

Determinants of Laplacians on non-compact surfaces

Clara L. Aldana

ABSTRACT. In this overview article we give a brief presentation of the definition of relative determinants of Laplace operators in the setting of surfaces with asymptotically cusp ends. We refer to renormalized determinants on surfaces that allow funnel ends in addition to cusp ends. We describe the extremal problem in these settings and the behavior of the determinant in the moduli space of surfaces with cusps.

1. Introduction

The purpose of this note is to give a brief presentation of some results about relative determinants of Laplace operators on surfaces with asymptotically cusp ends. We show how to define the relative determinant when comparing the Laplacian on the whole surface with the hyperbolic Laplacian on the cusp ends. Some of the results presented here are part of the doctoral dissertation of the author [3], and are presented in [5]. We also mention related results by other authors for surfaces that either have funnel ends, [10], or allow funnels and cusps, [2]. Determinants of operators have been widely studied in the last few decades, so even a modest review of the topic is out of our reach here.

In order to keep the presentation clear and condensed, we restrict ourselves to the main statements, giving only ideas of the proofs and skipping technical details.

Regularized determinants of elliptic operators were introduced by D. B. Ray and I. M. Singer in [36] in relation to R -torsion. On closed manifolds regularized determinants of pseudo-differential elliptic non-negative operators of positive order are defined via a zeta function regularization procedure; see M. Kontsevich and S. Vishik [25].

The main inspiration to work on this topic arose from the well known results by B. Osgood, R. Phillips and P. Sarnak (to whom we refer as OPS from now on) in [34] and [35] about extremals of determinants on surfaces and compactness of isospectral isometry classes of metrics on a closed surface, respectively. In [34], the authors consider the determinant as a functional on the space of metrics on the surface. They prove that given a surface with fixed genus, inside the conformal class of a metric, among all metrics of unit area, there exists a unique metric of constant curvature at which the regularized determinant attains a maximum. They

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also prove the corresponding statement for compact surfaces with boundary and suitable conditions at the boundary. Their proofs use Polyakov's formula for the determinant, which is obtained using a variational formula. Polyakov's formula is the key tool in the analysis of extrema of determinants on manifolds.

Relative determinants were introduced in a general setting by W. Müller in [31] as a way to generalize regularized determinants. Operators with continuous spectrum are instances of these settings. Previously, a relative determinant for admissible surfaces was introduced by R. Lundelius in [27], and for Dirac operators in \mathbb{R}^n by V. Bruneau in [13].

Regularized determinants have also played an important role in the index theory for families on certain bundles associated to punctured Riemann surfaces; see for example [40] and [6], [7].

This paper is organized as follows. In section 2, we introduce the definitions of regularized determinants and of relative determinants. In section 3, we give the definition of surfaces with cusps and their main properties. In section 4, we describe the proof that the relative determinant on surfaces with asymptotically hyperbolic cusp ends is well defined. In section 5 we describe other ways of defining determinants of Laplacians on surfaces that allow funnel ends. We mention the work of Borthwick, Judge and Perry in [10], and we describe in more detail the work of P. Albin, F. Rochon and the author in [2] about renormalized determinants. Finally, in section 6 we touch on the problem of studying the determinant as a function of the moduli space of hyperbolic metrics on a surface.

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2. Determinants of operators, recalling the definitions

2.1. Regularized determinant of the Laplacian on a closed manifold.

Let us recall the definition of the regularized determinant formulating it for this particular case.

Let M be a compact connected Riemannian manifold of dimension n without boundary. The spectrum of the Laplacian $\sigma(\Delta_g)$ is given by an increasing sequence of eigenvalues:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

This sequence obeys Weyl's law that establishes how the sequence grows:

$$N(\lambda) := \sum_{\lambda_j \leq \lambda} 1 \sim \frac{\omega_n \operatorname{vol}(M)}{(2\pi)^n} \lambda^{n/2},$$

as $\lambda \rightarrow \infty$, where $\omega_n = \frac{(2\pi)^{n/2}}{\Gamma(n/2)}$. Weyl's law implies that for $j \gg 1$,

$$\lambda_j^{n/2} \sim \frac{(2\pi)^n j}{\omega_n \operatorname{vol}(M)}.$$

The spectral zeta function associated to Δ is defined for $\operatorname{Re}(s) > n/2$ as

$$\zeta_{\Delta_g}(s) := \sum_{\lambda_j > 0} \lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{Tr}(e^{-t\Delta_g}) - 1) t^{s-1} dt.$$

The heat-operator trace is well known to have an asymptotic expansion for small values of t :

$$\text{Tr}(e^{-t\Delta_g}) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^j.$$

where the a_j are the heat invariants that are local quantities, see [19].

This expansion allows one to extend the spectral zeta function to a meromorphic function on the complex plane that is regular at $s = 0$, and the determinant of Δ_g can be defined as

$$(2.1) \quad \det \Delta_g = \exp \left(-\frac{d}{ds} \zeta_{\Delta}(s) \Big|_{s=0} \right).$$

The determinant of the Laplacian is a geometric invariant that depends only on the spectrum of the operator.

