A limit problem for degenerate quasilinear variational inequalities in cylinders

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ABSTRACT. We consider quasilinear degenerate variational inequalities with pointwise constraint on the values of the solutions. The limit problem as the domain becomes unbounded in some directions is exhibited.

1. Introduction and main results

We propose a contribution to the study of limit problems for variational inequalities set in cylinders becoming unbounded in some directions, as in [10] or [9], which develop a research already considered in [2], [3], [4], [5], [6], [7], [8], [11], [13]. However, in contrast to the previous papers, we consider *quasilinear* variational inequalities, also with nonlinear lower order terms, in the spirit of [12]. In particular, the presence of the *p*-Laplace operator in place of the usual Laplacian introduces several technicalities which don't let us obtain precise estimates as in the papers cited above. Nevertheless, a description of the limit problem is still possible.

Let us present the precise setting of the problem. Let $m, n \in \mathbb{N}$ and let $\omega_1 \subset \mathbb{R}^m$ and $\omega_2 \subset \mathbb{R}^n$ be two bounded open subsets such that

(1.1)
$$\omega_1$$
 is convex and contains 0.

For any $\ell > 0$ we introduce the cylinder

$$\Omega_{\ell} = \ell \omega_1 \times \omega_2 \subset \mathbb{R}^m \times \mathbb{R}^n,$$

whose points will be denoted by (x, y), so that x will denote a generic point in $\ell \omega_1$, while $y \in \omega_2$.

A general constrained problem can be the following: for any $y \in \omega_2$, let K(y) be a convex subset of $\mathbb{R} \times \mathbb{R}^{m+n}$. Finally, fixed $p \in (1, \infty)$ and $g \in W_0^{1,p}(\omega_2)$, we introduce the constrain set

$$K_{\ell} := \Big\{ v \in W^{1,p}(\Omega_{\ell}) : v = g \text{ on } \partial\Omega_{\ell}, (v, Dv)(x, y) \in K(y) \text{ a.e. in } \Omega_{\ell} \Big\},\$$

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which is a closed and convex subset of $W^{1,p}(\Omega_{\ell})$. Once fixed $f \in L^{p'}(\omega_2)$, we finally consider the nonlinear variational inequality

$$(P_{\ell}) \qquad \begin{cases} u_{\ell} \in K_{\ell} \\ \int_{\Omega_{\ell}} \left[|Du_{\ell}|^{p-2} Du_{\ell} \cdot D(v - u_{\ell}) + h(y, u_{\ell})(v - u_{\ell}) \right] dxdy \\ \geq \int_{\Omega_{\ell}} f(y)(v - u_{\ell}) dxdy \quad \forall v \in K_{\ell}. \end{cases}$$

REMARK 1.1. We can replace $f \in L^{p'}(\omega_2)$ with the less restrictive condition $f \in L^q(\omega_2)$, where q = pn/(pn - n + p) when p < n, but for the sake of simplicity we present all the results for $f \in L^{p'}(\omega_2)$.

Associated to (P_{ℓ}) there is a natural expected limit problem

$$(P_{\infty}) \qquad \begin{cases} u_{\infty} \in K_{\infty} \\ \int_{\omega_2} \left[|Du_{\infty}|^{p-2} Du_{\infty} \cdot D(v-u_{\infty}) + h(y,u_{\infty})(v-u_{\infty}) \right] dy \\ \geq \int_{\omega_2} f(y)(v-u_{\infty}) \, dy \quad \forall v \in K_{\infty}, \end{cases}$$

where

$$K_{\infty} := \left\{ u \in W^{1,p}(\omega_2) : (u, 0, D_y u)(y) \in K(y) \text{ for a.e. } y \in \omega_2 \right\}$$

and $D_y u = (\partial_{y_1} u, \dots, \partial_{y_n} u)$ is the gradient of u with respect to the y-variables.

In view of the asymptotic estimate found in [10], we concentrate on the case in which

g = 0 and K(y) is a closed interval of \mathbb{R} containing 0.

It is not clear whether (P_{ℓ}) and (P_{∞}) admit solutions, since the nonlinear term h may cause problems. For this we assume:

 $(h)(i) h : \omega_2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $a \in L^{p'}(\omega_2), b > 0$ and $q \ge 1$ such that

(1.2)
$$|h(y,s)| \le a(y) + b|s|^{q-1}$$
 for a.e. $y \in \omega_2$ and all $s \in \mathbb{R}$.

