Contemporary Mathematics
Volume **611**, 2014
http://dx.doi.org/10.1090/conm/611/12161

Finite coverings: A journey through groups, loops, rings and semigroups

Luise-Charlotte Kappe

Abstract. In 1975, Paul Erdös asked the question if there exists a finite bound on the cardinality of sets of pairwise noncommuting elements in a group provided every such set is finite. B.H. Neumann answered Erdös' question in the affirmative by showing that every such group is central-by-finite and that the converse also holds. In an unpublished result R. Baer had shown earlier that a group is the union of finitely many abelian subgroups if and only if it is central-by-finite. All the proofs rely heavily on Neumann's Lemma, stating that in a set of subgroups whose union is the whole group, all subgroups of infinite index can be removed and the union of the remaining subgroups is still the whole group.

The question by Erdös, Neumann's Lemma and finite coverings all make sense in other algebraic structures. The topic of this paper is a survey of analogues of the above results for groups in the case of loops, rings and semigroups. In addition some results for groups concerning finite coverings are mentioned because they appear to make interesting topics for investigation in other algebraic structures.

1. Introduction

A group is said to be covered by a collection of subsets if each element of the group belongs to at least one subset in the collection. Such a collection of subsets is called a covering of the group. The topic of this paper is to give a report on some analogues for loops, rings and semigroups of certain results on finite coverings by subgroups, specifically a question by Paul Erdös, an unpublished theorem by Reinhold Baer, and a lemma by Bernhard Neumann.

In 1975, Paul Erdös posed the following problem:

> Let G be a group in which every set of pairwise noncommuting elements is finite; is there then a finite bound on the cardinality of sets of pairwise noncommuting elements?

Baer's Theorem can be found as Theorem 4.16 in [**21**] and is stated as follows.

Baer's Theorem. *A group is central-by-finite if and only if it has a finite covering consisting of abelian subgroups.*

2010 *Mathematics Subject Classification.* Primary 20-02, 16-02; Secondary 20M99, 20N05.

Key words and phrases. Groups, loops, rings, semigroups, finite union of proper subgroups, subloops, subrings, subsemigroups.

In [**19**], B.H. Neumann answered Erdös' question in the affirmative, characterizing these groups as those with a center of finite index, i.e. tying it with Baer's Theorem. An essential tool in the proof is Neumann's Lemma which can be found in [**18**].

NEUMANN'S LEMMA. *Let* $G = \bigcup_{i=1}^{n} S_i g_i$ *where* $S_1, \ldots, S_n$ *are (not necessarily distinct) subgroups of* G*. Then we can omit from the union any* $S_i g_i$ *for which* $[G : S_i]$ *is finite.*

The question by Erdös makes sense in other algebraic structures, such as loops, rings and semigroups, and we may ask if such structures can be characterized in a similar way as in the case of groups. To that end, we make the following definition.

DEFINITION 1.1. A group is a Paul Erdös group, or a PE-group, if every set of pairwise noncommuting elements is finite.

The following theorem has served as the road map for investigations in rings [**2**], semigroups [**12**] and loops [**7**] in how far analogues of the group result hold in those structures, and in the case such analogues do not hold universally, what are the conditions to be imposed on the structure such that we obtain a result analogous to the characterization in groups.

THEOREM 1.2. *For a group G the following conditions are equivalent:*

(i) G *is a* PE*-group;*
(ii) G *is central-by-finite;*
(iii) G *is the union of finitely many abelian subgroups.*

But first we will take a look at groups to establish additional equivalences in Theorem 1.2 and to find possible extensions to other properties, a topic suggested by B.H. Neumann in [**17**]. In particular, we report on extensions to coverings by subgroups of bounded nilpotency class [**4**] and coverings by 2-Engel subgroups [**11**].

In Section 3 we consider rings in this context. The characterizations for rings in [**2**] are almost verbatim those for groups, just group replaced by ring. This is due to the fact that Neumann's Lemma holds in rings for the additive group.

In semigroups, investigated in [**12**], the answer to Erdös' question is not in the affirmative, as established by an example. However, under suitable conditions for the semigroup, such as being embeddable into a group or being isomorphic to the multiplicative semigroup of a ring, we obtain an analogue to Theorem 1.2.