2.2. Relative determinants. The relative determinant was introduced by W. Müller in [31]. It is defined for a pair of self-adjoint, nonnegative linear operators, H_1 and H_0 , in a separable Hilbert space \mathcal{H} . The operators should satisfy the following assumptions:

- (1) The relative heat operator $e^{-tH_1} - e^{-tH_0}$ should be trace class for all $t > 0$.
- (2) For small time, there should be an asymptotic expansion of the relative trace of the form

$$\text{Tr}(e^{-tH_1} - e^{-tH_0}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\ell_j} \log^k t,$$

where $-\infty < \ell_0 < \ell_1 < \dots$ and $\ell_k \rightarrow \infty$. Moreover, if $\ell_j = 0$ we assume that $a_{jk} = 0$ for $k > 0$.

- (3) For large time, an asymptotic expansion is also required. In our case, it suffices to have an expansion of the form $\text{Tr}(e^{-tH_1} - e^{-tH_0}) = h + O(e^{-ct})$ as $t \rightarrow \infty$, where $h = \dim \text{Ker } H_1 - \dim \text{Ker } H_0$.

The relative zeta function is then defined in the same way as the spectral zeta function but with the relative heat trace instead of the heat trace:

$$\zeta(s; H_1, H_0) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\text{Tr}(e^{-tH_1} - e^{-tH_0}) - h) t^{s-1} dt.$$

As in the compact case, the determinant is defined via the meromorphic continuation of the relative zeta function:

$$\det(H_1, H_0) := e^{-\zeta'(0; H_1, H_0)}.$$

3. Surfaces with cusps

3.1. Surfaces with cusps. A surface with cusps (swc) is a 2-dimensional Riemannian manifold that consists of the union of a compact part and a finite number of ends:

$$M = M_0 \cup Z_1 \cup \dots \cup Z_m,$$

where M_0 is a compact surface with smooth boundary and each Z_i , $i = 1, \dots, m$, is a topological cylinder that carries the hyperbolic metric on it:

$$(3.1) \quad Z_i \cong [a_i, \infty)_{y_i} \times S_{x_i}^1, \quad g|_{Z_i} = y_i^{-2}(dy_i^2 + dx_i^2), \quad a_i > 0.$$

The subsets Z_i are called cusps. Sometimes we denote Z_i by Z_{a_i} .

Examples of swc are quotients $\Gamma \backslash \mathbb{H}$ of the upper half plane \mathbb{H} by a Fuchsian group of the first kind $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ without elliptic elements, see [21].

3.2. Spectral theory of surfaces with cusps. Let (M, g) be a swc; then M is complete and the Laplace-Beltrami operator on functions Δ has a unique closed self-adjoint extension that we denote by Δ_g . The spectrum of the Laplace operator has a continuous part $\sigma_c(\Delta_g) = [1/4, \infty)$, with multiplicity m , and a discrete part $\sigma_d(\Delta_g)$. Associated to the continuous spectrum there are the generalized eigenfunctions, $E_j(z, s)$, $z \in M$, $s \in \mathbb{C}$, $1 \leq j \leq m$. They satisfy:

$$\begin{aligned} \Delta_g E_j(z, s) &= s(1-s)E_j(z, s), \\ E_i((y_j, x_j), s) &= \delta_{ij} y_j^s + C_{ij}(s) y_j^{1-s} + O(e^{-cy_j}), \text{ as } y_j \rightarrow \infty. \end{aligned}$$

The coefficients $C_{ij}(s)$ form a matrix, $C(s) = (C_{ij}(s))_{1 \leq i, j \leq m}$ called the scattering matrix. Its determinant $\phi(s) = \det C(s(1-s))$ is called the scattering phase. Details about the spectral theory of surfaces with cusps can be found in W. Müller [30], Y. Colin de Verdière [17], and the references therein.

Now, let $\Delta_{a,0}$ be the Dirichlet extension of the operator

$$-y^2 \frac{\partial^2}{\partial y^2} : C_c^\infty((a, \infty)) \rightarrow L^2([a, \infty), y^{-2} dy).$$

Let $\bar{\Delta}_{a,0} = \oplus_{j=1}^m \Delta_{a_j,0}$ be the direct sum of the operators $\Delta_{a_j,0}$ defined above. The self-adjoint operator $\bar{\Delta}_{a,0}$ acts on a subspace of $\oplus_{j=1}^m L^2([a_j, \infty), y_j^{-2} dy_j)$. Let $\Delta_{Z_a, D}$ be the Dirichlet Laplacian on the cusp Z_a with Dirichlet boundary conditions at $\{a\} \times S^1$.

The space $L^2(Z_a, dA_g)$ can be decomposed as the orthogonal direct sum

$$L^2(Z_a, dA_g) = L^2([a, \infty), y^{-2} dy) \oplus L_0^2(Z_a)$$

with $L_0^2(Z_a) = \{f \in L^2(Z_a, dA_g) \mid \int_{S^1} f(y, x) dx = 0 \text{ for a.e. } y \geq a\}$.

This decomposition is invariant under $\Delta_{Z_a, D}$. Therefore, the operator $\Delta_{Z_a, D}$ can be decomposed as $\Delta_{Z_a, D} = \Delta_{a,0} \oplus \Delta_{Z_a,1}$, where $\Delta_{a,0}$ was defined above and $\Delta_{Z_a,1}$ acts on $L_0^2(Z_a)$.