Here $q \in [1, p^*)$, where $p^* = \infty$ if $p \ge n$ and $p^* = pn/(n-p)$ if p < n. Moreover, we assume one of the following conditions:

(ii) h is non decreasing in the second variable, h(y,0) = 0 for a.e. $y \in \omega_2$ and in (1.2) q is allowed to vary in $[1, p^*]$ if p < n; or (iii) there exists $L \in [0, \mu_1)$ such that

$$|h(y,s_1) - h(y,s_2)| \le L|s_1 - s_2|^{p-1}$$
 for all $s_1, s_2 \in \mathbb{R}$ and for a.e. $y \in \omega_2$,

and in addition

$$\liminf_{|s|\to\infty}\frac{h(y,s)}{|s|^{p-2}s}:=\alpha(y)>-\min\{\lambda_{1,p},\mu_1\}.$$

Here μ_1 is the best constant in the Poincaré inequality in ω_2 :

(1.3)
$$\mu_1 \int_{\omega_2} |u|^p dy \le \int_{\omega_2} |D_y u|^p dx \text{ for all } u \in W_0^{1,p}(\omega_2).$$

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Moreover, $\lambda_{1,p}$ denotes the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\mathbb{R}^m \times \omega_2)$, i.e.

$$\lambda_{1,p} = \inf_{\substack{u \in W_0^{1,p}(\mathbb{R}^m \times \omega_2)\\ u \neq 0}} \frac{\int_{\mathbb{R}^m \times \omega_2} |Du|^p dx dy}{\int_{\mathbb{R}^m \times \omega_2} |u|^p dx dy},$$

which is a strictly positive number, see [1, Remark 9.21], and which guarantees the following Poincaré inequality:

(1.4)
$$\lambda_{1,p} \int_{\mathbb{R}^m \times \omega_2} |u|^p dx dy \le \int_{\mathbb{R}^m \times \omega_2} |Du|^p dx dy \text{ for all } u \in W_0^{1,p}(\mathbb{R}^m \times \omega_2).$$

Let us also remark that by an easy null extension argument, we find that for every $\ell > 0$

(1.5)
$$\lambda_{1,p} \leq \lambda_{1,p,\ell} := \inf_{\substack{u \in W_0^{1,p}(\Omega_\ell) \\ u \neq 0}} \frac{\int_{\Omega_\ell} |Du|^p dx dy}{\int_{\Omega_\ell} |u|^p dx dy},$$

for which there holds

(1.6)
$$\lambda_{1,p,\ell} \int_{\Omega_{\ell}} |u|^p dx dy \le \int_{\Omega_{\ell}} |Du|^p dx dy \text{ for all } u \in W_0^{1,p}(\Omega_{\ell}).$$

Simple arguments also show that $\mu_1 \leq \lambda_{1,p,\ell}$ for every $\ell > 0$.

LEMMA 1.1. If $f \in L^{p'}(\omega_2)$ and h satisfies (h)(i), (ii) or (h)(i), (iii), then problem (P_ℓ) has a solution for every $\ell > 0$, and problem (P_∞) has a solution, as well.

REMARK 1.2. The condition that h is Lipschitz continuous in the second variable uniformly in the first one, without any additional condition on the size of the Lipschitz constant is sufficient for the existence of a solution. However, in (h)(iii) we need the condition $L < \mu_1$ when dealing with the limit behaviour of the solutions.

With additional assumptions, also uniqueness is granted. In particular, the solution u_{ℓ} of (P_{ℓ}) is unique if $p \geq 2$ and an additional assumption on h is verified. Indeed, if $p \geq 2$, the operator $-\Delta_p$ is strongly monotone, which is a straightforward consequence of the following fact: if $p \geq 2$, there exist $C_p \geq c_p > 0$ such that

(1.7)
$$c_p |\xi - \zeta|^p \le \left(|\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta \right) \cdot \left(\xi - \zeta \right)$$

and

(1.8)
$$\left| |\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta \right| \le C_p(|\xi|^{p-2} + |\zeta|^{p-2})|\xi - \zeta|$$

for every $\xi, \zeta \in \mathbb{R}^k, k \in \mathbb{N}$.

LEMMA 1.2. If, in addition to the assumptions of Lemma 1.1, there exists a measurable function $\beta: \omega_2 \longrightarrow \mathbb{R}$ such that

$$\frac{h(y,s_1) - h(y,s_2)}{|s_1 - s_2|^{p-2}(s_1 - s_2)} \ge \beta(y) > -c_p \min\{\lambda_{1,p}, \mu_1\} \quad \forall s_2 \neq s_1 \text{ and for } a.e. x \in \omega_2,$$

then the solution of problem (P_{ℓ}) is unique if $p \geq 2$ for every $\ell > 0$, and the solution of (P_{∞}) is unique, as well.

If h is strictly increasing in the second variable, the solution is unique for every p > 1.

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The main result of this paper is that, as expected, u_{ℓ} converges to u_{∞} , in the following sense:

THEOREM 1.1. Let $p \ge 2$ be such that m(p-2) < 1. Then, under the assumptions of Lemmas 1.1 and 1.2,

$$\inf_{\ell,\tau} \int_{\Omega_{\tau}} |D(u_{\ell} - u_{\infty})|^p dx dy = 0$$

More precisely, if

(1.9)
$$\inf_{\ell} \int_{\Omega_{\ell}} |D(u_{\ell} - u_{\infty})|^p dx dy > 0,$$

then there exist constants $\mathcal{A}, \mathcal{B} > 0, \eta \in (0, 1)$ such that

(1.10)
$$\int_{\Omega_{\frac{\ell}{2}}} |D(u_{\ell} - u_{\infty})|^p dx dy \le \mathcal{A}e^{-\mathcal{B}\ell^{\eta}}.$$

In [10] the authors prove that, for p = 2 and h = 0 the following estimate holds:

(1.11)
$$\int_{\Omega_{\frac{\ell}{2}}} |D(u_{\ell} - u_{\infty})|^{p} dy \le c e^{-\alpha \ell} ||f||_{L^{p'}(\omega_{2})},$$

where $C, \alpha > 0$ are independent of ℓ . In our case we are not able to prove such an estimate, due to the presence of a remainder term which disappears only if p = 2, and which we can control only under the additional condition m(p-2) < 1.