Lastly, in loops the situation is not suitable for a general characterization as established by an example in [**7**]. The main reason is that cosets of a subloop do not necessarily form a partition of the loop. But imposing strong conditions on the coset decomposition of the subloops involved leads to an analogue of Neumann's Lemma for loops. This allows us to establish an analogue to Theorem 1.2 under very restrictive conditions.

In the last section we report on some further results in other structures such as a characterization of rings which are the union of three proper subrings [**15**], an analogue of Scorza's characterization of groups which are the union of three proper subgroups [**22**]. We conclude our journey with some group results worthy of investigation in loops, rings and semigroups.

For easier reference, we formulate here the axioms of a group from which the axioms of a loop, a semigroup or a quasigroup can be easily obtained by deleting one or the other axiom.

DEFINITION 1.3. A group is a nonempty set with a binary operation $G \times G \to G$, satisfying the following conditions:

$$(G) = \begin{cases} (1) & \text{associative;} \\ (2) & \text{identity } 1 \text{ with } \ 1 \cdot a = a \cdot 1 = a \ \text{ for all } \ a \in G; \\ (3) & \text{for } a, b \in G \text{ there exist unique } x, y \in G \text{ with } xa = b \text{ and } ay = b. \end{cases}$$

A loop then satisfies the axioms $(G) - (1)$, a semigroup $(G) - (2) - (3)$, and a quasigroup $(G) - (1) - (2)$.

2. Groups

Before exploring what the answer to Erdös' question is in other structures, we want to take a second look at groups to establish other equivalences in Theorem 1.2 and consider finite coverings by subgroups with other properties, such as 2-Engel and nilpotency of class c.

In [**4**], two more equivalencies are added to Theorem 1.2, B. Neumann's result in [**19**].

THEOREM 2.1. *For a group G, the following conditions are equivalent:*

(i) G is a PE-group;
(ii) G is central-by-finite;
(iii) there exist subgroups H_i, $i = 1, \ldots, k$, with $H_i' = 1$ such that $G = \bigcup_{i=1}^{k} H_i$;
(iv) G has only finitely many maximal abelian subgroups;
(v) every maximal abelian subgroup has finite index in G.

In [**17**], Bernhard Neumann suggested the following problem:

Given a group G covered by finitely many subgroups $H_1, \ldots, H_n$ with intersection D. If $H_1, \ldots, H_n$ possess a certain property $\mathfrak{E}$, what can be said about D in relation to G, or about G itself?

In the same paper, B. Neumann raises the question whether Baer's characterization of central-by-finite groups can be extended to finite coverings by k-Engel groups. An answer to this question is given in [**11**] for the case of 2-Engel groups.

We define $\epsilon_k(x, y) = [x, {}_k y] = [[x, {}_{k-1} y], y]$ as the k-Engel word, where $\epsilon_1(x, y) = [x, {}_1 y] = [x, y]$ is the commutator of x and y. An element a in a group G is a right k-Engel element if $[a, {}_k x] = 1$ for all x in G and a group is a k-Engel group if $[x, {}_k y] = 1$ for all $x, y \in G$. Let G be a group and let

$$L(G) = \{a \in G; [a, {}_2 x] = 1 \ \forall \, x \in G\}$$

be the set of right 2-Engel elements. In [**13**], W.P. Kappe has shown that $L(G)$ is a characteristic subgroup of G. For the property 2-Engel we have now a direct analogue of Baer's Theorem.

THEOREM 2.2. ([**11**]) *A group is the union of finitely many 2-Engel subgroups if and only if $G/L(G)$ is finite.*

A similar result cannot be expected for coverings by k-Engel subgroups, $k > 2$, since the right k-Engel elements in a group do not necessarily form a subgroup. The question arises, what other properties of subgroups lead to similar characterizations as in the abelian case, in case there exists a finite covering of the group by subgroups with this property. Obviously, if $Z_n(G)$, the n-th center of the group G, has finite index in G, then G has a finite covering by subgroups of nilpotency class n. However the converse is not true. In [**11**], a group G was constructed which has a finite covering by subgroups of nilpotency class 2 but $Z_2(G)$ has not finite index in G. Contrary to abelian and k-Engel, nilpotency of class n cannot be described by a property of pairs of elements. However, in [**4**] a special embedding property of commutators was introduced leading to a characterization of groups with $Z_n(G)$ having finite index of G in terms of coverings by certain subgroups of nilpotency class n.