The operator $\Delta_{Z_a,1}$ has compact resolvent; see [32, Lemma 7.3]. In addition, the counting function for it implies that the heat operator $e^{-t\Delta_{Z_a,1}}$ is trace class, see [17, Thm. 6].

3.3. Surfaces with asymptotically hyperbolic cusps. A surface with asymptotically cusp ends (swac) is a surface (M, h) where the metric h is a conformal transformation of the metric on a swc (M, g) , i.e. $h = e^{2\varphi} g$, where $\varphi \in C^\infty(M)$ and φ as well as some of its derivatives have a suitable decay in the cusps. We call the function φ the conformal factor.

Two metrics g_1, g_2 on a given manifold M are quasi-isometric if there exist constants $C_1, C_2 > 0$ such that for each $z \in M$,

$$C_1 g_1(z) \leq g_2(z) \leq C_2 g_1(z),$$

in the sense of positive definite forms. Quasi-isometric metrics have equivalent geodesic distances. The associated L^2 -spaces coincide as sets, though the inner product is not the same.

Let (M, h) be a swac, with $h = e^{2\varphi} g$ and (M, g) a swc. Since the function φ is bounded on M , it follows that the metrics g and h are quasi-isometric. Therefore the geodesic distances, d_g and d_h , are equivalent. Under these assumptions the

metric h is complete. If in addition, $\Delta_g \varphi = O(1)$ as $y \rightarrow \infty$, then (M, g) and (M, h) have the same injectivity radius, [32, Prop.2.1], that in a swc vanishes.

The domains of the Laplacians Δ_g and Δ_h lie in different Hilbert spaces. If we want to be accurate, we should consider a unitary map between the spaces $L^2(M, dA_g)$ and $L^2(M, dA_h)$ and include this map in all our computations. The unitary map is given by

$$(3.2) \quad T : L^2(M, dA_g) \rightarrow L^2(M, dA_h), \quad f \mapsto e^{-\varphi} f.$$

However, for the sake of simplicity in the presentation, we do not include this transformation in our statements.

3.4. Heat kernel estimates and Duhamel’s principle. One of the main tools that we use is Duhamel’s principle. In the case of a swc and a swac, Duhamel’s principle can be stated in terms of the heat operators as

$$(3.3) \quad e^{-t\Delta_h} - e^{-t\Delta_g} = \int_0^t e^{-s\Delta_h} (\Delta_g - \Delta_h) e^{-(t-s)\Delta_g} ds.$$

Another important tool is furnished by the upper bounds on the heat kernel and its derivatives. For this we refer to S. Y. Cheng, P. Li and S. T. Yau in [16], Theorems 4, 6 and 7. Let h, g and φ be as above; then the metrics are quasi-isometric, and the heat kernels K_h and K_g satisfy the same estimates:

$$K_*(z, z', t) \ll (i(z)i(z'))^{\frac{1}{2}} t^{-1} \exp\left(-\frac{c d_*^2(z, z')}{t}\right),$$

uniformly for $0 < t < T$, where $*$ denotes the metric g or h , $c > 0$ is a constant, and i is a function on M given by $i(z) = 1$, if $z \in M_0$, and by $i(y, x) = y$, if $z = (y, x)$ belongs to a cusp.

In [16], the authors explain how the derivatives of the heat kernel are expected to satisfy similar inequalities as those of the heat kernel itself, except for the powers of the time variable t which will be different; and the constants will depend on the curvature of M and its covariant derivatives:

$$|\nabla K_*(z, z', t)| \leq \tilde{c} (i(z)i(z'))^{1/2} t^{-3/2} \exp\left(-\frac{c_1 d_*^2(z, z')}{t}\right).$$

4. Relative determinants on surfaces with asymptotic cusps

In this section we describe the proof that the relative determinant of the Laplacian on a swac is well defined. That the determinant of the pair $(\Delta_g, \bar{\Delta}_0)$ is well defined was proved by W. Müller in [30] and [31]. We present here the conditions on φ that allow one to define the relative determinant of the pairs (Δ_h, Δ_g) and $(\Delta_h, \bar{\Delta}_0)$.

4.1. Trace class property of relative heat operators. In this section we verify that the conditions given in section 2.2 to define the relative determinant are fulfilled. Let us start with the trace class property for the relative heat operator.

