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Adjunction of Roots in Exponential A-Groups

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Abstract. The purpose of this note is to prove that if G is an exponential A-group with $g \in G$ and $\beta \in A$, then G can be partially A-embedded in a partial exponential A-group where g has a βth root. This is a variation of a theorem by B. H. Neumann.

1. Introduction

An *exponential A-group* is a group G, equipped with an action by a commutative ring A with unity, such that for all $g \in G$ and for all $\alpha \in A$, the element $g^\alpha \in G$ is uniquely defined and the following axioms hold:

(1) $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, and $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and α, $\beta \in A$;
(2) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all g, $h \in G$ and $\alpha \in A$;
(3) If g and h are commuting elements of G, then $(gh)^\alpha = g^\alpha h^\alpha$ for all $\alpha \in A$.

Examples of exponential A-groups include A-modules, R. Lyndon's free $\mathbb{Z}[x]$-group (see [**2**]), and P. Hall's nilpotent A-powered groups ([**1**], pages 27-28).

A group G is called a *partial A-group* if for some elements $g \in G$ and $\alpha \in A$, g^α is defined, and whenever it is defined it is uniquely defined; and each equality in the axioms of an exponential A-group holds whenever both sides of the axiom are defined.

In [**6**], B. H. Neumann considered the problem of adjoining roots to elements in groups. Using amalgamated free products, he showed that for any group G, an arbitrary element $g \in G$, and a positive integer m, G embeds in a group H where $h^m = g$ for some $h \in H$.

In this note, we carry out a similar construction in the context of exponential A-groups. The difficulty here is that the amalgamated free product of two exponential A-groups need not be an exponential A-group, not even a partial one (see [**5**]). However, the particular construction given in the proof of Theorem 3.2 admits the structure of a partial A-group, where A is assumed to be a PID.

2. Preliminaries

We begin by recalling some terminology and results that will be used throughout the paper.

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The collection of exponential A-groups forms a category. Thus, notions such as A-subgroup, normal A-subgroup, and A-homomorphism are defined in the obvious way (see [**5**]). If H is an A-subgroup (a normal A-subgroup) of an exponential A-group G, then we write $H \leq_A G$ ($H \trianglelefteq_A G$).

DEFINITION 2.1. Let G be an exponential A-group, and let $S = \{s_1, \ldots, s_j\}$ be a subset of G. Then

$$gp_A(s_1, \ldots, s_j) = \bigcap_{S \subset H_i \leq_A G} \{H_i\}$$

is called the *A-subgroup of G which is A-generated by S*.

Given a subset $S = \{s_1, \ldots, s_j\} \subseteq G$, if $gp_A(s_1, \ldots, s_j) = G$, then G is termed *finitely A-generated*. In case $j = 1$, we say that G is a *cyclic A-group*.

DEFINITION 2.2. Let G be an exponential A-group. An element $g \in G$ ($g \neq 1$) is called a *torsion element* if $g^\alpha = 1$ for some non-zero $\alpha \in A$. The *annihilator* or *order ideal* of g is the ideal

$$Ann(g) = \{\alpha \in A \mid g^\alpha = 1\}.$$

If A is a PID, then $Ann(g) = \langle \mu \rangle$ for some $\mu \in A$. We define μ to be the *order* of g, and denote it as $|g| = \mu$. Note that any other generator of $Ann(g)$ is an associate of μ in A, and the relation

$$\mu_1 \sim \mu_2 \text{ if and only if } \mu_1 = a\mu_2 \text{ for some unit } a \in A$$

is an equivalence relation on A. Thus, for any torsion element $g \in G$, $|g|$ is unique up to associates in A. This definition of $|g|$ is a slight revision of the one given in [**4**]. By convention, if $A = \mathbb{Z}$, the choice of generator for the order ideal of g is always a positive integer.

We remark that if g has torsion in the usual sense as a group element, say $|g| = m$ ($m \in \mathbb{Z}^+$), then $m \in Ann(g)$ and m is the least positive integer in $Ann(g)$.

Next we mention some well-known notions and facts about free products with amalgamation (see [**3**] for details). Let $P = G *_H T$ be the free product of G and T with H amalgamated. A sequence $c_1, c_2, \ldots, c_n$ in the ambient free product $G * T$ is termed *reduced* if

(1) each c_i lies in either factor G or T,
(2) for each i, c_i and c_{i+1} lie in distinct factors,
(3) if $n > 1$, no c_i lies in H, and
(4) if $n = 1$, $c_1 \neq 1$.

Given a reduced sequence $c_1, \ldots, c_n$, the product $w = c_1 \cdots c_n$ is an element of P. We call n the *reduced length* of w and write $l(w) = n$. It is clear that every element of P is equal to a product of elements in a reduced sequence.

