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On the regularity of a graph related to conjugacy classes of groups: Small valencies

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Dedicated to Prof. Daniela Nikolova on the occasion of her 60*th birthday*

Abstract. Given a finite group G, denote by $\Gamma(G)$ the simple undirected graph whose vertices are the distinct sizes of noncentral conjugacy classes of G, and set two vertices of $\Gamma(G)$ to be adjacent if and only if they are not coprime numbers. In this note we prove that, if $\Gamma(G)$ is a k-regular graph and either G is an $\mathbf{F}$-group or $k \leq 5$, then $\Gamma(G)$ is a complete graph with $k+1$ vertices.

Introduction

Given a finite group G, let $\Gamma(G)$ be the simple undirected graph whose vertices are the distinct sizes of *noncentral* conjugacy classes of G, two of them being adjacent if and only if they are not coprime numbers. The interplay between certain properties of this graph and the group structure of G has been widely studied in the past decades, and it is nowadays a classical topic in finite group theory (see, for instance, [**4**]). The present note is a contribution in this direction.

As stated in [**2**] we conjecture that, for every integer $k \geq 1$, the graph $\Gamma(G)$ is k-regular if and only if it is a complete graph with $k+1$ vertices. That paper settles the case $k \leq 3$, whereas here we develop a different approach which covers the case $k \leq 5$; a key step in our proof is Theorem 1.5, which provides an affirmative answer to the conjecture for the class of $\mathbf{F}$-*groups* (see Section 1).

Thus many regular graphs are excluded from the class of graphs occurring as $\Gamma(G)$ for some finite group G, but of course the problem in its full generality remains open.

Every group considered in the following discussion is tacitly assumed to be a finite group.

1. F-groups

A group G is called an $\mathbf{F}$-*group* if for every $x, y \in G \setminus \mathbf{Z}(G)$, the condition $\mathbf{C}_G(x) \leq \mathbf{C}_G(y)$ implies $\mathbf{C}_G(x) = \mathbf{C}_G(y)$. Nonabelian $\mathbf{F}$-groups were classified in [**6**, Theorem A], that we state next (see [**9**] for the original classification).

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THEOREM 1.1. *Let G be a nonabelian group and write $Z = \mathbf{Z}(G)$. Then G is a an $\mathbf{F}$-group if and only if it is of one of the following types.*

(a) *G has an abelian normal subgroup of prime index.*
(b) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, where K and L are abelian.*
(c) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, such that $K = PZ$, where P is a normal Sylow p-subgroup of G for some $p \in \pi(G)$, P is an $\mathbf{F}$-group, $\mathbf{Z}(P) = P \cap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*
(d) *$G/Z \simeq S_4$ and if V/Z is the Klein four-group in G/Z, then V is nonabelian.*
(e) *$G = P \times A$, where P is a nonabelian $\mathbf{F}$-group of prime-power order and A is abelian.*
(f) *$G/Z \simeq \mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$ and $G' \simeq \mathrm{SL}(2, p^n)$, where p is a prime and $p^n > 3$.*
(g) *$G/Z \simeq \mathrm{PSL}(2, 9)$ or $\mathrm{PGL}(2, 9)$ and G' is isomorphic to the Schur cover of $\mathrm{PSL}(2, 9)$.*

We call a graph *a star* if at least one of its vertices is adjacent to all the remaining vertices. A complete graph is certainly a star.

Our first result is the following theorem.

THEOREM 1.2. *Let G be a nonabelian $\mathbf{F}$-group. Then the graph $\Gamma(G)$ is either disconnected or a star.*

For the proof of Theorem 1.2 we need the following lemma.

LEMMA 1.3. *Let G be a nonabelian group and suppose that $\mathbf{Z}(G/\mathbf{Z}(G)) = 1$. If $\Gamma(G/\mathbf{Z}(G))$ is complete (resp., a star), then also $\Gamma(G)$ is complete (resp., a star).*

PROOF. Write $Z = \mathbf{Z}(G)$. Since $\mathbf{Z}(G/Z) = 1$, if $x \in G \setminus Z$ then the class $(xZ)^{G/Z}$ is noncentral in G/Z and we have

$$\left|(xZ)^{G/Z}\right| \,\Big|\, |x^G| \,.$$

The claim easily follows. □

We are now ready for the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let G be a nonabelian $\mathbf{F}$-group and write $Z = \mathbf{Z}(G)$. Then G satisfies one of the conditions (a)-(g) of Theorem 1.1. We shall deal now with each case one by one.