THEOREM 4.1. *Let (M, h) be a swac with $h = e^{2\varphi} g$, and assume that on each cusp Z the functions $\varphi(y, x)$, $|\nabla_g \varphi(y, x)|$ and $\Delta_g \varphi(y, x)$ are $O(y^{-\alpha})$ with $\alpha > 0$, as $y \rightarrow \infty$. Then for any $t > 0$ the operator $e^{-t\Delta_h} - e^{-t\Delta_g}$ is trace class.*

The tools used in the proof of Theorem 4.1 are Duhamel’s principle and the estimates on the heat kernels and their derivatives up to second order. The method is similar to the one used by U.Bunke in [14]. Let us describe how the proof goes. Note that it is not helpful to use the semigroup property of the heat operators directly because they are not Hilbert-Schmidt. To get around this problem, we use the same method as Müller and Salomonsen in [32]. We start by using Duhamel’s principle and estimating the trace norms:

$$\|e^{-t\Delta_h} - e^{-t\Delta_g}\|_1 \leq \int_0^{t/2} \|(\Delta_g - \Delta_h)e^{-(t-s)\Delta_g}\|_1 ds + \int_{t/2}^t \|e^{-s\Delta_h}(\Delta_g - \Delta_h)\|_1 ds.$$

Then we prove that for any $0 < a < b < \infty$, for $t \in [a, b]$, the operators

$$(\Delta_g - \Delta_h)e^{-t\Delta_g} \quad \text{and} \quad e^{-t\Delta_h}(\Delta_g - \Delta_h)$$

are trace class and each trace norm is uniformly bounded on $t \in [a, b]$. Let us consider the first operator. We write it as

$$(\Delta_g - \Delta_h)e^{-t\Delta_g} = ((\Delta_g - \Delta_h)e^{-(t/2)\Delta_g} M_\phi^{-1}) \circ (M_\phi e^{-(t/2)\Delta_g}),$$

where ϕ is a smooth positive function on M that satisfies

$$\phi(y, x) = y^{-\beta}, \quad (y, x) \in Z,$$

with $\beta = \alpha/2$, if $\alpha \in (0, 1)$, and $\beta = 1/2$ if $\alpha \geq 1$, and where M_ϕ and M_ϕ^{-1} denote the operators multiplication by ϕ and ϕ^{-1} , respectively. Then for each $t > 0$, $(\Delta_g - \Delta_h)e^{-t\Delta_g} M_\phi^{-1}$ and $M_\phi e^{-t\Delta_g}$ are Hilbert-Schmidt operators. Notice that the operator $e^{-t\Delta_g}$ needs an extra weight to be Hilbert-Schmidt; see [5] for further details.

4.2. Asymptotic expansion for small time. We have the following expansion:

THEOREM 4.2. *Let (M, h) be a swac with $h = e^{2\varphi}g$. Let $\nu \geq 1$. If the following conditions are satisfied:*

- (1) *On each cusp Z the functions $\varphi|_Z(z)$, $\Delta_g \varphi|_Z(z)$, and $|\nabla_g \varphi|_g|_Z(z)$ with $z = (y, x)$, are $O(y^{-k})$ as $y \rightarrow \infty$ with $k \geq 5\nu + 8$.*
- (2) *If $\nu \geq 3$ higher derivatives of the conformal factor should decay as well: for $2 \leq \ell \leq \nu$, $|\nabla^\ell \varphi|_g|_Z(z) = O(y^{-k})$ with $k \geq 5(\nu - 2) - 1$.*

Then there is an expansion up to order ν of the relative heat trace:

$$(4.1) \quad \text{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g}) = \sum_{\ell=0}^{\nu} a_\ell t^{\ell-1} + O(t^\nu), \quad \text{as } t \rightarrow 0.$$

REMARK 4.3. *Note that we only require the conformal factor to be smooth in the interior of M and to decay at infinity. In particular, there is no requirement of smoothness at the boundary. For example, conformal factors of the form $\varphi_1(y, x) = y^{-23/2}$ and $\varphi_2(y, x) = y^{-40/3}\psi(x)$ with $\psi \in C^\infty(S^1)$ are allowed and induce expansions up to order one. This fact shows that our results are not entirely covered by Vaillant’s construction [41] mentioned in section 5.2.*

REMARK 4.4. *The usual approach of integrating the local asymptotic expansion of the heat kernels $K_h(z, z, t)$ and $K_g(z, z, t)$ does not work here because the remainder terms of the expansion are not uniformly bounded in the space variable and do not integrate to something finite.*

In order to sketch the proof of Theorem 4.2 we first need to introduce some notation:

For the sake of simplicity we assume that (M, g) has only one cusp $Z \cong [1, \infty) \times S^1$ with the hyperbolic metric on it. Let g, h and φ be as in the statement of the theorem. The restriction of the metric h to the cusp Z can be extended to a metric on the complete cusp, or horn, $\tilde{Z} = \mathbb{R}^+ \times S^1$ in the following way: On \tilde{Z} we have the hyperbolic metric $g_0 = y^{-2}(dy^2 + dx^2)$, and $g|_Z = g_0$. We extend the function $\varphi|_Z$ to a smooth function $\tilde{\varphi}$ on \tilde{Z} that vanishes in a small neighborhood of zero. Then on $(0, \infty) \times S^1$ we define h as $h := e^{2\tilde{\varphi}}g_0$. It is a complete metric and $h = g_0$ close to the boundary $\{0\} \times S^1$. In this way we can define the Laplacian on (\tilde{Z}, h) , and we denote its unique self-adjoint extension by $\Delta_{1,h}$. The heat kernel associated to $\Delta_{1,h}$ is denoted by $K_{1,h}(z, z', t)$, for $z, z' \in \tilde{Z}$ and $t > 0$.