2. Proofs of the Lemmas

PROOF OF LEMMA 1.1. We concentrate on (P_{ℓ}) , the proof for (P_{∞}) being the same. Consider the functional $I: W^{1,p}(\Omega_{\ell}) \to (-\infty, \infty]$ defined as

$$I(u) = \begin{cases} \frac{1}{p} \int_{\Omega_{\ell}} |Du|^{p} dx dy + \int_{\Omega_{\ell}} H(y, u) dx dy - \int_{\Omega_{\ell}} f(y) u dx dy & \text{if } u \in K_{\ell}, \\ +\infty & \text{elsewhere,} \end{cases}$$

where $H(y, u) = \int_0^u h(y, s) ds$. Note that by the general assumption on h, I needs not be convex. However, we will show that I has a minimum.

First, let us assume that (h)(ii) holds, and let $(u_n)_n \subset K_\ell$ be a minimizing sequence. Since h is non decreasing, then H is nonnegative, so that it is readily seen that $(u_n)_n$ is bounded. Thus, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega_\ell)$ and a.e. in Ω_ℓ . Of course, $u \in K_\ell$. By the semicontinuity of the $W^{1,p}$ and L^{p^*} -norms (or the continuity of the L^q -norm), we find that I has a minimum point in K_ℓ , and so a solution of (P_ℓ) is given.

If (h)(iii) holds, we proceed as follows. Fixed $\varepsilon > 0$, there exists M > 0 such that

$$\frac{h(y,s)}{|s|^{p-2}s} - \alpha(y) > -\varepsilon \quad \forall |s| > M \text{ and a.e. } y \in \omega_2.$$

Integrating we get

(2.12)
$$H(y,s) - H(y,M) > \frac{\alpha(y) - \varepsilon}{p} (|s|^p - M^p) \quad \forall |s| > M \text{ and a.e. } y \in \omega_2,$$

while (1.2) implies

(2.13)
$$|H(y,s)| \le \left(a(y) + \frac{b}{p}M^{q-1}\right)M \quad \forall |s| \le M \text{ and a.e. } y \in \omega_2.$$

Then, by (2.12) and (2.13) there exists $C_M > 0$ such that

$$\frac{H(y,s)}{|s|^p} = \frac{\int_0^M h(y,\sigma) \, d\sigma + \int_M^s h(y,\sigma) \, d\sigma}{|s|^p} > \frac{C_M + \frac{\alpha(y) - \varepsilon}{p} (|s|^p - M^p)}{|s|^p},$$

so that

$$\liminf_{|s|\to\infty} \frac{H(y,s)}{|s|^p} \ge \frac{\alpha(y)-\varepsilon}{p}$$

for every $\varepsilon > 0$, i.e.

(2.14)
$$\liminf_{|s|\to\infty} \frac{H(y,s)}{|s|^p} \ge \frac{\alpha(y)}{p}.$$

Now let us show

(2.15)
$$\liminf_{\substack{\|u\|\to\infty\\u\in K_{\ell}}} \frac{I(u)}{\|u\|^p} > 0.$$

Take $(u_n)_n$ in K_ℓ such that $||u_n|| \to \infty$. Up to a subsequence we can assume that $v_n := \frac{u_n}{||u_n||}$ converges to a function $u \in K_\ell$ weakly in $W_0^{1,p}(\Omega_\ell)$, strongly in $L^p(\Omega_\ell)$ and a.e. in Ω_ℓ . Moreover $||u|| \le 1$ and

(2.16)
$$\frac{|H(y,u_n)|}{\|u_n\|^p} \le \frac{a(y)|u_n| + b|u_n|^p/p}{\|u_n\|^p} \longrightarrow \frac{b}{p}|u|^p \text{ in } L^1(\Omega_\ell)$$

We recall the following generalized Fatou's Lemma: if $(\phi_n)_n$ and $(\psi_n)_n$ are two sequences of measurable functions on a measurable space (X, μ) such that

$$\begin{aligned} \phi_n \geq \psi_n & \mu\text{-a.e. in } X, \\ \psi_n \rightarrow \psi & \mu\text{-a.e. in } X \end{aligned}$$

and

$$\lim_{n \to \infty} \int_X \psi_n d\mu = \int_X \lim_{n \to \infty} \psi_n d\mu \in \mathbb{R},$$

then

$$\int_X \liminf_{n \to \infty} \phi_n \, d\mu \le \liminf_{n \to \infty} \int_X \phi_n \, d\mu.$$

The proof of the statement is obtained by applying the Fatou Lemma to the functions $\theta_n = \phi_n - \psi_n$.