THEOREM 2.3. *Let $w = c_1 \cdots c_n$ be an element in $G * T$, where $c_1, \ldots, c_n$ is a reduced sequence. Then $w \neq 1$ in P.*

It follows that G and T are embedded in P by their canonical maps.

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3. Results

It is mentioned in [**5**] that a free product with amalgamation of two exponential A-groups may not admit the structure of a partial A-group. Theorem 3.1 provides an instance when such a partial A-action is definable. This result plays a central role in the proof of Theorem 3.2.

THEOREM 3.1. *Let G and T be exponential A-groups with $g_0 \in G$ and $t_0 \in T$, and suppose $gp_A(g_0) \cong_A gp_A(t_0)$. Let $P = (G*T)/N$, where N is the normal closure of the set $\{g_0^\alpha t_0^{-\alpha} \mid \alpha \in A\}$ in $G*T$. Then P is a partial A-group and the canonical maps $G \to P$ and $T \to P$ are partial A-embeddings.*

PROOF. Consider the isomorphism from $gp_A(g_0)$ to $gp_A(t_0)$ defined by $g_0^\alpha \mapsto t_0^\alpha$ for every $\alpha \in A$. Observe that this isomorphism can be realized as an A-isomorphism via the mapping $g_0 \mapsto t_0$. If $P = (G*T)/N$ is defined as in the statement of the theorem, then P can be realized as a free product with amalgamation $G*_H T$, where

$$H \cong gp_A(\,g_0\,) \cong gp_A(\,t_0\,).$$

Let $\overline{G} = GN/N$ and $\overline{T} = TN/N$ be the embedded copies of G and T in P, respectively. We define a partial A-action on P by defining an A-action on both $\overline{G}$ and $\overline{T}$ (but on no elements of P that lie outside of $\overline{G}$ or $\overline{T}$) as follows:

If $\overline{g} \in \overline{G}$, $\overline{t} \in \overline{T}$, and $\alpha \in A$, then

$$\overline{g}^\alpha = \overline{g^\alpha} \quad \text{and} \quad \overline{t}^\alpha = \overline{t^\alpha}.$$

To see that this is a well-defined partial action of A on P, note first that $\overline{g_0}^{\,\alpha}\overline{t_0}^{\,-\alpha} = g_0^\alpha t_0^{-\alpha} N = N = \overline{1}$. Hence, $\overline{g_0}^{\,\alpha} = \overline{t_0}^{\,\alpha}$.

Now consider $\overline{G}$. Since G embeds in P, the action is well-defined on $\overline{G}$. Let $\overline{g}$, $\overline{g_1}$, $\overline{g_2} \in \overline{G}$ and α, $\beta \in A$. The following facts are inherited from the assumption that G is an exponential A-group:

(i) $\overline{g}^1 = \overline{g}$, $\overline{g}^\alpha \overline{g}^\beta = \overline{g}^{\alpha+\beta}$, and $(\overline{g}^\alpha)^\beta = \overline{g}^{\alpha\beta}$;
(ii) if $\overline{g_1}$ and $\overline{g_2}$ commute, then $(\overline{g_1}\,\overline{g_2})^\alpha = \overline{g_1}^{\,\alpha}\overline{g_2}^{\,\alpha}$.

It suffices to check the conjugation axiom. Let $\overline{g}$ be a nontrivial element of $\overline{G}$. In order for this axiom to make sense we assume, in turn, that either $w^{-1}\overline{g}w \in \overline{G}$, or $w^{-1}\overline{g}w \in \overline{T}$ for some $w \in P$. Suppose first that $w^{-1}\overline{g}w = \overline{x} \in \overline{G}$. Then $w \in \overline{G}$. Otherwise, w can be written as the product of a reduced sequence of one of the following forms:

(i) $w = \overline{t_1}\,v$;
(ii) $w = \overline{g_1}\,\overline{t_1}\,v$;

where $\overline{t_1} \in \overline{T} \setminus H$, $\overline{g_1} \in \overline{G} \setminus H$, and v is either the identity or the product of a reduced sequence beginning with an element from $\overline{G} \setminus H$. If w begins with $\overline{t_1}$ as in the first case, it is easy to see that $w^{-1}\overline{g}w\overline{x}^{-1}$ is a reduced product equal to 1, contradicting Theorem 2.3. Suppose then that w is of form (ii). We get

$$w^{-1}\overline{g}w\overline{x}^{-1} = v^{-1}\overline{t_1}^{\,-1}\,\overline{g_1}^{\,-1}\,\overline{g}\,\overline{g_1}\,\overline{t_1}\,v\overline{x}^{-1} = 1.$$

The only way that this product could collapse is if $\overline{g_1}^{\,-1}\,\overline{g}\,\overline{g_1} = 1$, hence, $\overline{g} = 1$; a contradiction. If it does not collapse, we get yet another reduced product equal to 1, again contradicting Theorem 2.3. Thus, $w \in \overline{G}$.