For $n > 1$, we consider the following sets:

$$M_n := M_0 \cup ([1, n] \times S^1), \quad Z'_n = [1, n] \times S^1, \quad Z_n = [n, \infty) \times S^1.$$

The idea of the proof of Theorem 4.2 is the following: We first replace each heat kernel in the trace by a parametrix defined in the standard way; for example for $K_h(z, w, t)$, we consider

$$Q_h(z, w, t) = \varphi_1(z)K_{W,h}(z, w, t)\psi_1(w) + \varphi_2(z)K_{1,h}(z, w, t)\psi_2(w),$$

where $K_{W,h}$ is the heat kernel on a closed manifold W that contains (M_2, h) isometrically, $K_{1,h}$ is the heat kernel defined above, and the functions $\varphi_l, \psi_l, l = 1, 2$, are suitable gluing functions. We perform the corresponding construction for the heat kernel K_g . Then, using again Duhamel's principle and the estimates on the heat kernels, one proves that there exist constants $C, c > 0$ such that for $0 < t < 1$ the following estimate holds:

$$\left| \int_M (K_h e^{2\varphi} - K_g) - (Q_h e^{2\varphi} - Q_g) dA_g \right| \leq C e^{-c/t}.$$

In this way, in order to prove an asymptotic expansion for small t , we may replace each heat kernel by its corresponding parametrix.

The next step is to consider an $a > 0$, and to split the integral into three parts:

$$\int_M (Q_h e^{2\varphi} - Q_g) dA_g = I_0(t) + I_1(t) + I_2(t),$$

where

- I_0 is the integral over the compact part M_0 .
- $I_1(t) = \int_{[1,a] \times S^1} \psi_2(z)(K_{1,h}e^{2\varphi} - K_{1,g})dA_g$
- $I_2(t) = \int_{Z_a} \psi_2(z)(K_{1,h}e^{2\varphi} - K_{1,g})dA_g(z)$.

Each of the integrals I_0 and I_1 has a complete asymptotic expansion. The expansion of $I_0(t)$ is obvious. But for I_1 and I_2 we need to work. The proof of the theorem is complete after proving the following propositions:

PROPOSITION 4.5. *Under the assumptions of Theorem 4.2, there is an asymptotic expansion as $t \rightarrow 0$ of the integral $I_1(t)$ above, with $a = t^{-1/5}$. For $N \geq 1$, the asymptotic expansion has the following form:*

$$\int_{[1,a] \times S^1} \psi_2(K_{1,h}e^{2\varphi} - K_{1,g}) dA_g = t^{-1} \sum_{j=0}^N \hat{a}_j t^j + O(t^N),$$

where the remainder term $O(t^N)$ includes a $O(e^{-c/a^4t})$ with $c > 0$ that determines the condition $a = t^{-1/5}$.

The idea of the proof is to pass to the universal covering of the horn \tilde{Z} and use the corresponding estimates of the heat kernel and its local expansion. Then we put $a = t^{-1/5}$, and estimate the remainder terms independently of a . Finally we make sure that the asymptotic expansion is preserved when we replace $a = t^{-1/5}$ in the region of integration of the integrals.

PROPOSITION 4.6. *Let $\nu \geq 1/2$, and let $\varphi|_Z(z)$, $\Delta_g\varphi|_Z(z)$, and $|\nabla_g\varphi|_g|_Z(z)$, with $z = (y, x)$, be $O(y^{-k})$ as $y \rightarrow \infty$, with $k \geq 1$. For $0 < t \leq 1$, and for $a = t^{-1/5}$, if $k \geq 5\nu + 8$ we have that*

$$|\text{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}))| \ll t^\nu.$$

Ideally we should have found a complete expansion of the integral $I_2(t)$, but this task turned out to be too complicated. Instead we prove that we can make $|I_2(t)|$ as small as we want, if we allow more decay of the conformal factor and its derivatives up to order 2. The proof relies on a trick which consists of realizing $I_2(t)$ as the trace of an operator $B(t)$ that involves the heat kernels on the cusp corresponding to the Dirichlet Laplacians $\Delta_{Z,h}$ and $\Delta_{Z,g}$. We then assume that $\varphi(y, x)$ and $\Delta_g\varphi(y, x)$ are $O(y^{-k})$, for $k \geq 1$, and proceed using Duhamel’s principle and the estimates on the heat kernel. We have to split the resulting integral into three parts, dealing with them in a similar way as in the proof of the trace class property in Theorem 4.1. We finally obtain

$$|\text{Tr}(B(t))| \ll a^{-k+1/2}t^{-3/2}.$$

The assumption $a = t^{-1/5}$ in the proof of Proposition 4.5 implies that $k \geq 5\nu + 8$; see [5] for all the details.

REMARK 4.7. *If we take $N = 1$ in Proposition 4.5 and $\nu = 1/2$ in Proposition 4.6, we obtain an expansion of the form*

$$(4.2) \quad \text{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g}) = a_0t^{-1} + a_1 + O(\sqrt{t}), \text{ as } t \rightarrow 0,$$

and we only need to require that φ and its derivatives up to order two have a decay of order k with $k \geq 11$.

4.3. The relative determinant. In order to define the relative determinant of the pair (Δ_h, Δ_g) we need to fulfill the conditions stated in Section 2.2. We have already proven the trace class property of the relative heat operator for all $t > 0$, and the existence of an expansion for small t of the relative heat trace under suitable conditions on the decay of the conformal factor.

