (4.12). □

Now we are ready to complete the proof of Proposition 4.1. We shall show that $U^B := U' + |\Theta|$ satisfies the inequalities in the proposition. First, we consider the case when $\kappa\gamma \le 1/10$. By (4.12), it follows that

$$(W^2)^{1/2}_{B_R^+(\hat{y}_0)} \le N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)} + N(F^2)^{1/2}_{\Omega_R(\hat{y}_0)}. \tag{4.13}$$

Noting that $B_r^+(y_0) \subset B_R^+(\hat{y}_0)$ and, since $\kappa\gamma \le 1/10$, $|B_R^+(\hat{y}_0)|/|B_r^+(y_0)| \le N(d)\kappa^d$, we obtain from (4.13)

$$(W^2)^{1/2}_{B_r^+(y_0)} \le N\kappa^{\frac{d}{2}} \left(\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)} + (F^2)^{1/2}_{\Omega_R(\hat{y}_0)}\right). \tag{4.14}$$

Next we denote

$$\mathcal{D}_1 = \Omega_r(y_0) \cap \{y_1 < 0\} = \Omega^*, \quad \mathcal{D}_2 = B_r^+(y_0), \quad \mathcal{D}_3 = B_{R/4}^+(\hat{y}_0).$$

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Since $\boldsymbol{v}$ in (4.11) is infinitely differentiable, by applying Lemma 3.7 to the system (4.11) with a scaling argument, we compute

$$\begin{aligned}(4.15)\quad \big(|V'-(V')_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}+\big(|\hat{\Theta}-(\hat{\Theta})_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}&\le Nr^{1/2}\big([V']_{C^{1/2}(\mathcal{D}_2)}+[\hat{\Theta}]_{C^{1/2}(\mathcal{D}_2)}\big)\\ &\le Nr^{1/2}\big([V']_{C^{1/2}(\mathcal{D}_3)}+[\hat{\Theta}]_{C^{1/2}(\mathcal{D}_3)}\big)\le N\kappa^{-1/2}(V^2)^{1/2}_{B^+_{R/2}(\hat{y}_0)}.\end{aligned}$$

Thanks to the fact that $\kappa\gamma\le 1/10$, we have

$$(4.16)\qquad |\mathcal{D}_1|\le N\kappa\gamma|\mathcal{D}_2|,\quad |\Omega_R(\hat{y}_0)|\le N\kappa^d|\mathcal{D}_2|.$$

By combining (4.13), (4.14), and (4.15), we get

$$\begin{aligned}&\big(|U^B-(U^B)_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}\\ &\le N\big(|V'-(V')_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}+N\big(|\hat{\Theta}-(\hat{\Theta})_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}+N\big(W\big)_{\mathcal{D}_2}\\ &\le N\kappa^{-1/2}(U^2)^{1/2}_{B^+_{R/2}(\hat{y}_0)}+N\kappa^{\frac{d}{2}}\left(\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}+(F^2)^{1/2}_{\Omega_R(\hat{y}_0)}\right).\end{aligned}$$

By using the triangle inequality and the assumption $\kappa\gamma\le 1/10$,

$$\begin{aligned}\big(|U^B-(U^B)_{\Omega_r(y_0)}|\big)_{\Omega_r(y_0)}&\le N\big(|U^B-(U^B)_{\mathcal{D}_2}|\big)_{\mathcal{D}_2}\\ &+N\kappa\gamma(U^B)_{\mathcal{D}_2}+N(1_{\mathcal{D}_1}U^B)_{\Omega_r(y_0)}.\end{aligned}$$

We use (3.5), (4.16), and Hölder's inequality to bound the last two terms on the right-hand side above as follows:

$$\kappa\gamma(U^B)_{\mathcal{D}_2}\le N\kappa\gamma(|U|^2)^{1/2}_{\mathcal{D}_2}\le N\kappa\gamma\kappa^{d/2}(|U|^2)^{1/2}_{\Omega_R(\hat{y}_0)},$$

$$(1_{\mathcal{D}_1}U^B)_{\Omega_r(y_0)}\le(1_{\mathcal{D}_1})^{1/2}_{\Omega_r(y_0)}(|U|^2)^{1/2}_{\Omega_r(y_0)}\le N(\kappa\gamma)^{1/2}\kappa^{d/2}(|U|^2)^{1/2}_{\Omega_R(\hat{y}_0)}.$$

Therefore,

$$\begin{aligned}(4.17)\quad \big(|U^B-(U^B)_{\Omega_r(y_0)}|\big)_{\Omega_r(y_0)}&\le N\big(\kappa^{-1/2}+(\kappa\gamma)^{1/2}\kappa^{d/2}\big)(|U|^2)^{1/2}_{\Omega_R(\hat{y}_0)}\\ &+N\kappa^{\frac{d}{2}}\left(\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{\Omega_R(\hat{y}_0)}+(F^2)^{1/2}_{\Omega_R(\hat{y}_0)}\right).\end{aligned}$$

In the remaining case when $\kappa\gamma>1/10$, by (3.5) and (4.6),

$$\begin{aligned}\big(|U^B-(U^B)_{\Omega_r(y_0)}|\big)_{\Omega_r(y_0)}&\le N(U^B)_{\Omega_r(y_0)}\le N(U)_{\Omega_r(y_0)}\\ &\le N(|U|^2)^{1/2}_{\Omega_r(y_0)}\le N\kappa^{d/2}(|U|^2)^{1/2}_{\Omega_R(\hat{y}_0)},\end{aligned}$$

where in the last inequality, we used the obvious inequality $|\Omega_R(\hat{y}_0)|\le N\kappa^d|\Omega_r(y_0)|$. Therefore, in this case, (4.17) still holds. Finally, we transform the obtained inequality back to the original coordinates to get the inequality (4.2). This completes the proof of Proposition 4.1.