Hence, by (2.16) we immediately find

(2.17)
$$\liminf_{n \to \infty} \int_{\Omega_{\ell}} \frac{H(y, u_n)}{\|u_n\|^p} \, dx dy \ge \int_{\Omega_{\ell}} \liminf_{n \to \infty} \frac{H(y, u_n)}{\|u_n\|^p} \, dx dy.$$

But

$$\Omega_{\ell} = \{ z \in \Omega_{\ell} : u_n(z) \text{ is bounded} \} \cup \{ z \in \Omega_{\ell} : |u_n(z)| \text{ is unbounded} \},\$$

and $\frac{H(y,u_n)}{\|u_n\|^p} \to 0$ in the set $\{z \in \Omega_\ell : u_n(z) \text{ is bounded}\}$, while in the set $\{z \in \Omega_\ell : |u_n(z)| \text{ is unbounded}\}$ we have

$$\liminf_{n \to \infty} \frac{H(y, u_n)}{\|u_n\|^p} = \liminf_{n \to \infty} \frac{H(y, u_n)}{\|u_n\|^p} \frac{\|u_n\|^p}{\|u_n\|^p} \ge \frac{\alpha(y)}{p} |u(y)|^p$$

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by (2.14). Therefore (2.17) gives

$$\liminf_{n \to \infty} \int_{\Omega_{\ell}} \frac{H(y, u_n)}{\|u_n\|^p} \ge \int_{\Omega_{\ell}} \frac{\alpha(y)}{p} |u|^p \begin{cases} > -\frac{\lambda_{1, p}}{p} \int_{\Omega_{\ell}} |u|^p \, dx dy & \text{if } u \neq 0, \\ = 0 & \text{if } u = 0, \end{cases}$$

so that

$$\liminf_{n \to \infty} \frac{I(u_n)}{\|u_n\|^p} > \begin{cases} \frac{1}{p} - \frac{\lambda_{1,p}}{p} \int_{\Omega_\ell} |u|^p \, dx dy & \text{if } u \neq 0, \\ \frac{1}{p} & \text{if } u = 0. \end{cases}$$

By the fact that $||u|| \le 1$, (1.5) and the Poincaré inequality (1.6), we get

$$\liminf_{n \to \infty} \frac{I(u_n)}{\|u_n\|^p} > \begin{cases} \frac{1}{p} - \frac{1}{p} \int_{\Omega_\ell} |Du|^p \, dx \ge 0 & \text{if } u \neq 0, \\ \frac{1}{p} & \text{if } u = 0, \end{cases}$$

and (2.15) follows.

As a consequence, I is coercive and obviously sequentially weakly lower semicontinuous in K_{ℓ} . Hence, by the Weierstrass Theorem, there exists a minimum of I on K_{ℓ} , which is a solution of problem (P_{ℓ}) .

PROOF OF LEMMA 1.2. As before, we prove the uniqueness result only for (P_{ℓ}) . Assume u_1, u_2 are two solutions of problem (P_{ℓ}) ; then, choosing u_2 as test function in (P_{ℓ}) when u_1 is considered as solution and u_1 as test function when u_2 is considered as solution, and summing up, we immediately find

(2.18)
$$\int_{\Omega_{\ell}} \left(|Du_1|^{p-2} Du_1 - |Du_2|^{p-2} Du_2 \right) \cdot \left(Du_2 - Du_1 \right) dxdy + \int_{\Omega_{\ell}} \left[h(x, u_1) - h(x, u_2) \right] (u_2 - u_1) dxdy \ge 0.$$

Then, from (1.7) and the additional hypothesis on h, we find

$$0 \le -c_p ||u_1 - u_2||^p - \int_{\Omega_\ell} \beta |u_1 - u_2|^p dx dy,$$

and, if $u_1 \neq u_2$, by (1.6), we would find

$$0 < -c_p \lambda_{1,p,\ell} \int_{\Omega_\ell} |u_1 - u_2|^p dx dy + c_p \lambda_{1,p} \int_{\Omega_\ell} |u_1 - u_2|^p dx dy$$

$$\leq -c_p \lambda_{1,p,\ell} \int_{\Omega_\ell} |u_1 - u_2|^p dx dy + c_p \lambda_{1,p,\ell} \int_{\Omega_\ell} |u_1 - u_2|^p dx dy = 0.$$

If h is strictly increasing in the second variable, from (2.18) we obtain, $-\Delta_p$ being monotone for every p > 1,

$$0 = \int_{\Omega_{\ell}} \left[h(x, u_1) - h(x, u_2) \right] (u_2 - u_1) dx dy,$$

from which $u_1 = u_2$ by the strict monotonicity.