The third condition in section 2.2, about the behavior of the relative heat trace for large t , is fulfilled thanks to the trace class property and the gap at zero of the continuous spectrum of the operator Δ_g ; see [31, Lemma 2.22]. We obtain that there exists a $C > 0$ such that

$$(4.3) \quad \text{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g}) = O(e^{-Ct}), \text{ as } t \rightarrow \infty.$$

In order to define the relative determinant, it suffices to have an expansion as in equation (4.2). Therefore, after Remark 4.7, we restrict ourselves to metrics whose

conformal factor lies in the following set:

$$\mathcal{F}_{11} := \{ \psi \in C^\infty(M) \mid \psi(z), |\nabla_g \psi|, \Delta_g \psi(z) \text{ are } O(y^{-11}), \\ \text{as } y \rightarrow \infty \text{ on each cusp } Z \}.$$

In this way we obtain that for a swc (M, g) and a metric $h = e^{2\varphi}g$, with $\varphi \in \mathcal{F}_{11}$, the relative determinant $\det(\Delta_h, \Delta_g)$ is well defined. As an immediate consequence of the trace expansion proved by W. Müller in [30, Thm.8.20],

$$\text{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}}) = \frac{A_g}{4\pi} t^{-1} + \left(\frac{\gamma m}{2} + \sum_{j=1}^m \log(a_j) \right) \frac{1}{\sqrt{4\pi t}} + \frac{m \log(t)}{2\sqrt{4\pi t}} + \frac{\chi(M)}{6} \\ + \frac{m}{4} + O(\sqrt{t}) \text{ as } t \rightarrow 0,$$

we have that the relative determinant $\det(\Delta_h, \bar{\Delta}_0)$ is also well defined.

We can now follow the same lines as OPS in [34] and prove Polyakov’s formula for the relative determinant that is valid for metrics whose conformal factor belongs to \mathcal{F}_{11} . The problem of integrating the remainder term appears again but we can deal with it in the same way as before. We obtain the following formula:

$$\log \det(\Delta_h, \Delta_{1,0}) = -\frac{1}{12\pi} \int_M |\nabla_g \varphi|^2 dA_g - \frac{1}{6\pi} \int_M K_g \varphi dA_g \\ + \log A_h + \log \det(\Delta_g, \Delta_{1,0}).$$

Using this equation it is easy to see that a maximizer of the relative determinant will be attained at the metric of constant curvature. However, the equation for the conformal change of the curvature in the cusps leads to the equation

$$-e^{2\varphi} = \Delta_g \varphi - 1.$$

This equation implies that in the cusps the function φ should decay as y^{-1} , which will sadly be outside the conformal class under consideration.

5. Other determinants on non-compact surfaces

Another natural geometric setup is furnished by surfaces with funnel ends. A funnel end corresponds to the end of the hyperbolic horn complementary to the cusp. The horn is given by $\tilde{Z} = (0, \infty) \times S^1$ with the hyperbolic metric on it. A cusp corresponds to $[1, \infty) \times S^1$ and has finite area; and the funnel corresponds to $(0, 1] \times S^1$ and has infinite area. Funnel ends can also be asymptotically hyperbolic. The spectral properties of a surface with funnel ends are different to the ones for surfaces with cusps, but they have elements in common, see [9]. For example, the continuous spectrum is $[1/4, \infty)$ but now it has infinite multiplicity. The spectral theory and scattering theory for this kind of surfaces has also been widely studied. It is out of the scope of this note to give an account of the results and references on these topics. However, we want to mention two results that involve determinants.

5.1. Relative determinants on surfaces with infinite area. In [10], D. Borthwick, C. Judge and P. Perry define a determinant for surfaces with funnels that are hyperbolic near infinity. They use the 0-calculus, the extension of the resolvent in [28], and Mazzeo-Taylor uniformization in [29] to define a relative determinant $D_{g,\tau}(s)$ as $\det(\Delta_g + s(1-s), \Delta_\tau + s(1-s))$. The relative determinant of the operators (Δ_g, Δ_τ) is then defined as $D_{g,\tau}(1)$. They prove some nice properties

of this determinant. For example, they prove that the relative determinant has zeroes at the eigenvalues and resonances of Δ_g , and poles at the eigenvalues and resonances of Δ_τ . They also prove Polyakov’s formula for this relative determinant and use it to prove compactness of isopolar metrics on the surface. In a later work [12] D. Borthwick and P. Perry extend the results in [10] to higher dimensions and improve the result in [10] for surfaces with funnels. The relative determinant of the Laplacian plays an important role in their discussion.

5.2. Renormalized determinants. In [2], P. Albin, F. Rochon and the author consider renormalized determinant on surfaces with asymptotically hyperbolic cusps and funnels. These metrics are called funnel-cusp metrics.

In this case it is convenient to describe the surface in terms of a boundary defining function, i.e. a function that vanishes to first order at the boundary and is positive everywhere else. This function is used as a coordinate as well.