Consider the following set of subgroups of a group G and n a positive integer:

$$\mathcal{H}(G,n) = \{H \leq G; H' \leq Z_{n-1}(G)\}.$$

Obviously, $H \in \mathcal{H}(G,n)$ implies $H \in \mathfrak{N}_n$, the groups of nilpotency class not exceeding n. One can easily see that $\mathcal{H}(G,n)$ has maximal elements. We denote this subset of $\mathcal{H}(G,n)$ by $\mathcal{M}(G,n)$. The following theorem characterizes groups with $G/Z_n(G)$ finite.

THEOREM 2.3. ([**4**], Theorem 2.2) *For a group G and a positive integer n the following are equivalent:*

(i) Any subset S of G, where $[x,y] \notin Z_{n-1}(G)$ for distinct $x, y \in S$, is finite;
(ii) $G/Z_n(G)$ is finite;
(iii) $G = \bigcup_{i=1}^{k} H_i$, $H_i \in \mathcal{H}(G,n)$;
(iv) $\mathcal{M}(G,n)$ is finite;
(v) the elements of $\mathcal{M}(G,n)$ have finite index in G.

Since $Z_0(G) = 1$, it can be seen that the elements of $\mathcal{M}(G,1)$ coincide with the maximal abelian subgroups of the group. (Note that this is no longer the case for $n > 1$.) It follows that Theorem 2.1 is a special case of Theorem 2.3 when $n = 1$.

3. Rings

When it comes to analogues in other algebraic structures to the answer of Erdös' question for groups, it turns out that the answer for rings is pretty much the same as for groups as one can see from [**2**]. One of the reasons is that Neumann's Lemma holds for the additive group of a ring, due to a basic ring-theoretic result by Lewin, which appeared as recently as 1967.

LEMMA 3.1. ([**14**], Lemma 1) *Let R be a ring. If S is any subring of finite index, then S contains a two-sided ideal I of R which is also of finite index.*

Defining a PE-ring as in the case of groups, i.e. every subset of pairwise non-commutating elements is finite, we obtain the following characterization of PE-rings.

THEOREM 3.2. ([**2**], Theorem 2.8) *For a ring R, the following are equivalent:*

(i) R is a PE-ring;

(ii) *$Z(R)$ has finite index in R;*
(iii) *R is the union of finitely many commutative subrings;*
(iv) *R has only finitely many maximal commutative subrings;*
(v) *every maximal commutative subring has finite index in R;*
(vi) *there exists a central ideal I such that $[R : I] < \infty$.*

Comparing Theorem 3.2 with Theorem 2.1, the corresponding result on groups, we observe that the equivalence conditions (i) through (v) are literally the same, just group is replaced by ring. Condition (vi) of Theorem 3.2 is just in the nature of rings, since the center of a ring is not necessarily an ideal.

4. Semigroups

In [**12**], semigroups were investigated in context with Erdös' question. As in the case of groups, we say a semigroup is a PE-semigroup if every set of pairwise noncommuting elements is finite. We start our discussion with an example.

EXAMPLE 4.1. ([**12**], Example 1.2) Let S_n be the symmetric group on n letters, $n \geq 1$. Consider the disjoint union $S = \bigcup_{n=1}^{\infty} S_n \cup \{0\}$ with a product defined as follows: For $a \in S_n$, $b \in S_m$, let $a \cdot b$ be as in S_n if $n = m$, and $a \cdot b = 0$ if $n \neq m$; also $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. Then S is a PE-semigroup but the size of sets of pairwise noncommuting elements is not bounded.

As this example shows, the answer to Erdös' question in the case of semigroups is not in the affirmative. The next question which arises is under what conditions is the size of sets of pairwise noncommuting elements in a PE-semigroup bounded. As we will see, a connection of the PE-semigroup to a ring will assure an affirmative answer.

Using results from [**2**], we obtain the following theorem.

THEOREM 4.2. ([**12**], Theorem 1.3) *For every PE-semigroup which can occur as the multiplicative semigroup of a ring there exists a bound on the size of sets of pairwise noncommuting elements.*

In case a semigroup is embeddable into a group, we get an analogue of Theorem 1.2 and as a consequence, the size of sets of pairwise noncommuting elements in a PE-semigroup which can be embedded into a group is bounded.