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3. Proof of the Theorem

We start as in [10]. Take $0 < \ell_1 < \ell - 1$ and a function $\phi \in C_C^{\infty}(\mathbb{R}^m)$ such that

$$0 \le \phi \le 1$$
, $\phi = 1$ on $\ell_1 \omega_1$, $\phi = 0$ on $\mathbb{R}^m \setminus (\ell_1 + 1)\omega_1$, $|D\phi| \le c$

for some constant c independent of ℓ_1 and ℓ . Then $u_{\ell} - (u_{\infty} - u_{\ell})\phi \in K_{\ell}$, so that from (P_{ℓ}) we get

(3.19)
$$-\int_{\Omega_{\ell}} |Du_{\ell}|^{p-2} Du_{\ell} \cdot D((u_{\ell}-u_{\infty})\phi) dx dy -\int_{\Omega_{\ell}} h(y,u_{\ell})(u_{\ell}-u_{\infty})\phi \, dx dy \ge -\int_{\Omega_{\ell}} f(y)(u_{\ell}-u_{\infty})\phi \, dx dy.$$

In an analogous way, since $u_{\infty} + (u_{\ell}(x, \cdot) - u_{\infty})\phi \in K_{\infty}$ for a.e. $x \in \ell \omega_1$, from (P_{∞}) we find that for a.e. $x \in \ell \omega_1$

$$\begin{split} &\int_{\omega_2} |D_y u_{\infty}|^{p-2} D_y u_{\infty} \cdot D_y ((u_{\ell}(x,y) - u_{\infty})\phi) dy \\ &+ \int_{\omega_2} h(y,u_{\infty}) (u_{\ell}(x,y) - u_{\infty})\phi \, dy \geq \int_{\omega_2} f(y) (u_{\ell}(x,y) - u_{\infty})\phi \, dy. \end{split}$$

Integrating the previous inequality in x, u_{∞} and ϕ being independent of x, we find

(3.20)
$$\int_{\Omega_{\ell}} |Du_{\infty}|^{p-2} Du_{\infty} \cdot D((u_{\ell} - u_{\infty})\phi) dx dy + \int_{\Omega_{\ell}} h(y, u_{\infty})(u_{\ell} - u_{\infty})\phi \, dx dy \ge \int_{\Omega_{\ell}} f(y)(u_{\ell} - u_{\infty})\phi \, dx dy.$$

Summing up both sides of (3.19) and (3.20), we get

(3.21)
$$\int_{\Omega_{\ell}} (|Du_{\ell}|^{p-2} Du_{\ell} - |Du_{\infty}|^{p-2} Du_{\infty}) \cdot D((u_{\ell} - u_{\infty})\phi) dx dy$$
$$\leq \int_{\Omega_{\ell}} [h(y, u_{\infty}) - h(y, u_{\ell})](u_{\ell} - u_{\infty}))\phi \, dx dy.$$

Now, if h is non decreasing in the second variable, the right hand side of (3.21) is non positive. Otherwise, if (h)(iii) holds, we can estimate (3.21) with

(3.22)
$$L\int_{\Omega_{\ell}}\phi|u_{\ell}-u_{\infty}|^{p}dxdy \leq L\int_{\Omega_{\ell_{1}+1}}|u_{\ell}-u_{\infty}|^{p}dxdy.$$

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Hence, recalling that $D\phi = 0$ in the complementary set of $\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}$, (3.21) and (1.8) imply

$$\begin{split} &\int_{\Omega_{\ell}} (|Du_{\ell}|^{p-2}Du_{\ell} - |Du_{\infty}|^{p-2}Du_{\infty}) \cdot D(u_{\ell} - u_{\infty})\phi dxdy \\ &\leq c \int_{\Omega_{\ell_{1}+1} \backslash \Omega_{\ell_{1}}} |u_{\ell} - u_{\infty}| \Big| |Du_{\ell}|^{p-2}Du_{\ell} - |Du_{\infty}|^{p-2}Du_{\infty} \Big| \, dxdy \\ &+ L \int_{\Omega_{\ell_{1}+1}} |u_{\ell} - u_{\infty}|^{p}dxdy \\ &\leq cC_{p} \int_{\Omega_{\ell_{1}+1} \backslash \Omega_{\ell_{1}}} |u_{\ell} - u_{\infty}| (|Du_{\ell}|^{p-2} + |Du_{\infty}|^{p-2}) |D(u_{\ell} - u_{\infty})| \, dxdy \\ &+ L \int_{\Omega_{\ell_{1}+1}} |u_{\ell} - u_{\infty}|^{p}dxdy, \end{split}$$

where we allow the value L = 0 if h is non decreasing.