The renormalized determinant is defined in a similar way as the relative determinant, via a zeta function regularization that involves renormalized integrals. The renormalized zeta function is defined by the equation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{R} \operatorname{Tr} (e^{-t\Delta} - P_{\operatorname{Ker}(\Delta_g)}) \frac{dt}{t},$$

where $P_{\operatorname{Ker}(\Delta_g)}$ is the projection on the kernel of Δ_g , and $\operatorname{R} \operatorname{Tr} (e^{-t\Delta} - P_{\operatorname{Ker}(\Delta_g)})$ is the renormalized trace, given by the renormalized integral of the heat kernel restricted to the diagonal. Before we proceed, let us explain how renormalized integrals are defined.

There are two classical ways of renormalizing integrals: the Hadamard renormalization and the Riesz renormalization. These two methods coincide under certain conditions on the metric that are assumed in [2]; see also [1]. Let us recall the definition of the Riesz renormalization. Let f be log-polyhomogeneous on M , i.e. f has an asymptotic expansion close to the boundary in terms of the boundary defining function; for a definition of log-polyhomogeneous functions see [26] and the references therein. Let $z \in \mathbb{C}$ with $\operatorname{Re}(z)$ big enough and consider the integral $\int_M x^z f \mu$. It has a meromorphic extension as function of z ; the renormalized integral is defined as the finite part of this extension at $z = 0$:

$$\int_M^R f \mu = \operatorname{FP}_{z=0} \int_M x^z f \mu.$$

As an example, let us compute the renormalized area of a funnel $F = (0, a]_x \times S^1_\theta$, $a > 0$ with the metric $g(x, \theta) = x^{-2}(dx^2 + d\theta^2)$:

$$A_F = \operatorname{FP}_{z=0} \int_{S^1} \int_0^a x^z \frac{dx d\theta}{x^2} = 2\pi \operatorname{FP}_{z=0} \frac{1}{z-1} a^{z-1} = -2\pi a^{-1}.$$

The renormalized determinant is well defined if the renormalized trace of the heat operator has a log-polyhomogeneous expansion for small t . This follows from the expansions of the heat kernel in the space and the time variables in the heat space. We refer to the work by P. Albin in [1] and by B.Vaillant in [41]. In each case, an expansion of the heat kernel for a metric h that is asymptotically hyperbolic at infinity is constructed, for manifolds with funnel ends by P. Albin and for manifolds with fiber cusp metrics by B. Vaillant. However, it is not trivial to extract the asymptotic expansion for the heat-operator trace from the corresponding heat

kernel constructions. A pushforward theorem is necessary. This process is carried out and nicely explained by Albin and Rochon in [6]. The renormalized heat trace then has an expansion of the form

$$(5.1) \quad {}^R \text{Tr} (e^{-t\Delta_h}) \sim \sum_{k \geq -2} a_k t^{k/2} + \sum_{k \geq -1} \tilde{a}_k t^{k/2} \log t, \text{ as } t \rightarrow 0.$$

In this way, the renormalized zeta function has a meromorphic extension that is regular at $s = 0$, and the renormalized determinant is defined in the same way as before.

REMARK 5.1. *Let us note here that, in order to have the asymptotic expansion in equation (5.1), the conformal factor needs to be smooth up to the boundary and only decay as a constant at the boundary. The conditions are different than the conditions required in Theorem 4.2. They are weaker in the sense that no strong decay on the conformal factor is required, but it should still have a power series expansion in terms of the boundary defining function at “infinity.” On the other hand, the renormalized trace coincides with the trace on trace class operators. Therefore*

$$\text{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g}) = {}^R \text{Tr} (e^{-t\Delta_h}) - {}^R \text{Tr} (e^{-t\Delta_g}).$$

Then, if the expansion for each trace is defined, they coincide. Otherwise, one could use one asymptotic expansion to obtain the other. It would be interesting to further understand how the different techniques and tools induce different conditions.

The renormalized determinant also satisfies a Polyakov formula, which is given by two different expressions depending on whether there are funnels or not, i.e. depending on whether the area is finite or infinite. The study of extremals inside a conformal class then splits into these two cases. In both cases, the existence of the maximizer is proven using normalized Ricci flow (NRF).

The existence of a unique solution of normalized Ricci flow with initial data for funnel-cusp metrics is one of the main results of [2]. The main tool in the proof is the construction of a potential function, in a very similar way as Hamilton’s existence for closed surfaces [20], and Ji-Mazzeo-Sesum’s existence [22] for surfaces with cusps.

The relevant properties are that the flow preserves the conformal class of the initial metric g_0 , and, for a convenient choice of constant, it preserves the renormalized area. The decay of the conformal factor is also preserved under the flow; this allows one to have the renormalized determinant defined for all time. In addition, the limiting metric is smooth and has constant curvature; see [2] for all the details.

Roughly speaking, one uses Polyakov’s formula to prove that the determinant increases under the flow. Since the flow converges to the unique metric of constant curvature with the same area as the initial metric, it follows that the determinant is maximal at the metric of constant curvature.

The idea of flowing the determinant was already in the proof for closed surfaces by OPS. The use of NRF to prove that the maximizer in that case is attained at the metric of constant curvature was done in [24].