(a) There exists $A \trianglelefteq G$ such that $|G : A| = p$ for some prime p and $Z < A$. If $x \in G \setminus Z$, then either $x \in A$, $\mathbf{C}_G(x) = A$ and $|x^G| = p$, or $x \in G \setminus A$, $\mathbf{C}_G(x) = \langle x \rangle Z$ and $|x^G| = |A : Z|$. Hence the graph $\Gamma(G)$ is either complete or disconnected.
(b) By [**1**] the graph $\Gamma(G)$ is disconnected.
(c) If $x \in G \setminus Z$, then either $x \in K \setminus Z$, $\mathbf{C}_G(x) = \mathbf{C}_P(x)Z$ and $|x^G|$ is divisible by $|P : \mathbf{C}_P(x)| > 1$, or x is conjugate to an element of the abelian group L, whence $|\mathbf{C}_G(x)| = |L|$ and $|x^G| = |P|$. Hence the graph $\Gamma(G)$ is complete.
(d) The vertices of the graph $\Gamma(S_4)$ are $\{3, 6, 8\}$. Hence $\Gamma(S_4)$ is a star and it follows by Lemma 1.3 that also the graph $\Gamma(G)$ is a star.
(e) For each $x \in G \setminus Z$, $|x^G|$ is a power of p and it follows that $\Gamma(G)$ is complete.

(f),(g) If $G/Z \simeq PSL(2, p^n)$ for $p^n > 3$ or

$$G/Z \simeq PGL(2, 2^n) \simeq PSL(2, 2^n) \quad \text{for } 2^n > 3\ ,$$

then G/Z is a nonabelian simple group, and by [**7**] the graph $\Gamma(G/Z)$ is complete. Thus, in view of Lemma 1.3, also the graph $\Gamma(G)$ is complete. If $G/Z \simeq PGL(2, p^n)$ for $p^n > 3$ and p odd, then the vertices of the graph $\Gamma(G/Z)$ are $\{q^2 - 1, q(q+1), q(q+1)/2, q(q-1), q(q-1)/2\}$, where $q = p^n$. Hence the graph $\Gamma(G/Z)$ is complete and it follows by Lemma 1.3 that also the graph $\Gamma(G)$ is complete.

□

After the next proposition we will be ready to show that, given an **F**-group G, the graph $\Gamma(G)$ is never regular unless it is complete.

Proposition 1.4. *Let Γ be a k-regular graph. If $k \geq 1$ and there exists a group G such that $\Gamma = \Gamma(G)$, then Γ is a connected graph.*

Proof. If Γ is not connected then, by [**1**] and [**8**], it consists of two isolated vertices and hence it is not k-regular for $k \geq 1$. □

Theorem 1.5. *Let G be a nonabelian* **F***-group. Then, for $k \geq 1$, the graph $\Gamma(G)$ is k-regular if and only if it is a complete graph with $k+1$ vertices.*

Proof. We need only to prove the "only if" part. So suppose that G is an **F**-group and $\Gamma(G)$ is k-regular. Then, by the previous proposition, the graph $\Gamma(G)$ is connected and by Theorem 1.2 it is a star. But the k-regularity of $\Gamma(G)$ implies that $\Gamma(G)$ is complete, as claimed. □

2. Graphs of diameter 3

Recall that, for every group G, the diameter of the graph $\Gamma(G)$ is at most 3 (see [**5**]). The following proposition shows that no graph of this kind can be regular of diameter 3.

Proposition 2.1. *Let Γ be a regular graph. If there exists a nonabelian group G such that $\Gamma = \Gamma(G)$, then the diameter of Γ is at most* 2.

Proof. The groups G such that the diameter of $\Gamma(G)$ is 3 are classified in [**8**]: they are direct products $F \times H$ where $(|F|, |H|) = 1$, the graph $\Gamma(F)$ consists of two isolated vertices, and $\Gamma(H)$ is not the empty graph. Now, let X be a vertex of $\Gamma(F)$, Y a vertex of $\Gamma(H)$, and consider the vertices Y, XY of $\Gamma(G)$; since $Y \mid XY$, every neighbor of Y (except XY) is a neighbor of XY as well. But XY is adjacent to X, whereas Y is not, therefore the valency of Y is strictly smaller than that of XY. We conclude that if $\Gamma(G)$ has diameter 3, then it is not regular. □

3. Prime graphs with no complete vertices

In this section, we focus on another graph which is usually attached to the set of class sizes of a group G, that is, the *prime graph* $\Delta(G)$. In this case, the vertices are the primes dividing some class size of G, and two vertices p, q are adjacent if there exists a class size of G that is divisible by pq. Our key ingredient for this section is the following result, which is Theorem C of [**3**].