THEOREM 4.3. ([**12**], Theorem 3.7) *Let S be a semigroup embeddable into a group. Then the following conditions are equivalent:*

(i) *S is a PE-semigroup;*
(ii) *$\langle S \rangle$, the subgroup generated by S, is central-by-finite;*
(iii) *S is the set-theoretic union of finitely many commutative subsemigroups.*

Not all semigroups can be embedded into groups. Malcev in [**16**] gave examples of cancellative semigroups which cannot be embedded into groups. However in [**12**] it was shown that a cancellative PE-semigroup can be embedded into a group. Thus we have the following corollary to Theorem 4.3.

COROLLARY 4.4. ([**12**], Corollary 4.4) *The size of sets of pairwise noncommuting elements in a cancellative PE-semigroup is bounded.*

5. Loops

In this section, we consider finite coverings of loops by subloops or subgroups, the focus of the investigations of [**7**]. Loops behave quite differently than groups or rings, or even semigroups. For instance

$$C(L) = \{a \in L : ax = xa \;\forall\, x \in L\},$$

the literal analogue of the center of a group and called the *centrum* of the loop, is not necessarily a subloop. Such a situation makes it difficult to find e.g. an analogue of central-by-finite for loops. We start with an example which is further evidence of the different situation we encounter in loops.

Example 5.1. ([**7**], Example 4.1) Consider a field $\mathbb{F}$ with multiplicative group $\mathbb{F}^*$ and the idempotent quasigroup with binary operation $\odot$ given in the table below:

$\odot$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Let $\mathcal{L}^{(3)}(\mathbb{F}) = \{a_i(x) : x \in \mathbb{F}^* \text{ and } i = 1,2,3\} \cup \{\mathbf{1}\}$ (i.e. each element of the form $a_i(x)$ in this set is double indexed by i and x). We define a binary operation on $\mathcal{L}^{(3)}(\mathbb{F})$ as follows:

(i) for any $l \in \mathcal{L}^{(3)}(\mathbb{F})$, $\mathbf{1} \cdot l = l \cdot \mathbf{1} = l$;

(ii) for $x, y \in \mathbb{F}^*$, $a_i(x)a_i(y) = \begin{cases} a_i(x+y) & \text{if } x+y \neq 0, \\ \mathbf{1} & \text{otherwise;} \end{cases}$

(iii) for $x, y \in \mathbb{F}^*$, $a_i(x)a_j(y) = \begin{cases} a_{i\odot j}(xy) & \text{for } i<j, \\ a_{i\odot j}(-xy) & \text{for } i>j. \end{cases}$

Then $\mathcal{L}^{(3)}(\mathbb{F})$ is a loop.

For convenience in light of (ii), we will also denote $\mathbf{1}$ by $a_i(0)$, where $i \in \{1,2,3\}$, and thus if $x + (-x) = 0$ we get $a_i(x)a_i(-x) = a_i(0) = \mathbf{1}$.

If char $\mathbb{F} = 0$, then $\mathcal{L}^{(3)}(\mathbb{F})$, $\mathcal{L}$ for short, has some pretty interesting properties in as far as coverings by subloops are concerned. We observe that $A_i = \{a_i(x); x \in \mathbb{F}\}$, $i = 1,2,3$, are abelian subgroups of $\mathcal{L}$ and $\mathcal{L} = A_1 \cup A_2 \cup A_3$ and $A_i \cap A_j = \{\mathbf{1}\}$, $i \neq j$. Also, $\mathcal{L} = A_i\mathbf{1} \cup A_i a_j(1) \cup A_i a_k(1)$, where i, j and k are distinct, i.e. it is the union of three cosets of A_i. On the other hand, there exist infinite subsets Y of $\mathcal{L}$ with $\mathcal{L} = \bigcup_{y\in Y} yA_i$, but no proper subset of Y gives a covering of $\mathcal{L}$. Furthermore, $\mathcal{L}$ has a trivial centrum and no normal subloop of finite index. For the details we refer to Propositions 4.2 and 4.4 in [**7**].

The evidence given by Example 5.1 suggests that there appears to be no connection between the existence of the finite covering by commutative subloops and the properties of the center of the loop. One of the obstacles in establishing such a connection is the absence of an analogue of Neumann's Lemma for loops. Before we go into the discussion of what conditions have to be imposed on a loop to assure that an analogue of Neumann's Lemma holds, we want to mention that the loop $\mathcal{L}$ of Example 5.1 is a PE-loop, in fact any set of pairwise noncommuting elements contains at most three elements, one from each A_i. Not much is known about the answer to Erdös' question in loops and it might be a topic for further investigations.