4.2. Proof of Theorem 2.2. We finish the proof of Theorem 2.2 in this subsection. First we observe that by taking a sufficiently large λ_0 and using interpolation inequalities, we can move all the lower-order terms of $\mathcal{L}\boldsymbol{u}$ to the right-hand side. Thus in the sequel we assume that all the lower-order coefficients of $\mathcal{L}$ are zero.

Recall the definition of $\mathbb{C}_l, l\in\mathbb{Z}$ above Theorem 3.9. Notice that if $x\in C\in\mathbb{C}_l$, then for the smallest $r>0$ such that $C\subset B_r(x)$ we have

$$\fint_C\fint_C|g(y)-g(z)|\,dy\,dz\le N(d)\fint_{B^+_r(x)}\fint_{B^+_r(x)}|g(y)-g(z)|\,dy\,dz.$$

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We use this inequality in the proof of the following corollary.

COROLLARY 4.4. *Let $\gamma \in (0,1/4)$, $\lambda > 0$, $\nu \in (2,\infty)$, $\nu' = 2\nu/(\nu-2)$, and $z_0 \in \overline{\mathbb{R}^d_+}$. Assume that $\boldsymbol{u} \in W^m_{\nu,loc}(\overline{\mathbb{R}^d_+})$ vanishes outside $B_{\gamma R_0}(z_0)$ and satisfies*

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha|\le m} D^\alpha \boldsymbol{f}_\alpha$$

locally in $\mathbb{R}^d_+$ with the conormal derivative condition on $\partial\mathbb{R}^d_+$, where $\boldsymbol{f}_\alpha \in L_{2,loc}(\overline{\mathbb{R}^d_+})$. Then under Assumption 2.1 (γ), for each $l \in \mathbb{Z}$, $C \in \mathbb{C}_l$, and $\kappa \ge 64$, there exists a function U^C depending on C such that $N^{-1}U \le U^C \le NU$ and

$$\left(|U^C - (U^C)_C|\right)_C \le N\,(F_\kappa)_C\,, \tag{4.18}$$

where $N = N(d,\delta,m,n,\tau)$ and

$$\begin{aligned} F_\kappa = {} & \left(\kappa^{-1/2} + (\kappa\gamma)^{1/2}\kappa^{d/2}\right)\left(\mathcal{M}(1_{\mathbb{R}^d_+}U^2)\right)^{1/2} \\ & + \kappa^{d/2}\left[\left(\mathcal{M}(1_{\mathbb{R}^d_+}F^2)\right)^{1/2} + \gamma^{1/\nu'}(\mathcal{M}(1_{\mathbb{R}^d_+}U^\nu))^{1/\nu}\right]. \end{aligned}$$

PROOF. For each $\kappa \ge 64$ and $C \in \mathbb{C}_l$, let $B_r(x_0)$ be the smallest ball containing C. Clearly, $x_0 \in \overline{\mathbb{R}^d_+}$.

If $\kappa r > R_0$, then we take $U^C = U$. Note that the volumes of C, $B_r(x_0)$, and $B^+_r(x_0)$ are comparable, and $C \subset B^+_r(x_0)$. Then by the triangle inequality and Hölder's inequality, the left-hand side of (4.18) is less than

$$\begin{aligned} N\left(U^2\right)^{1/2}_{B^+_r(x_0)} &\le N\kappa^{d/2}\left(1_{B^+_{\gamma R_0}(z_0)}U^2\right)^{1/2}_{B_{\kappa r}(x_0)} \\ &\le N\kappa^{d/2}\left(1_{B_{\gamma R_0(z_0)}}\right)^{1/\nu'}_{B_{\kappa r}(x_0)}\left(1_{\mathbb{R}^d_+}U^\nu\right)^{1/\nu}_{B_{\kappa r}(x_0)} \le N\kappa^{d/2}\gamma^{1/\nu'}\left(1_{\mathbb{R}^d_+}U^\nu\right)^{1/\nu}_{B_{\kappa r}(x_0)}. \end{aligned}$$

Here the first inequality is because $|B_{\kappa r}(x_0)| \le 2\kappa^d |B^+_r(x_0)|$ and U vanishes outside $B_{\gamma R_0}(z_0)$. The second inequality follows from Hölder's inequality. In the last inequality we used $\kappa r > R_0$ and $\gamma^d \le \gamma$. Note that

$$\left(1_{\mathbb{R}^d_+}U^\nu\right)^{1/\nu}_{B_{\kappa r}(x_0)} \le \mathcal{M}^{1/\nu}\left(1_{\mathbb{R}^d_+}U^\nu\right)(x) \tag{4.19}$$

for all $x \in C$. Hence the inequality (4.18) follows.