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Since $\overline{G}$ is an exponential A-group, we know that

$$(w^{-1}\overline{g}w)^{\alpha} = w^{-1}\overline{g}^{\alpha}w. \tag{3.1}$$

Next assume that $w^{-1}\overline{g}w \in \overline{T} \setminus H$; that is, $w^{-1}\overline{g}w = \overline{t}$ for some $\overline{t} \in \overline{T} \setminus H$. Applying Theorem 2.3 again, we conclude in this case that both $\overline{g}$ and w must lie in $\overline{T}$. Otherwise, since w clearly cannot lie in $\overline{G}$, we can write w as the product of a reduced sequence either of the form $w = \overline{t_1}\,\overline{g_1}\,v$ or of the form $w = \overline{g_1}\,\overline{t_1}\,v$, where $\overline{t_1} \in \overline{T} \setminus H$, $\overline{g_1} \in \overline{G} \setminus H$, and v is either the identity or the product of a reduced sequence. It is easy to see that if w begins with $\overline{t_1}$ and $\overline{g}$ is nontrivial, then $w^{-1}\overline{g}w\overline{t}^{-1}$ is a reduced product equal to 1, contradicting Theorem 2.3. If $w = \overline{g_1}\,\overline{t_1}\,v$, then we can write

$$w^{-1}\overline{g}w\overline{t}^{-1} = v^{-1}\overline{t_1}^{\,-1}\,\overline{g_1}^{\,-1}\,\overline{g}\,\overline{g_1}\,\overline{t_1}\,v\overline{t}^{-1} = 1.$$

Although $\overline{t_1}\,v\,\overline{t}^{\,-1}$ could collapse at the end, $v^{-1}\overline{t_1}^{\,-1}$ at the beginning is not trivial. Again we get a contradiction to Theorem 2.3; and we conclude that $w \in \overline{T}$. It immediately follows that $\overline{g} \in \overline{T}$. Since $\overline{T}$ is an exponential A-group, (3.1) also holds.

Clearly, then, the embedding $G \hookrightarrow P$ is a partial A-embedding. By the same reasoning, the action of A on $\overline{T}$ is well-defined, satisfies the required axioms, and the embedding $T \hookrightarrow P$ is a partial A-embedding. □

THEOREM 3.2. *Let G be an exponential A-group, where A is a PID, and suppose $g \in G$ and $\beta \in A$ are arbitrary. Then G can be partially A-embedded in a partial A-group in which g has a βth root.*

PROOF. Let G, g, and β be as stated in the hypothesis. Define T to be a cyclic A-group A-generated by t, where $T \cap G = 1$ and

$$|t| = \begin{cases} \alpha\beta & \text{if } |g| = \alpha \neq \infty \\ \infty & \text{if } |g| = \infty. \end{cases}$$

Observe that $gp_A(g)$ is A-isomorphic to $gp_A(t^{\beta})$. (Note that in particular, g and t^{β} have the same order, in the usual group sense.) If N denotes the normal closure of the set

$$\{g^{\gamma}t^{-\beta\gamma} \mid \gamma \in A\}$$

in $G * T$, then Theorem 3.1 shows that $P = (G * T)/N$ is a partial A-group in which both G and T may be partially A-embedded in such a way that t^{β} and g are identified, and therefore t is a βth root of g in P. □

References

[1] P. Hall, *Nilpotent Groups*, (Canad. Math. Congress, Summer Seminar, University of Alberta, 1957)

[2] R. C. Lyndon, *Groups with parametric exponents*, (Trans. of the Amer. Math. Soc., 1960), 96, 518 - 533 MR0151502 (27:1487)

[3] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, (Springer-Verlag, Berlin-Heidelberg-New York, 1977) MR0577064 (58:28182)

[4] S. Majewicz and M. Zyman, *Power-commutative nilpotent R-powered groups*, (Groups-Complexity-Cryptology, 2009), Vol. 1, No. 2, 297 - 309 MR2598996 (2011e:20049)

[5] A. Myasnikov and V. Remeslennikov, *Groups with Exponents. I: Fundamentals of the theory and tensor completions*, (Siberian Math. Journal, 1994), 35 (5), 986 - 996 MR1308240 (95m:20047)

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[6] B. H. Neumann, *Adjunction of elements to groups*, (J. Lond. Math. Soc., 1943), 18, 4 - 11 MR0008808 (5:58s)

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