

# CONTEMPORARY MATHEMATICS

589

## Algebraic and Combinatorial Aspects of Tropical Geometry

CIEM Workshop  
Tropical Geometry  
December 12–16, 2011  
International Centre for Mathematical Meetings  
Castro Urdiales, Spain

Erwan Brugallé  
Mariá Angélica Cueto  
Alicia Dickenstein  
Eva-Maria Feichtner  
Ilia Itenberg  
Editors



American Mathematical Society

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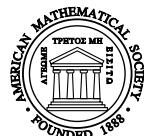
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## **Dedication**

To the memory of our dear friend and colleague Mikael Passare.  
(1959–2011)



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## Preface

This volume compiles the proceedings of the CIEM workshop on Tropical Geometry, which took place at the International Centre for Mathematical Meetings (CIEM), Castro Urdiales, Spain, December 12–16, 2011. The main goals of the meeting were to bring together the leading experts in order to discuss the most recent developments in the field, and to promote tropical geometry within the Iberian mathematical community. Over the five days of the conference, twenty-two research talks and seventeen posters were presented; a software session and an informal open problem discussion took place. Some of the articles in the present volume report on material that was presented at the workshop, while others grew out of the exchange of ideas at CIEM, thus reflecting the lively atmosphere of the meeting.

One possible point of view on tropical geometry interprets it as algebraic geometry over the tropical semifield, i.e., the set  $\mathbb{R} \cup \{-\infty\}$  endowed with the arithmetic operations  $x \oplus y = \max\{x, y\}$  (tropical addition) and  $x \odot y = x+y$  (tropical multiplication). Tropical geometry can be seen as a polyhedral version of algebraic geometry. Complex algebraic varieties are replaced by considerably simpler piecewise-linear objects after passing to the tropical limit. These limiting objects retain relevant information about classical algebraic varieties.

The foundational ideas in the field can already be retrieved, under very different names, from the pioneering texts of G. Bergman (1971), R. Bieri and J. Groves (1984), V. Maslov and G. Litvinov (90's), as well as in O. Viro's patchworking construction (1979). In spite of this, it was only about 10 years ago that the theory began to consolidate itself. The name *Tropical Geometry* was suggested by G. Mikhalkin and B. Sturmfels in 2002, adopting the adjective *tropical*, which had been coined by French mathematicians and computer scientists in honor of their Brazilian colleague Imre Simon. Recent years have seen a tremendous development in tropical geometry that both established the field as an area of its own right, and unveiled its deep connections to numerous branches of pure and applied mathematics. To a large extent, this development was initiated by fruitful applications of tropical geometry to complex and real enumerative geometry, based on G. Mikhalkin's correspondence theorem.

The continued growth in publications, doctoral dissertations, and meetings in tropical geometry over the last ten years attests to the extensive activity in this field of mathematics. *Contemporary Mathematics* has already devoted four volumes to the tropical realm. They cover the following four topics: idempotent mathematics, tropical linear algebra and tropical convex geometry, relations of tropical geometry to mirror symmetry, and connections between tropical geometry and integrable systems. The present volume is mainly focused on the geometric side

of the tropical world laying the emphasis on relations between tropical geometry, algebraic geometry, and combinatorics. The following topics are covered:

- Berkovich analytic spaces and non-Archimedean aspects of tropical geometry (M. Chan & B. Sturmfels; J. Draisma & B. Frenk; W. Gubler; E. Katz; P. Popescu-Pampu & D. Stepanov),
- complex and real algebraic aspects (B. Bertrand, L. López de Medrano & J.-J. Risler, M. Chan, M. Melo & F. Viviani, K. Shaw),
- tropical intersection theory (B. Bertrand & F. Bihan, K. Shaw),
- tropical moduli spaces (M. Chan, M. Melo & F. Viviani),
- combinatorial aspects (F. Santos),
- tropical algebra (Z. Izhakian, M. Knebusch & L. Rowen).

The CIEM workshop on Tropical Geometry would not have been possible without the help of many people. Erwan Brugallé, María Angélica Cueto and Eva-Maria Feichtner thank their co-organizers Tomás Recio, Martín Sombra, Luis Felipe Tabera, and Josephine Yu for their tremendous energy. The organizers of the CIEM workshop kindly acknowledge support from the Spanish Ministerio de Educación y Cultura, Proyecto ‘Ingenio Mathematica (i-MATH)’ No. CSD2006-00032 (Consolider Ingenio 2010, actividad FUT-C6-0403 : Workshop on tropical geometry), Ministerio de Ciencia e Innovación (acción complementaria MTM2010-12192-E: Workshop on tropical geometry), Centro Internacional de Encuentros Matemáticos CIEM, Consejería de Educación, Comunidad de Cantabria “Ayudas a la Difusión de la Investigación”, l’Université Pierre et Marie Curie, the Georgia Institute of Technology and the National Science Foundation (Conference Grant DMS-1138935). We also thank all the speakers, poster presenters and contributors to this volume, as well as the referees for their generous work.

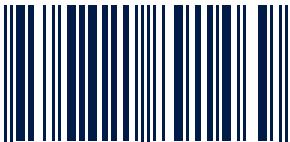
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This volume contains the proceedings of the CIEM workshop on Tropical Geometry, held December 12–16, 2011, at the International Centre for Mathematical Meetings (CIEM), Castro Urdiales, Spain.

Tropical geometry is a new and rapidly developing field of mathematics which has deep connections with various areas of mathematics and physics, such as algebraic geometry, symplectic geometry, complex analysis, dynamical systems, combinatorics, statistical physics, and string theory. As reflected by the content of this volume, this meeting was mainly focused on the geometric side of the tropical world with an emphasis on relations between tropical geometry, algebraic geometry, and combinatorics.

This volume provides an overview of current trends concerning algebraic and combinatorial aspects of tropical geometry through eleven papers combining expository parts and development of modern techniques and tools.

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