If $\kappa r \le R_0$, from Proposition 4.1, we find U^B with $B^+ = B^+_{\kappa r}(x_0)$. Take $U^C = U^B$. Then by Proposition 4.1 we have

$$\left(|U^C - \left(U^C\right)_C|\right)_C \le N(d)I, \tag{4.20}$$

where I is the right-hand side of the inequality (4.2). Note that, for example,

$$\left(U^2\right)_{B^+_{\kappa r}(x_0)} \le N(d)\left(1_{\mathbb{R}^d_+}U^2\right)_{B_{\kappa r}(x_0)}.$$

Using this and inequalities like (4.19), we see that (4.20) implies the desired inequality (4.18). □

THEOREM 4.5. *Let $p \in (2,\infty)$, $\lambda > 0$, $z_0 \in \overline{\mathbb{R}^d_+}$, and $\boldsymbol{f}_\alpha \in L_p(\mathbb{R}^d_+)$. There exist positive constants $\gamma \in (0,1/4)$ and N, depending only on d, δ, m, n, p, such that under Assumption 2.1 (γ), for $\boldsymbol{u} \in W^m_p(\mathbb{R}^d_+)$ vanishing outside $B_{\gamma R_0}(z_0)$ and satisfying*

$$(-1)^m \mathcal{L}\boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha|\le m} D^\alpha \boldsymbol{f}_\alpha$$

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in $\mathbb{R}^d_+$ *with the conormal derivative condition on* $\partial\mathbb{R}^d_+$*, we have*

$$\|U\|_{L_p(\mathbb{R}^d_+)} \le N\|F\|_{L_p(\mathbb{R}^d_+)},$$

where $N = N(d, \delta, m, n, p)$.

Proof. Let $\gamma > 0$ and $\kappa \ge 64$ be constants to be specified below. Take a constant ν such that $p > \nu > 2$. Then we see that $\boldsymbol{u} \in W^m_{\nu,\mathrm{loc}}(\overline{\mathbb{R}^d_+})$ and all the conditions in Corollary 4.4 are satisfied.

For each $l \in \mathbb{Z}$ and $C \in \mathbb{C}_l$, let U^C be the function from Corollary 4.4. Then by Corollary 4.4 and Theorem 3.9 we have

$$\|U\|^p_{L_p(\mathbb{R}^d_+)} \le N\|F_\kappa\|_{L_p(\mathbb{R}^d_+)}\|U\|^{p-1}_{L_p(\mathbb{R}^d_+)}.$$

The implies that

$$\|U\|_{L_p(\mathbb{R}^d_+)} \le N\|F_\kappa\|_{L_p(\mathbb{R}^d_+)}.$$

Now we observe that by the Hardy–Littlewood maximal function theorem

$$\begin{aligned}\|F_\kappa\|_{L_p(\mathbb{R}^d_+)} \le \|F_\kappa\|_{L_p(\mathbb{R}^d)} &\le N\big(\kappa^{-1/2} + (\kappa\gamma)^{1/2}\kappa^{d/2}\big)\|1_{\mathbb{R}^d_+}U\|_{L_p(\mathbb{R}^d)} \\ &\quad + N\kappa^{d/2}\|1_{\mathbb{R}^d_+}F\|_{L_p(\mathbb{R}^d)} + N\kappa^{d/2}\gamma^{1/\nu'}\|1_{\mathbb{R}^d_+}U\|_{L_p(\mathbb{R}^d)}.\end{aligned}$$

To complete the proof, it remains to choose a sufficiently large κ, and then a sufficiently small γ so that

$$N\big(\kappa^{-1/2} + (\kappa\gamma)^{1/2}\kappa^{d/2}\big) + N\kappa^{d/2}\gamma^{1/\nu'} < 1/2.$$

□

Proof of Theorem 2.2. We treat the following three cases separately.

Case 1: $p = 2$. In this case, the theorem follows from Theorem 3.1.

Case 2: $p \in (2, \infty)$. Assertion (i) follows from Theorem 4.5 and the standard partition of unity argument. Then Assertion (ii) is derived from Assertion (i) by using the method of continuity. Finally, Assertion (iii) is due to a standard scaling argument.

Case 3: $p \in (1, 2)$. In this case, Assertion (i) is a consequence of the duality argument and the W^m_q-solvability obtained above for $q = p/(p-1) \in (2, \infty)$. With the a priori estimate, the remaining part of the theorem is proved in the same way as in Case 2. The theorem is proved. □

5. Systems on a Reifenberg flat domain

In this section, we consider elliptic systems on a Reifenberg flat domain. The crucial ingredients of the proofs below are the interior and the boundary estimates established in Sections 3, a result in [**28, 19**] on the "crawling of ink drops", and an idea in [**6**].

By a scaling, in the sequel we may assume $R_0 = 1$ in Assumption 2.3. Recall the definitions of U and F in Sections 3 and 4.