One of the obstacles to establishing an analogue of Neumann's Lemma in loops is the fact that left (right) cosets modulo a subloop do not necessarily form a partition of the loop. To that end, the following definition is made:

DEFINITION 5.2. ([**20**], Definition I.2.10) A loop L has a left (right) coset decomposition modulo a subloop H if the left (right) cosets form a partition. If L has left and right coset decomposition modulo H, then we say that L has a coset decomposition modulo H.

According to Theorem I.2.12 in [**20**], subloops having a coset decomposition in a loop can be characterized as follows.

PROPOSITION 5.3. *A loop L has a left coset decomposition modulo a subloop H if and only if for any $x \in L$ and $h \in H$, $(xh)H = xH$.*

In [**23**], an alternate proof of Neumann's Lemma for groups is given. Almost to the end it looks like this proof can be adapted to the case of loops, just requiring that the loop has left (right) coset decomposition for all subloops involved. However at the very end, we need to conclude that $x(yH) = (xy)H$ for all $x, y \in L$ and L having a coset decomposition modulo H. This leads to the following definition.

DEFINITION 5.4. A loop L has a strong left (right) coset decomposition modulo H, where H is a subloop of L, if $x(yH) = (xy)H$ for all $x, y \in L$. If L has a strong left and right coset decomposition modulo H, then we say that L has a strong coset decomposition modulo H.

Denoting with $[L : H]_l$ the index of a left coset decomposition of L modulo H, we obtain now the following loop analogue of Neumann's Lemma.

THEOREM 5.5. ([**7**], Theorem 6.4) *Let L be a loop with $L = \bigcup_{i=1}^{n} g_i H_i$, where $H_1, \dots, H_n$ are (not necessarily distinct) subloops of L, and L having strong left coset decompositions modulo H_i, $i = 1, \dots, n$. Then all cosets in the union for which the corresponding index $[L : H_i]_l$ is infinite can be omitted from the union and the remaining cosets still cover the loop.*

In [**7**], there are several interesting corollaries of Theorem 5.5 addressing various types of finite coverings of a loop. In context with our theme of finite coverings by commutative substructures, we want to mention here one of these. Note that we say that a loop L has an n-covering, if there exist subloops H_i, $i \in \Omega$, an index set, such that for every $\{x_1, \dots, x_n\} \subseteq L$ there exists an $i \in \Omega$ with $\{x_1, \dots, x_n\} \subseteq H_i$. As mentioned earlier, $C(L)$, the centrum of a loop, is not necessarily a subloop. However, $Z(L) = C(L) \cap Nuc(L)$, the center of a loop, is always a normal subloop. Here $Nuc(L)$ is the nucleus of a loop defined as

$$Nuc(L) = Nuc_l(L) \cap Nuc_m(L) \cap Nuc_r(L),$$

where

$$Nuc_l(L) = \{x \in L : x(yz) = (xy)z \ \forall\, y, z \in L\},$$
$$Nuc_m(L) = \{y \in L : x(yz) = (xy)z \ \forall\, x, z \in L\},$$
$$Nuc_r(L) = \{z \in L : x(yz) = (xy)z \ \forall\, x, y \in L\},$$

the left, middle, and right nucleus of a loop, respectively (see e.g. [**7**]).

With this definition we obtain now a partial analogue of Baer's Theorem for groups.

Corollary 5.6. ([**7**], Corollary 6.7) *Given a loop L with a finite 2-covering by abelian subgroups H_i, $i = 1, \ldots, n$, such that L has a strong left coset decomposition modulo H_i for all i, then $Z(L)$ has finite index in L.*

In view of Example 5.1, we cannot expect that the converse of this statement holds true too.

6. Further results and some open problems

In this section we want to address some further results in other structures which are analogues to results for groups concerning finite coverings. However they are not directly related to the main theme of this paper, namely, Neumann's Lemma, finite coverings by abelian subgroups and Erdös' question. In addition we will list some open questions in context with these results and those of the preceding sections. We do not claim that our list is in any way complete.

No group is the union of two proper subgroups, in fact it can be shown that no quasigroup is the union of two proper subquasigroups (see e.g. [**7**]). Thus the analogous result will hold for loops. Since the additive group of a ring can't have a covering by two proper subgroups, a ring as well can't be the union of two proper subrings. However, a semigroup can be the union of two proper subsemigroups, e.g. the semigroup of nonzero integers under multiplication is the union of the subsemigroups of odd and even integers.