On the other hand, again by (1.7), we deduce

$$(3.23) \qquad c_p \int_{\Omega_{\ell_1}} |D(u_\ell - u_\infty)|^p dx dy \le c_p \int_{\Omega_{\ell}} \phi |D(u_\ell - u_\infty)|^p dx dy$$
$$(4.23) \qquad \le c C_p \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |u_\ell - u_\infty| (|Du_\ell|^{p-2} + |Du_\infty|^{p-2}) |D(u_\ell - u_\infty)| \, dx dy$$
$$+ L \int_{\Omega_{\ell_1+1}} |u_\ell - u_\infty|^p dx dy.$$

By Young's and Hölder's inequalities, for every $\varepsilon > 0$ we have (3.24)

$$\begin{split} &\leq \varepsilon c C_p \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |u_{\ell} - u_{\infty}|^p dx dy \\ &+ \frac{c C_p}{\varepsilon^{\frac{1}{p-1}}} \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (|Du_{\ell}|^{p-2} + |Du_{\infty}|^{p-2})^{p/(p-1)} |D(u_{\ell} - u_{\infty})|^{p/(p-1)} dx dy \right] \\ &+ \frac{L}{\varepsilon^{\frac{1}{p-1}}} \int_{\Omega_{\ell_1+1}} |u_{\ell} - u_{\infty}|^p dx dy \\ &\leq \varepsilon c C_p \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |u_{\ell} - u_{\infty}|^p dx dy \\ &+ \frac{c C_p}{\varepsilon^{\frac{1}{p-1}}} \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (|Du_{\ell}|^{p-2} + |Du_{\infty}|^{p-2})^{p/(p-2)} dx dy \right]^{\frac{p-2}{p-1}} \\ &\cdot \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dx dy \right]^{\frac{1}{p-1}} + \frac{L}{\varepsilon^{\frac{1}{p-1}}} \int_{\Omega_{\ell_1+1}} |u_{\ell} - u_{\infty}|^p dx dy. \end{split}$$

Now, by the Poincaré inequality (1.3), we have that for a.e. $x \in \omega_1$

$$\int_{\omega_2} |u_\ell - u_\infty|^p dy \le \frac{1}{\mu_1} \int_{\omega_2} |D_y(u_\ell - u_\infty)|^p dy.$$

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Integrating over $(\ell_1 + 1)\omega_1 \setminus \ell_1\omega_1$, we immediately find

(3.25)
$$\int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |u_\ell - u_\infty|^p dxdy \le \frac{1}{\mu_1} \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |D(u_\ell - u_\infty)|^p dxdy,$$

while integrating over $(\ell_1 + 1)\omega_1$ gives

(3.26)
$$\int_{\Omega_{\ell_1+1}} |u_{\ell} - u_{\infty}|^p dx dy \le \frac{1}{\mu_1} \int_{\Omega_{\ell_1+1}} |D(u_{\ell} - u_{\infty})|^p dx dy$$

Hence, by (3.23) and (3.24), using (3.25) and (3.26), we easily obtain (3.27)

$$\begin{split} \int_{\Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dxdy &\leq \frac{\varepsilon cC_p + L\varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon cC_p} \int_{\Omega_{\ell_1+1}} |D(u_{\ell} - u_{\infty})|^p dxdy \\ &+ \frac{cC_p \mu_1 \varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon cC_p} \\ &\cdot \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (|Du_{\ell}|^{p-2} + |Du_{\infty}|^{p-2})^{p/(p-2)} dxdy \right]^{\frac{p-2}{p-1}} \\ &\cdot \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dxdy \right]^{\frac{1}{p-1}}. \end{split}$$

LEMMA 3.1. There exists M > 0 such that

$$\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (|Du_{\ell}|^{p-2} + |Du_{\infty}|^{p-2})^{p/(p-2)} dx dy \le M\ell_1^m$$

and

$$\int_{\Omega_{\ell_1}} (|Du_\ell|^p + |Du_\infty|^p) dx dy \le M\ell_1^m$$

for every $\ell_1 \geq 1$.

PROOF. Let us start from u_{∞} . Taking v = 0 in (P_{∞}) , we find $\int |Du|^p du + \int h(u|u_{\infty}) u du \leq \int f(u) u du \leq ||f||_{\infty} ||u_{\infty}||u|$

$$\int_{\omega_2} |Du_{\infty}|^p dy + \int_{\omega_2} h(y, u_{\infty}) u_{\infty} dy \leq \int_{\omega_2} f(y) u_{\infty} dy \leq ||f||_{L^{p'}(\omega_2)} ||u_{\infty}||_{L^p(\omega_2)}.$$

If $(h)(\text{ii})$ holds, by (1.3) we obtain

$$(3.28) \quad \int_{\omega_2} |Du_{\infty}|^p dy \le \|f\|_{L^{p'}(\omega_2)} \|u_{\infty}\|_{L^p(\omega_2)} \le \frac{\|f\|_{L^{p'}(\omega_2)}}{\mu_1^{1/p}} \left(\int_{\omega_2} |Du_{\infty}|^p dy\right)^{1/p},$$

while, if (h)(iii) is in force, we get

(3.29)
$$\int_{\omega_2} |Du_{\infty}|^p dy \le ||f||_{L^{p'}(\omega_2)} ||u_{\infty}||_{L^p(\omega_2)} + L \int_{\omega_2} |u_{\infty}|^p dy.$$

From (3.28), integrating over $(\ell_1 + 1)\omega_1 \setminus \ell_1\omega_1$, we find