Lemma 5.1. *Let* $\gamma \in (0, 1/50)$*,* $R \in (0, 1]$*,* $\lambda \in (0, \infty)$*,* $\nu \in (2, \infty)$*,* $\nu' = 2\nu/(\nu - 2)$*,* $\boldsymbol{f}_\alpha = (f^1_\alpha, \dots, f^n_\alpha)^{tr} \in L_{2,loc}(\overline{\Omega})$*,* $|\alpha| \le m$*. Assume that* $a_{\alpha\beta} \equiv 0$ *for any* α, β *satisfying* $|\alpha| + |\beta| < 2m$ *and that* $\boldsymbol{u} \in W^m_{\nu,loc}(\overline{\Omega})$ *satisfies* (2.6) *locally in* Ω *with the conormal derivative condition on* $\partial\Omega$*. Then the following hold true.*

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(i) Suppose $0 \in \Omega$, *dist*$(0, \partial\Omega) \geq R$, *and Assumption* 2.3 *(*γ*) (i) holds at the origin. Then there exist nonnegative functions* V *and* W *in* B_R *such that* $U \leq V + W$ *in* B_R, *and* V *and* W *satisfy*

$$(W^2)^{1/2}_{B_R} \leq N\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{B_R} + N(F^2)^{1/2}_{B_R}$$

and

$$\|V\|_{L_\infty(B_{R/4})} \leq N\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{B_R} + N(F^2)^{1/2}_{B_R} + N(U^2)^{1/2}_{B_R},$$

where $N = N(d, n, m, \delta, \nu) > 0$ *is a constant.*
(ii) Suppose $0 \in \partial\Omega$ *and Assumption* 2.3 *(*γ*) (ii) holds at the origin. Then there exist nonnegative functions* V *and* W *in* Ω_R *such that* $U \leq V + W$ *in* Ω_R, *and* W *and* V *satisfy*

$$(W^2)^{1/2}_{\Omega_R} \leq N\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{\Omega_R} + N(F^2)^{1/2}_{\Omega_R} \tag{5.1}$$

and

$$\|V\|_{L_\infty(\Omega_{R/4})} \leq N\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{\Omega_R} + N(F^2)^{1/2}_{\Omega_R} + N(U^2)^{1/2}_{\Omega_R}, \tag{5.2}$$

where $N = N(d, n, m, \delta, \nu) > 0$ *is a constant.*

Proof. The proof is similar to that of Proposition 4.1 with some modifications. We assume that Assumption 2.3 holds in the original coordinates. Without loss of generality, we may further assume that the coefficients $\bar{a}_{\alpha\beta}$ are infinitely differentiable.

Assertion (i) is basically an interior estimate which does not involve boundary conditions, so the proof is exactly the same as that of Assertion (i) in [**15**, Lemma 8.3].

Next, we prove Assertion (ii). Due to Assumption 2.3, by shifting the origin upward, we can assume that

$$B^+_R(x_0) \subset \Omega_R(x_0) \subset \{(x_1, x') : -2\gamma R < x_1\} \cap B_R(x_0)$$

where $x_0 \in \partial\Omega$ (see Figure 3). Define $\bar{a}_{\alpha\beta}$ as in Section 4. Then $\boldsymbol{u}$ satisfies (4.8)

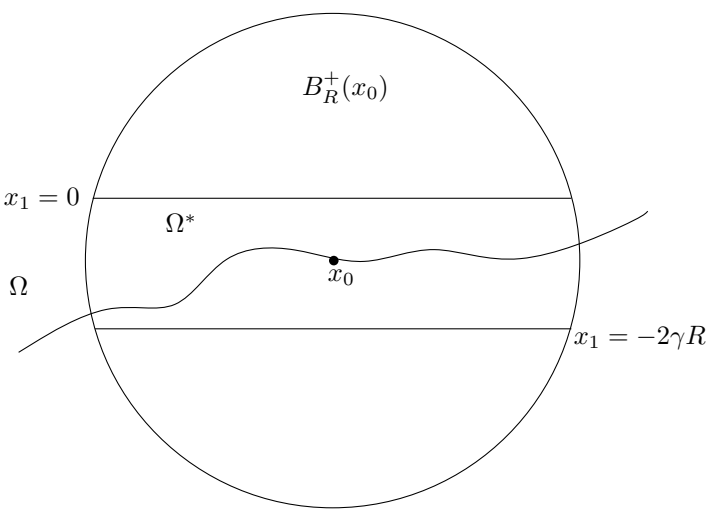

Figure 3

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in $B_R^+(x_0)$ with the conormal derivative condition on $\Gamma_R(x_0)$. By following the argument in the proof of Proposition 4.1, we can find $\boldsymbol{w} \in W_2^m(\mathbb{R}_+^d)$ and $\boldsymbol{v} \in W_2^m(B_R^+(x_0))$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{v}$, the function $\boldsymbol{w}$ satisfies

$$\sum_{k=0}^{m} \lambda^{\frac{1}{2}-\frac{k}{2m}} (|D^k \boldsymbol{w}|^2)^{1/2}_{B_R^+(x_0)} \le N\gamma^{1/\nu'} (U^\nu)^{1/\nu}_{\Omega_R(x_0)} + N(F^2)^{1/2}_{\Omega_R(x_0)}, \tag{5.3}$$

and the function $\boldsymbol{v}$ satisfies

$$(-1)^m \mathcal{L}\boldsymbol{v} + \lambda \boldsymbol{v} = 0$$

in $B_{R/2}^+(x_0)$ with the conormal derivative condition on $\Gamma_{R/2}(x_0)$. We define V and W in $B_R^+(x_0)$ as in Section 4. As noted in the proof of Proposition 4.1, we can assume that $\boldsymbol{v}$ is infinitely differentiable. Applying Lemma 3.7, we get

$$\|V\|_{L_\infty(B_{R/4}^+(x_0))} \le N(V^2)^{1/2}_{B_{R/2}^+(x_0)}. \tag{5.4}$$