What about groups which are the union of three proper subgroups? In his seminal paper [**22**], Scorza characterized the groups which are the union of three proper subgroups as those groups having a homomorphic image isomorphic to the Klein-Four group. In [**3**], a characterization of groups which are the union of finitely many proper normal subgroups was given as those groups with a homomorphic image isomorphic to an elementary abelian p-group of rank 2 for some prime p. With this result, Scorza's characterization can be expanded as follows.

Theorem 6.1. *For a group G the following conditions are equivalent:*
(i) G has a homomorphic image isomorphic to the Klein-Four group;
(ii) G is the set theoretic union of three proper normal subgroups;
(iii) G is the set theoretic union of three proper subgroups.

In [**15**], Lucchini and Maróti explore the ring analogue of Scorza's theorem. It is easy to see that there exist rings which are the union of three proper subrings, e.g. consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_p$ denotes the field of integers modulo a prime p. However the solution is less simple than in the group case. As shown in [**15**], a ring is the union of three proper subrings if and only if it has a homomorphic image isomorphic to one of five rings. For details we refer to Theorem 1.2 in [**15**]. However for coverings by three proper ideals, we have the following analogue of the group result.

Theorem 6.2. *A ring R is the union of three proper ideals if and only if there exists an ideal I in R such that R/I is either the zero ring on $\mathbb{Z}_2 \times \mathbb{Z}_2$ or R/I is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the direct sum of two Galois fields of order 2.*

This theorem is an easy corollary to Theorem 1.2 in [**15**]. The question arises, if rings which are the union of finitely many proper ideals can be characterized in a similar way as groups which are the union of finitely many proper normal subgroups.

In [**10**], Greco characterizes all groups G which are the union of four proper subgroups. In view of the results in [**15**] for the case of the union of three proper subrings, this does not seem to be a feasible problem for rings. However the covering number $\sigma(G)$, as introduced by Cohn in [**5**], seems to be an object worthy of investigations in other structures. Let G be a group which has a covering by n proper subgroups. We say the covering is minimal if no covering of G has fewer than n subgroups. The size of a minimal cover is denoted by $\sigma(G)$ and called the covering number of the group. Cohn conjectured that the covering number of a solvable group is congruent to one modulo a prime power. This conjecture was later confirmed by Tomkinson in [**24**]. There it was also shown that there exists no group G with $\sigma(G) = 7$ and Tomkinson conjectured that there exist no groups with covering number 11 or 13. In [**6**], Tomkinson's conjecture was confirmed for $n = 11$. However, in [**1**] it was shown that $\sigma(S_6) = 13$. For further results we refer to [**9**], where Garonzi determines all $n < 27$, such that n is not a covering number. It should be mentioned here that it is not known, if the set of such integers is finite or infinite.

All questions about covering numbers should be of interest in other structures too, in particular in rings, where the question about covering numbers reduces to finite rings as in the case of groups. An interesting question would be if there are integers > 2 that are not the covering number of a ring.

For loops one can show that there exist loops with covering number n for every $n > 2$. This can be achieved by using an idempotent quasigroup of order n for the construction of the loop instead of the one of order 3 in Example 5.1. Nevertheless it is of interest to find the covering numbers of certain families of loops as was done by Gagola in [**8**] for the smallest Paige loop.

In the case of semigroups, there seem to be so far no investigations concerning covering numbers. At least one could make a start by finding the covering number for some finite semigroups, perhaps those embeddable into groups.

The list of open problems on coverings of loops, rings and semigroups given here is just a beginning. The hope is that it will lead into research just as rich and interesting as the one in group theory.