In higher dimensions the problem needs some modifications. There are many results about spectral zeta functions and zeta determinants for operators that change conformally. Some of these results are by A.S. Chang *et al*, see for example [15]. Another approach is taken by W. Müller and K. Wendland in [33] where they

work with Kähler manifolds and they define an analog to the determinant using the Kähler structure.

Steve Rosenberg has also contributed to the subject. Let us mention here his work in [37], where he proves that, in odd dimensions, the determinant of a conformally covariant operator is a conformal invariant.

6. The relative determinant as a function on the moduli space of hyperbolic surfaces with cusps

Another interesting problem is the study of the relative determinant as a function on the moduli space of hyperbolic surfaces.

Let us mention a conjecture given by P. Sarnak in [39]. The conjecture states that the determinant, as a function on the space of metrics (up to isometry) on a surface of fixed genus and fixed area, is a Morse function and has a unique global maximum. P. Sarnak points out that, if the conjecture is true, it implies that the determinant would be a Morse function on the moduli space of hyperbolic surfaces of fixed genus and fixed area. There are few results in this direction, and not yet a complete answer. For example, OPS proved that the height $h(u) = -\log(\Delta_u)$ goes to infinity as u approaches the boundary of the moduli space of uniform metrics on a surface of genus p and n boundaries, with $pn = 0$, see [38].

We present the corresponding result on $\mathcal{M}_{p,m}$, the moduli space of compact Riemann surfaces of genus p with m punctures, where we think of it as a space of complete hyperbolic metrics on a topological surface of genus p with m punctures. The result that we present here is published in [4]. We define the free Laplacian as the Laplacian $\bar{\Delta}_{1,0}$ associated to the union of m cusps all starting at $a_i = 1$ as in equation (3.1). The Laplacian $\bar{\Delta}_{1,0}$ acts on a subspace of $\oplus_{j=1}^m L^2([1, \infty), y_j^{-2} dy_j)$. If (M, g) can be decomposed as $M = M_0 \cup Z_{a_1} \cup \cdots \cup Z_{a_m}$, with $a_j \geq 1$; then the difference $e^{-t\Delta_g} - e^{-t\bar{\Delta}_{1,0}}$ is taken in the extended L^2 space given by

$$L^2(M, dA_\tau) \oplus \oplus_{j=1}^m L^2([1, a_j], y^{-2} dy) = \\ L^2(M_0, dA_\tau) \oplus \oplus_{j=1}^m (L_0^2(Z_{a_j}) \oplus L^2([1, \infty), y^{-2} dy)).$$

The relative determinant defines a function on the moduli space in the same way as in the compact case: $[g] \in \mathcal{M}_{p,m} \mapsto \det(\Delta_g, \bar{\Delta}_{1,0}) \in \mathbb{R}^+$, where $g \in [g]$ is hyperbolic. If the metric is hyperbolic, the surface can be realized as a quotient $\Gamma \backslash \mathbb{H} = M$, where Γ is a Fuchsian group of the first kind. We use Selberg's trace formula to find a relation between the relative determinant and the hyperbolic determinant $\det_{\text{hyp}} \Delta_g$ by J. Jorgenson and R. Lundelius defined in [23]. It is well known that each point of the boundary of the moduli space can be reached through a degenerating family of metrics. The degeneration arises from closed geodesics whose length converges to zero, see [8]. Then we used the results in [23] and those by S.A. Wolpert in [42] to prove the following theorem:

THEOREM 6.1. *Let $\mathcal{M}_{p,m}$ be the moduli space of hyperbolic surfaces with cusps. Consider the relative determinant $\det(\Delta_g, \bar{\Delta}_{1,0})$ as a function on $\mathcal{M}_{p,m}$. As $[g]$ approaches $\overline{\mathcal{M}}_{m,p} \backslash \mathcal{M}_{p,m}$, the boundary of the moduli space, the relative determinant $\det(\Delta_g, \bar{\Delta}_{1,0})$ tends to zero.*

To finish, let us mention that we can relate the relative determinant with the Selberg zeta function $Z(s)$:

$$\det(\Delta_g, \bar{\Delta}_{1,0}) = A \det_{\text{hyp}} \Delta_g = A Z'_M(1) e^{\chi(M) \left(\frac{1}{2} \log 2\pi - 2\zeta'_R(-1) + \frac{1}{4} \right)},$$

where ζ_R denotes the Riemann zeta function, and A is a constant that depends only on the number of cusps, [4]. The second equality was proven in [23] as a generalization of the corresponding formula on compact Riemann surfaces given in [18] and [39]. A similar result holds for hyperbolic surfaces with cusps and funnels. If there is at least one funnel, the renormalized determinant is given by

$$\det(\Delta) = C_{F,c} Z(1) = e^{\chi(M) \left(\frac{1}{2} \log 2\pi - 2\zeta'_R(-1) + \frac{1}{4} \right)} (2\pi)^{-\chi(M)} (\sqrt{2\pi})^{-n_c} Z(1),$$

where n_c is the number of cusps. This result was proven in [2], using the results in [11].

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MAX PLANCK INSTITUT FÜR GRAVITATIONSPHYSIK, D-14476 GOLM, GERMANY
E-mail address: clara.aldana@aei.mpg.de