Now we extend W and V on $\Omega^* = \mathbb{R}_-^d \cap \Omega_R(x_0)$ by setting $W = U$ and $V = 0$, respectively. Then we see that by Hölder's inequality and (5.3)

$$\begin{aligned}
\left(W^2\right)^{1/2}_{\Omega_R(x_0)} &= \left[\frac{1}{|\Omega_R(x_0)|}\int_{B_R^+(x_0)} W^2\,dx + \frac{1}{|\Omega_R(x_0)|}\int_{\Omega^*} U^2\,dx\right]^{1/2} \\
&\le N\left(W^2\right)^{1/2}_{B_R^+(x_0)} + N\left(1_{\Omega^*}\right)^{1/\nu'}_{\Omega_R(x_0)} (U^\nu)^{1/\nu}_{\Omega_R(x_0)} \\
&\le N\gamma^{1/\nu'}(U^\nu)^{1/\nu}_{\Omega_R(x_0)} + N(F^2)^{1/2}_{\Omega_R(x_0)}.
\end{aligned}$$

Upon recalling that the origin was shifted from x_0, we arrive at (5.1). To prove (5.2) we observe that by (5.4) and the fact that $V \le U + W$

$$\|V\|_{L_\infty(\Omega_{R/4}(x_0))} = \|V\|_{L_\infty(B_{R/4}^+(x_0))} \le N\left(V^2\right)^{1/2}_{B_{R/2}^+(x_0)}$$

$$\le N\left(V^2\right)^{1/2}_{\Omega_R(x_0)} \le N\left(W^2\right)^{1/2}_{\Omega_R(x_0)} + N\left(U^2\right)^{1/2}_{\Omega_R(x_0)}.$$

This together with (5.1) gives (5.2). This completes the proof of the lemma. □

For a function f on a set $\mathcal{D} \subset \mathbb{R}^d$, we define its maximal function $\mathcal{M}f$ by $\mathcal{M}f = \mathcal{M}(I_{\mathcal{D}} f)$. For any $s > 0$, we introduce two level sets

$$\mathcal{A}(s) = \{x \in \Omega : U > s\},$$

$$\mathcal{B}(s) = \left\{x \in \Omega : \gamma^{-1/\nu'}\left(\mathcal{M}(F^2)\right)^{1/2} + \left(\mathcal{M}(U^\nu)\right)^{1/\nu} > s\right\}.$$

With Lemma 5.1 in hand, we get the following corollary.

Corollary 5.2. *Under the assumptions of Lemma 5.1, suppose $0 \in \overline{\Omega}$ and Assumption 2.3 (γ) holds. Let $s \in (0,\infty)$ be a constant. Then there exists a constant $\kappa \in (1,\infty)$, depending only on d, n, m, δ, and ν, such that the following holds. If*

$$\left|\Omega_{R/32} \cap \mathcal{A}(\kappa s)\right| > \gamma^{2/\nu'}|\Omega_{R/32}|, \tag{5.5}$$

then we have $\Omega_{R/32} \subset \mathcal{B}(s)$.

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PROOF. By dividing $\boldsymbol{u}$ and $\boldsymbol{f}$ by s, we may assume $s=1$. We prove by contradiction. Suppose at a point $x \in \Omega_{R/32}$, we have

$$\gamma^{-1/\nu'}\big(\mathcal{M}(F^2)(x)\big)^{1/2} + \big(\mathcal{M}(U^\nu)(x)\big)^{1/\nu} \le 1. \tag{5.6}$$

Let us consider two cases.

First we consider the case when $\operatorname{dist}(0,\partial\Omega) \ge R/8$. Notice that

$$x \in \Omega_{R/32} = B_{R/32} \subset B_{R/8} \subset \Omega.$$

Due to Lemma 5.1 (i), we have $U \le V + W$ and, by (5.6),

$$\|V\|_{L_\infty(B_{R/32})} \le N_1, \quad (W^2)^{1/2}_{B_{R/8}} \le N_1\gamma^{1/\nu'}, \tag{5.7}$$

where N_1 and constants N_i below depend only on d, n, m, δ, and ν. By (5.7), the triangle inequality and Chebyshev's inequality, we get

$$\begin{aligned} \big|\Omega_{R/32} \cap \mathcal{A}(\kappa)\big| &= \big|\{x \in \Omega_{R/32} : U > \kappa\}\big| \\ &\le \big|\{x \in \Omega_{R/32} : W > \kappa - N_1\}\big| \le (\kappa - N_1)^{-2} N_1^2 \gamma^{2/\nu'} |B_{R/8}|, \end{aligned} \tag{5.8}$$

which contradicts with (5.5) if we choose κ sufficiently large.

Next we consider the case when $\operatorname{dist}(0,\partial\Omega) < R/8$. We take $y \in \partial\Omega$ such that $|y| = \operatorname{dist}(0,\partial\Omega)$. Notice that in this case we have

$$x \in \Omega_{R/32} \subset \Omega_{R/4}(y) \subset \Omega_R(y).$$

Due to Lemma 5.1 (ii), we have $U \le V + W$ in $\Omega_R(y)$ and, by (5.6),

$$\|V\|_{L_\infty(\Omega_{R/32})} \le \|V\|_{L_\infty(\Omega_{R/4}(y))} \le N_2, \quad (W^2)^{1/2}_{\Omega_R(y)} \le N_2\gamma^{1/\nu'}. \tag{5.9}$$

By (5.9), the triangle inequality and Chebyshev's inequality, we still get (5.8) with N_2 in place of N_1, which contradicts with (5.5) if we choose κ sufficiently large. □

THEOREM 5.3. *Let $p \in (2,\infty)$, $\lambda > 0$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{f}_\alpha \in L_p(\Omega)$. Suppose that $a_{\alpha\beta} \equiv 0$ for any α, β satisfying $|\alpha| + |\beta| < 2m$, and $\boldsymbol{u} \in W^m_p(\Omega)$ is supported on $B_\gamma(x_0) \cap \overline{\Omega}$ and satisfies (2.6) in Ω with the conormal derivative boundary condition. There exist positive constants $\gamma \in (0, 1/50)$ and N, depending only on d, δ, m, n, p, such that, under Assumption 2.3 (γ) we have*

$$\|U\|_{L_p(\Omega)} \le N\|F\|_{L_p(\Omega)}, \tag{5.10}$$

where $N = N(d,\delta,m,n,p)$.