(3.30)
$$\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |Du_{\infty}|^p dx dy \le A \ell_1^{m-1}$$

for some constant A > 0. On the other hand, starting from (3.29), by (1.3), using the fact that $L < \mu_1$, we obtain

(3.31)
$$\int_{\omega_2} |Du_{\infty}|^p dy \le B$$

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$$\int_{\Omega\ell_1+1\setminus\Omega\ell_1} |Du_{\infty}|^p dxdy \le B\ell_1^{m-1}$$

Concerning u_{ℓ} , choosing v = 0 in (P_{ℓ}) , in an analogous way, we find

$$\|u_{\ell}\|_{W_{0}^{1,p}(\Omega_{\ell_{1}})}^{p-1} \leq \frac{|\omega_{1}|\|f\|_{L^{p'}(\omega_{2})}}{\mu_{1}^{p}}\ell_{1}^{m/p'} \leq \frac{|\omega_{1}|\|f\|_{L^{p'}(\omega_{2})}}{\mu_{1}^{p}}\ell_{1}^{m}$$

if (h)(ii) holds (recall that p' > 1), while, under assumption (h)(iii), we find

$$\|u_{\ell}\|_{W_0^{1,p}(\Omega_{\ell_1})}^p \le C\ell_1^m$$

for some constant C > 0.

Proceeding as above and integrating over $\ell_1 \omega_1$, the conclusions easily follow. \Box

Starting from (3.27), using Lemma 3.1, we find

$$\begin{split} &\int_{\Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dx dy \leq \frac{\varepsilon c C_p + L\varepsilon^{-\frac{1}{p-1}}}{\mu_1 c_p + \varepsilon c C_p} \int_{\Omega_{\ell_1+1}} |D(u_{\ell} - u_{\infty})|^p dx dy \\ &+ \frac{c C_p \mu_1 \varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon c C_p} \left[M\ell^m \right]^{\frac{p-2}{p-1}} \left[\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dx dy \right]^{\frac{1}{p-1}}. \end{split}$$

We need the following inequality, whose proof is very easy: if $a \ge b \ge 0$ and $\alpha \in [0, 1]$, then

$$(a-b)^{\alpha} \le 2^{1-\alpha}a^{\alpha} - b^{\alpha}.$$

As a consequence, we get

$$\begin{split} &\int_{\Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dxdy \leq \frac{\varepsilon cC_p + L\varepsilon^{-\frac{1}{p-1}}}{\mu_1 c_p + \varepsilon cC_p} \int_{\Omega_{\ell_1+1}} |D(u_{\ell} - u_{\infty})|^p dxdy \\ &+ \frac{cC_p \mu_1 \varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon cC_p} \left[M\ell^m \right]^{\frac{p-2}{p-1}} 2^{\frac{p-2}{p-1}} \left[\int_{\Omega_{\ell_1+1}} |D(u_{\ell} - u_{\infty})|^p dxdy \right]^{\frac{1}{p-1}} \\ &- \frac{cC_p \mu_1 \varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon cC_p} \left[M\ell^m \right]^{\frac{p-2}{p-1}} \left[\int_{\Omega_{\ell_1}} |D(u_{\ell} - u_{\infty})|^p dxdy \right]^{\frac{1}{p-1}}. \end{split}$$

Setting

$$f(\ell,\tau) = \int_{\Omega_{\tau}} |D(u_{\ell} - u_{\infty})|^p dxdy$$

and

(3.32)
$$k = \frac{\varepsilon cC_p + L\varepsilon^{-\frac{1}{p-1}}}{\mu_1 c_p + \varepsilon cC_p} \text{ and } A = \frac{cC_p \mu_1 \varepsilon^{-1/(p-1)}}{\mu_1 c_p + \varepsilon cC_p} \left[M\ell^m \right]^{\frac{p-2}{p-1}},$$

this means that

(3.33)
$$f(\ell,\ell_1) + Af(\ell,\ell_1)^{\frac{1}{p-1}} \le kf(\ell,\ell_1+1) + 2^{\frac{p-2}{p-1}}Af(\ell,\ell_1+1)^{\frac{1}{p-1}}.$$

First, assume by contradiction that

$$\inf f = \beta > 0.$$

Then we claim that there exists $\lambda \in (k, 1)$ such that

(3.34)
$$f(\ell,\ell_1) + Af(\ell,\ell_1)^{\frac{1}{p-1}} \le \lambda \left(f(\ell,\ell_1+1) + Af(\ell,\ell_1+1)^{\frac{1}{p-1}} \right).$$

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Indeed, we prove that

$$kf(\ell,\ell_1+1) + 2^{\frac{p-2}{p-1}}Af(\ell,\ell_1+1)^{\frac{1}{p-1}} \le \lambda \left(f(\ell,\ell_1+1) + Af(\ell,\ell_1+1)^{\frac{1}{p-1}}\right),$$

or, equivalently,

(3.35)
$$\lambda - k \ge (2^{\frac{p-2}{p-1}} - \lambda) A f(\ell, \ell_1 + 1)^{-\frac{p}{p-1}}.$$