References

[1] A. Abdollahi, F. Ashraf, and S. M. Shaker, *The symmetric group of degree six can be covered by 13 and no fewer proper subgroups*, Bull. Malays. Math. Sci. Soc. (2) **30** (2007), no. 1, 57–58. MR2330635 (2008b:20006)

[2] H. E. Bell, A. A. Klein, and Luise-Charlotte Kappe, *An analogue for rings of a group problem of P. Erdős and B. H. Neumann*, Acta Math. Hungar. **77** (1997), no. 1-2, 57–67, DOI 10.1023/A:1006531605931. MR1485586 (98k:16047)

[3] M. A. Brodie, R. F. Chamberlain, and L.-C. Kappe, *Finite coverings by normal subgroups*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 669–674, DOI 10.2307/2046770. MR964841 (89i:20062)

[4] Marc A. Brodie and Luise-Charlotte Kappe, *Finite coverings by subgroups with a given property*, Glasgow Math. J. **35** (1993), no. 2, 179–188, DOI 10.1017/S0017089500009733. MR1220560 (94f:20077)

[5] J. H. E. Cohn, *On n-sum groups*, Math. Scand. **75** (1994), no. 1, 44–58. MR1308936 (95k:20026)

[6] Eloisa Detomi and Andrea Lucchini, *On the structure of primitive n-sum groups*, Cubo **10** (2008), no. 3, 195–210 (English, with English and Spanish summaries). MR2467921 (2009k:20051)

[7] Tuval Foguel and Luise-Charlotte Kappe, *On loops covered by subloops*, Expo. Math. **23** (2005), no. 3, 255–270, DOI 10.1016/j.exmath.2005.01.009. MR2164777 (2006d:20121)

[8] Stephen M. Gagola III, *Coverings of the smallest Paige loop*, J. Algebraic Combin. **34** (2011), no. 4, 607–615, DOI 10.1007/s10801-011-0284-6. MR2842912 (2012h:20151)

[9] Martino Garonzi, *Finite groups that are the union of at most 25 proper subgroups*, J. Algebra Appl. **12** (2013), no. 4, 1350002, 11, DOI 10.1142/S0219498810003872. MR3037278

[10] Donato Greco, *I gruppi finiti che sono somma di quattro sottogruppi*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 74–85 (1952) (Italian). MR0051223 (14,445b)

[11] Luise-Charlotte Kappe, *Finite coverings by* 2*-Engel groups*, Bull. Austral. Math. Soc. **38** (1988), no. 1, 141–150, DOI 10.1017/S0004972700027350. MR968238 (89k:20052)

[12] Luise-Charlotte Kappe, John C. Lennox, and James Wiegold, *An analogue for semigroups of a group problem of P. Erdős and B. H. Neumann*, Bull. Austral. Math. Soc. **63** (2001), no. 1, 59–66, DOI 10.1017/S0004972700019109. MR1812308 (2001j:20094)

[13] Wolfgang Kappe, *Die A-Norm einer Gruppe*, Illinois J. Math. **5** (1961), 187–197 (German). MR0121399 (22 #12137)

[14] Jacques Lewin, *Subrings of finite index in finitely generated rings*, J. Algebra **5** (1967), 84–88. MR0200297 (34 #196)

[15] Andrea Lucchini and Attila Maróti, *Rings as the unions of proper subrings*, Algebr. Represent. Theory **15** (2012), no. 6, 1035–1047, DOI 10.1007/s10468-011-9277-3. MR2994015

[16] A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann. **113** (1937), no. 1, 686–691, DOI 10.1007/BF01571659. MR1513116

[17] B. H. Neumann, *Groups covered by finitely many cosets*, Publ. Math. Debrecen **3** (1954), 227–242 (1955). MR0072138 (17,234f)

[18] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248. MR0062122 (15,931b)

[19] B. H. Neumann, *A problem of Paul Erdős on groups*, J. Austral. Math. Soc. Ser. A **21** (1976), no. 4, 467–472. MR0419283 (54 #7306)

[20] Hala O. Pflugfelder, *Quasigroups and loops: introduction*, Sigma Series in Pure Mathematics, vol. 7, Heldermann Verlag, Berlin, 1990. MR1125767 (93g:20132)

[21] D.J.S. Robinson, *Finiteness conditions and generalized solvable groups, Part I*, Springer Verlag 1972.

[22] G. Scorza, *I gruppi che possone pensarzi come somma di tre lori sottogruppi*, Boll. Un. Mat. Ital., **5** (1926), 216-218.

[23] Jack Sonn, *Groups that are the union of finitely many proper subgroups*, Amer. Math. Monthly **83** (1976), no. 4, 263–265. MR0399279 (53 #3130)

[24] M. J. Tomkinson, *Groups as the union of proper subgroups*, Math. Scand. **81** (1997), no. 2, 191–198. MR1613772 (99g:20042)

Department of Mathematical Sciences, Binghamton University, Binghamton, New York 13902-6000

E-mail address: menger@math.binghamton.edu