PROOF. We fix $\nu = p/2 + 1$ and let $\nu' = 2\nu/(\nu-2)$. Then we see that $\boldsymbol{u} \in W^m_{\nu,\mathrm{loc}}(\overline{\Omega})$. Let κ be the constant in Corollary 5.2. Recall the elementary identity:

$$\|f\|^p_{L_p(\mathcal{D})} = p\int_0^\infty \big|\{x \in \mathcal{D} : |f(x)| > s\}\big| s^{p-1}\, ds,$$

which implies that

$$\|U\|^p_{L_p(\Omega)} = p\kappa^p \int_0^\infty |\mathcal{A}(\kappa s)| s^{p-1}\, ds. \tag{5.11}$$

Thus, to obtain (5.10) we need to estimate $\mathcal{A}(\kappa s)$. First, we note that by Chebyshev's inequality

$$|\mathcal{A}(\kappa s)| \le (\kappa s)^{-2}\|U\|^2_{L_2(\Omega)}. \tag{5.12}$$

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When $\kappa s \geq \gamma^{-1/\nu'}\|U\|_{L_2(\Omega)}$, this indicates that

$$|\mathcal{A}(\kappa s)| \leq \gamma^{2/\nu'}.$$

With the above inequality and Corollary 5.2 in hand, we see that all the conditions of the "crawling of ink drops" lemma are satisfied; see [**28**], [**19**, Sect. 2], or [**3**, Lemma 3] for the lemma. Hence we have

$$|\mathcal{A}(\kappa s)| \leq N_4\gamma^{2/\nu'}|\mathcal{B}(s)|. \tag{5.13}$$

Now we estimate $\|U\|^p_{L_p(\Omega)}$ in (5.11) by splitting the integral into two depending on the range of s. If $\kappa s \geq \gamma^{-1/\nu'}\|U\|_{L_2(\Omega)}$, we use (5.13). Otherwise, we use (5.12). Then it follows that

$$\begin{aligned}\|U\|^p_{L_p(\Omega)} &\leq N_5\gamma^{(2-p)/\nu'}\big(\|U\|^p_{L_2(\Omega)} + \big\|\big(\mathcal{M}(F^2)\big)^{1/2}\big\|^p_{L_p(\Omega)}\big)\\ &\quad + N_5\gamma^{2/\nu'}\big\|\big(\mathcal{M}(U^\nu)\big)^{1/\nu}\big\|^p_{L_p(\Omega)}\\ &\leq N_5\gamma^{(2-p)/\nu'}\|U\|^p_{L_2(\Omega)} + N_6\gamma^{(2-p)/\nu'}\|F\|^p_{L_p(\Omega)} + N_6\gamma^{2/\nu'}\|U\|^p_{L_p(\Omega)},\end{aligned}$$

where we used the Hardy–Littlewood maximal function theorem in the last inequality. By Hölder's inequality,

$$\|U\|_{L_2(\Omega)} = \|U\|_{L_2(B_\gamma(x_0)\cap\Omega)} \leq N\|U\|_{L_p(\Omega)}\gamma^{d(1/2-1/p)}. \tag{5.14}$$

By the choice of ν, $d(p/2-1) + (2-p)/\nu' > 2/\nu'$. Thus, we get

$$\|U\|^p_{L_p(\Omega)} \leq N_6\gamma^{(2-p)/\nu'}\|F\|^p_{L_p(\Omega)} + N_6\gamma^{2/\nu'}\|U\|^p_{L_p(\Omega)}.$$

To get the estimate (5.10), it suffices to take $\gamma = \gamma(d,n,m,\delta,p) \in (0,1/50]$ sufficiently small such that $N_6\gamma^{2/\nu'} \leq 1/2$. □

Proof of Theorem 2.4. We again consider the following three cases separately.

Case 1: $p = 2$. In this case, the theorem follows directly from the well-known Lax–Milgram lemma.