Condition (3.35) is guaranteed, for example, if

$$\lambda - k \ge (2^{\frac{p-2}{p-1}} - \lambda)A\beta^{-\frac{p}{p-1}}$$

that is

(3.36)
$$\lambda \ge \frac{k + 2^{\frac{p-2}{p-1}} A \beta^{-\frac{p}{p-1}}}{1 + A \beta^{-\frac{p}{p-1}}}.$$

We now choose $\varepsilon = \ell^{\gamma}$ with $\gamma > (m-1)(p-2)/(p-1)$, so that, recalling (3.32), (3.36) reads

$$\lambda \ge \frac{cC_p \ell^{\gamma} + L\ell^{-\frac{\gamma}{p-1}} + 2^{\frac{p-2}{p-1}} cC_p \beta^{-\frac{p}{p-1}} \mu_1 M^{\frac{p-2}{p-1}} \ell^{\frac{m(p-2)-\gamma}{p-1}}}{\mu_1 c_p + cC_p \ell^{\gamma} + cC_p \beta^{-\frac{p}{p-1}} \mu_1 M^{\frac{p-2}{p-1}} \ell^{\frac{m(p-2)-\gamma}{p-1}}} := \lambda_0$$

Note that $\lambda_0 < 1$ if and only if we choose $\gamma > m(p-2)$ and ℓ large. In this case

$$\lim_{\ell \to \infty} \lambda_0 = 1^-.$$

Thus, we can take $\lambda = \lambda_0 < 1$, and starting from (3.34), once set $g = f + A f^{\frac{1}{p-1}}$, we find

(3.37)
$$g(\ell, \ell_1) \le \lambda_0 g(\ell, \ell_1 + 1).$$

Choosing $\ell_1 = \ell/2$ and iterating, we easily get

$$g\left(\ell,\frac{\ell}{2}\right) \leq \lambda_0^{\left[\frac{\ell}{2}\right]}g\left(\ell,\frac{\ell}{2}+\left[\frac{\ell}{2}\right]\right).$$

Recalling that $\frac{\ell}{2} - 1 \leq \left[\frac{\ell}{2}\right] \leq \frac{\ell}{2}$, we finally obtain

(3.38)
$$g\left(\ell,\frac{\ell}{2}\right) \le e^{\left(\frac{\ell}{2}-1\right)\ln\lambda_0}g(\ell,\ell) = \frac{1}{\lambda_0}e^{\frac{\ell}{2}\ln\lambda_0}g(\ell,\ell).$$

Since $\lambda_0 \to 1^-$ as $\ell \to \infty$, we take the first order expansion of the right hand side of (3.38), so that by Lemma 3.1, we find

$$\exp\left(-\frac{\mu_1 c_p}{2cC_p}\ell^{1-\gamma}\right)D\ell^m$$

for some constant D > 0. Taking γ also such that $\gamma < 1$ (which is possible, since m(p-2) < 1), we can find $\mathcal{A}, B > 0$ and $\eta \in (0, 1)$ such that

$$g\left(\ell, \frac{\ell}{2}\right) \leq \mathcal{A}e^{-\mathcal{B}\ell^{\eta}} \to 0 \text{ as } \ell \to \infty,$$

against the assumption that $\inf f > 0$, which implies $\inf g > 0$.

Hence $\inf g = \inf f = 0$. Now, if $\inf_{\ell} g(\ell, \ell) = \beta > 0$, we can proceed as we did to obtain (3.37) from (3.36), starting with $\ell_1 = \ell - 1$ and finding

$$g(\ell, \ell-1) \leq \lambda_0 g(\ell, \ell),$$

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which implies

$$g\left(\ell - \left[\frac{\ell}{2}\right], \ell - \left[\frac{\ell}{2}\right] - 1\right) \leq \lambda_0^{\left[\frac{\ell}{2}\right]+1} g(\ell, \ell),$$

and, as before, we can find $\mathcal{A}, B > 0$ and $\eta \in (0, 1)$ such that

$$g\left(\frac{\ell}{2} - \left[\frac{\ell}{2}\right], \frac{\ell}{2} - 1\right) \le \mathcal{A}e^{-\mathcal{B}\ell^{\eta}} \to 0 \text{ as } \ell \to \infty.$$

Setting $\kappa = \frac{\ell}{2} - \left[\frac{\ell}{2}\right]$, we have $\frac{\ell}{2} \le \kappa \le \frac{\ell}{2} + 1$ and thus

$$g(\kappa, \kappa - 2) \le g\left(k, \frac{\kappa}{2} + \frac{1}{2}\left[\frac{\ell}{2}\right] - 1\right) \le \mathcal{A}e^{-\mathcal{B}\kappa^{\eta}}$$

and the theorem is completely proved.

REMARK 3.1. In contrast to [10], we are not able to prove an estimate of the form (1.11). However, we believe that it is coherent with (3.33) when $p \neq 2$. Indeed, for instance, the function ℓ^{-1} satisfies (3.33), but, obviously, has no exponential decay.

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