Case 2: $p \in (2,\infty)$. Assertion (i) follows from Theorem 5.3 and the standard partition of unity argument. By the method of continuity, for Assertion (ii) it suffices to prove the solvability for the operator $\mathcal{L}_1 := \delta_{ij}\Delta^m$, which is not immediate because the domain Ω is irregular. We approximate Ω by regular domains. Recall the definition of Ω^ε in (3.10). By Lemma 3.12, for any $\varepsilon \in (0,1/4)$, Ω^ε satisfies Assumption 3.11 ($N_0\gamma^{1/2}$) with a constant $R_1(\varepsilon) > 0$. Thanks to Lemma 3.13, there is a sequence of expanding smooth domains $\Omega^{\varepsilon,k}$ which converges to Ω^ε as $k \to \infty$. Moreover, $\Omega^{\varepsilon,k}$ satisfies Assumption 3.11 ($N_0\gamma^{1/2}$) with the constant $R_1(\varepsilon)/2$ which is independent of k. In particular, $\Omega^{\varepsilon,k}$ satisfies Assumption 2.3 ($N_0\gamma^{1/2}$) with the constant $R_1(\varepsilon)/2$. By the classical result, there is a constant $\lambda_\varepsilon = \lambda_\varepsilon(d,n,m,p,\delta) \geq \lambda_0$ such that, for any $\lambda > \lambda_\varepsilon$, the equation

$$(-1)^m\mathcal{L}_1\boldsymbol{u} + \lambda\boldsymbol{u} = \sum_{|\alpha|\leq m} D^\alpha\boldsymbol{f}_\alpha \quad \text{in } \Omega^{\varepsilon,k}$$

with the conformal derivative boundary condition has a unique solution $\boldsymbol{u}^{\varepsilon,k} \in W^m_p(\Omega^{\varepsilon,k})$. The a priori estimate above gives

$$\|\boldsymbol{u}^{\varepsilon,k}\|_{W^m_p(\Omega^{\varepsilon,k})} \leq N_\varepsilon, \tag{5.15}$$

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where $N_\varepsilon > 0$ is a constant independent of k. By the weak compactness, there is a subsequence, which is still denoted by $\boldsymbol{u}^{\varepsilon,k}$, and functions $\boldsymbol{v}^\varepsilon, \boldsymbol{v}^\varepsilon_\alpha \in L_p(\Omega^\varepsilon), 1 \le |\alpha| \le m$, such that weakly in $L_p(\Omega^\varepsilon)$,

$$\boldsymbol{u}^{\varepsilon,k} I_{\Omega^{\varepsilon,k}} \rightharpoonup \boldsymbol{v}^\varepsilon, \quad D^\alpha \boldsymbol{u}^{\varepsilon,k} I_{\Omega^{\varepsilon,k}} \rightharpoonup \boldsymbol{v}^\varepsilon_\alpha \quad \forall\, \alpha,\, 1 \le |\alpha| \le m.$$

It is easily seen that $\boldsymbol{v}^\varepsilon_\alpha = D^\alpha \boldsymbol{v}^\varepsilon$ in the sense of distributions. Thus, by (5.15) and the weak convergence, $\boldsymbol{v}^\varepsilon \in W^m_p(\Omega^\varepsilon)$ is a solution to

$$(-1)^m \mathcal{L}_1 \boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha| \le m} D^\alpha \boldsymbol{f}_\alpha \quad \text{in } \Omega^\varepsilon \tag{5.16}$$

with the conormal derivative boundary condition. We have proved the solvability for any $\lambda > \lambda_\varepsilon$. Recall that, by Lemma 3.12, Ω^ε satisfies Assumption 2.3 ($N_0\gamma^{1/2}$) with $R_0 = 1/2$. By the a priori estimate in Assertion (i) and the method of continuity, for any $\lambda > \lambda_0$ there is a unique solution $\boldsymbol{u}^\varepsilon \in W^m_p(\Omega^\varepsilon)$ to (5.16) with the conormal derivative boundary condition. Moreover, we have

$$\|\boldsymbol{u}^\varepsilon\|_{W^m_p(\Omega^\varepsilon)} \le N, \tag{5.17}$$

where N is a constant independent of ε. Again by the weak compactness, there is a subsequence $\boldsymbol{u}^{\varepsilon_j}$, and functions $\boldsymbol{u}, \boldsymbol{u}_\alpha \in L_p(\Omega), 1 \le |\alpha| \le m$, such that weakly in $L_p(\Omega)$,

$$\boldsymbol{u}^{\varepsilon_j} I_{\Omega^{\varepsilon_j}} \rightharpoonup \boldsymbol{u}, \quad D^\alpha \boldsymbol{u}^{\varepsilon_j} I_{\Omega^{\varepsilon_j}} \rightharpoonup \boldsymbol{u}_\alpha \quad \forall\, \alpha,\, 1 \le |\alpha| \le m.$$

It is easily seen that $\boldsymbol{u}_\alpha = D^\alpha \boldsymbol{u}$ in the sense of distributions. Thus, by (5.17) and the weak convergence, $\boldsymbol{u} \in W^m_p(\Omega)$ is a solution to

$$(-1)^m \mathcal{L}_1 \boldsymbol{u} + \lambda \boldsymbol{u} = \sum_{|\alpha| \le m} D^\alpha \boldsymbol{f}_\alpha \quad \text{in } \Omega$$

with the conormal derivative boundary condition. The uniqueness then follows from the a priori estimate. This completes the proof of Assertion (ii).

Case 3: $p \in (1,2)$. The a priori estimate in Assertion (i) is a directly consequence of the solvability when $p \in (2,\infty)$ and the duality argument. Then the solvability in Assertion (ii) follows from the a priori estimate by using the same argument as in Case 2.

The theorem is proved. □

We now give the proofs of Corollary 2.5 and Theorem 2.6.

Proof of Corollary 2.5. *Case 1:* $p = 2$. We define a Hilbert space

$$H := \{\boldsymbol{u} \in W^m_2(\Omega) \mid (\boldsymbol{u})_\Omega = (D\boldsymbol{u})_\Omega = \ldots = (D^{m-1}\boldsymbol{u})_\Omega = 0\}.$$

By the Lax–Milgram lemma, there is a unique $\boldsymbol{u} \in H$ such that for any $\boldsymbol{v} \in H$,

$$\int_\Omega a_{\alpha\beta} D^\beta \boldsymbol{u} D^\alpha \boldsymbol{v}\, dx = \sum_{|\alpha| = m} \int_\Omega (-1)^{|\alpha|} \boldsymbol{f}_\alpha D^\alpha \boldsymbol{v} \tag{5.18}$$

and

$$\|D^m \boldsymbol{u}\|_{L_2(\Omega)} \le N \sum_{|\alpha| = m} \|\boldsymbol{f}_\alpha\|_{L_2(\Omega)}.$$

Note that any function $\boldsymbol{v} \in W^m_2(\Omega)$ can be decomposed as a sum of a function in H and a polynomial of degree at most $m-1$. Therefore, (5.18) also holds for any $\boldsymbol{v} \in W^m_2(\Omega)$. This implies that $\boldsymbol{u} \in W^m_2(\Omega)$ is a solution to the original equation.

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On the other hand, by the uniqueness of the solution in H, any solution $\boldsymbol{w} \in W_2^m(\Omega)$ can only differ from $\boldsymbol{u}$ by a polynomial of order at most $m-1$.

Case 2: $p \in (2, \infty)$. First we suppose that p satisfies $1/p > 1/2 - 1/d$. Since Ω is bounded, $\boldsymbol{f} \in L_2(\Omega)$. Let $\boldsymbol{u}$ be the unique H-solution to the equation. By Theorem 2.4, there is a unique solution $\boldsymbol{v} \in W_p^m(\Omega)$ to the equation

$$(-1)^m \mathcal{L}\boldsymbol{v} + (\lambda_0 + 1)\boldsymbol{v} = \sum_{|\alpha|=m} D^\alpha \boldsymbol{f}_\alpha + (\lambda_0 + 1)\boldsymbol{u} \quad \text{in } \Omega \tag{5.19}$$

with the conormal derivative boundary condition. Moreover, we have

$$\|\boldsymbol{v}\|_{W_p^m(\Omega)} \le N \sum_{|\alpha|=m} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)} + N\|\boldsymbol{u}\|_{L_p(\Omega)}. \tag{5.20}$$

By the Sobolev imbedding theorem and the W_2^m estimate, we have

$$\|\boldsymbol{u}\|_{L_p(\Omega)} \le N\|\boldsymbol{u}\|_{W_2^1(\Omega)} \le N \sum_{|\alpha|=m} \|\boldsymbol{f}_\alpha\|_{L_2(\Omega)} \le N \sum_{|\alpha|=m} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)},$$

which together with (5.20) gives

$$\|\boldsymbol{v}\|_{W_p^m(\Omega)} \le N \sum_{|\alpha|=m} \|\boldsymbol{f}_\alpha\|_{L_p(\Omega)}.$$

Since both $\boldsymbol{v}$ and $\boldsymbol{u}$ are $W_2^m(\Omega)$-solutions to (5.19), by Theorem 2.4 we have $\boldsymbol{u} = \boldsymbol{v}$. Therefore, the solvability is proved under the assumption $1/p > 1/2 - 1/d$. The general case follows by using a bootstrap argument. The uniqueness is due to the uniqueness of W_2^m-solutions.

Case 3: $p \in (1, 2)$. By the duality argument and Case 2, we have the a priori estimate (2.8) for any $\boldsymbol{u} \in W_p^m(\Omega)$ satisfying (2.7) with the conormal derivative boundary condition. For the solvability, we take a sequence

$$\boldsymbol{f}_\alpha^k = \min\{\max\{\boldsymbol{f}_\alpha, -k\}, k\} \in L_2(\Omega)$$

which converges to $\boldsymbol{f}_\alpha$ in $L_p(\Omega)$. Let $\boldsymbol{u}^k$ be the H-solution to the equation with the right-hand side $\boldsymbol{f}_\alpha^k$. Since Ω is bounded, we have $\boldsymbol{u}^k \in W_p^m(\Omega)$. By the a priori estimate, $\boldsymbol{u}^k$ is a Cauchy sequence in $W_p^m(\Omega)$. Then it is easily seen that the limit $\boldsymbol{u}$ is the $W_p^m(\Omega)$-solution to the original equation. Next we show the uniqueness. Let $\boldsymbol{u}_1$ be another $W_p^m(\Omega)$-solution to the equation. Then $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}_1 \in W_p^m(\Omega)$ satisfies the equation with the zero right-hand side. Following the bootstrap argument in Case 2, we infer that $\boldsymbol{v} \in W_2^m(\Omega)$. Therefore, by Case 1, $\boldsymbol{v}$ must be a polynomial of degree at most $m-1$.

The corollary is proved. □

Proof of Theorem 2.6. The theorem is proved in the same way as Corollary 2.5 in Cases 2 and 3 by using the classical W_2^1-estimate of the conormal derivative problem on a domain with a finite measure; see Theorem 13 of [**12**]. We remark that although in Theorem 13 (i) of [**12**] it is assumed that $b_i = c = 0$, the same proof works under the relaxed condition $-D_i b_i + c = 0$ in Ω and $b_i n_i = 0$ on $\partial\Omega$ in the weak sense, i.e., Assumption (H*). □

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