THEOREM 3.1. *Let G be a group. Assume that no vertex of $\Delta(G)$ is adjacent to all the other vertices. Then (up to an abelian direct factor) G is a semidirect product of $K \trianglelefteq G$ and $H \leq G$, where K and H are abelian of coprime orders.*

The next lemma will also turn out to be useful.

LEMMA 3.2. *Let G be a nonabelian group. Assume that no vertex of $\Delta(G)$ is adjacent to all the other vertices, and that the graph $\Gamma(G)$ is connected. Then there exist three vertices A, B, C of $\Gamma(G)$ such that A and B are nonadjacent, and they both divide C.*

PROOF. By Theorem 3.1, there exist $K \trianglelefteq G$ and $H \leq G$ with the following properties: up to an abelian direct factor we have $G = KH$, where K and H are abelian and $(|K|, |H|) = 1$. Since G is nonabelian, setting $Z = \mathbf{Z}(G)$, we have $K, H \not\leq Z$ and consequently there exist $k \in K \setminus Z$ and $h \in H \setminus Z$. Since $K \leq \mathbf{C}_G(k)$ and $H \leq \mathbf{C}_G(h)$, it follows that $|k^G| \bigm| |H|$ and $|h^G| \bigm| |K|$. In particular, $(|k^G|, |h^G|) = 1$. Since $\Gamma(G)$ is connected, there exists $g \in G \setminus Z$ such that

$$(|g^G|, |K|) > 1 \quad \text{and} \quad (|g^G|, |H|) > 1 \ .$$

Denote by n the order of gZ in G/Z. Then

$$n = n_K \cdot n_H \quad \text{where} \quad n_K \bigm| |K| \ \text{ and } \ n_H \bigm| |H| \ .$$

If $n_K = 1$, then $g \in H^y Z$ for some $y \in G$ and $H^y \leq \mathbf{C}_G(g)$, in contradiction to $(|g^G|, |H|) > 1$. So $n_K > 1$ and similarly $n_H > 1$. Thus $g^{n_K} \in H^y Z \setminus Z$ for some $y \in G$ and $g^{n_H} \in KZ \setminus Z$, implying that $|(g^{n_K})^G| \bigm| |K|$ and $|(g^{n_H})^G| \bigm| |H|$.

Set $|g^G| = C$, $\left|(g^{n_K})^G\right| = A$ and $\left|(g^{n_H})^G\right| = B$. Then $(A, B) = 1$ and since $\mathbf{C}_G(g) \leq \mathbf{C}_G(g^{n_K}) \cap \mathbf{C}_G(g^{n_H})$, we have $A \mid C$ and $B \mid C$. The proof is complete. □

We are now in a position to prove the following result.

THEOREM 3.3. *Let G be a nonabelian group. Assume that no vertex of $\Delta(G)$ is adjacent to all the other vertices. Then $\Gamma(G)$ is not k-regular for any $k \geq 1$.*

PROOF. By Proposition 1.4, we can assume that $\Gamma(G)$ is connected. So, consider three vertices A, B, and C as in Lemma 3.2, so A and B are nonadjacent in $\Gamma(G)$ and each of them is adjacent to C. Denote by e the number of vertices of $\Gamma(G)$ adjacent to A and by f the number of vertices adjacent to C. The vertices of $\Gamma(G)$ adjacent to A are C and $e - 1$ vertices $D_1, \ldots, D_{e-1}$ distinct from B. Since each vertex of $\Gamma(G)$ adjacent to A is also adjacent to C, the following distinct vertices of $\Gamma(G)$ are adjacent to C: $A, B, D_1, \ldots, D_{e-1}$. Hence $f \geq e + 1$, implying that the graph $\Gamma(G)$ is nonregular. □

4. Proof of the main result

In this section we prove that the conjecture mentioned in the Introduction is true for "small" valencies. We start with a preliminary lemma.

LEMMA 4.1. *Let Γ be a regular graph, and assume that there exists a group G with $\Gamma = \Gamma(G)$. Assume also that there exist two distinct vertices X and Y of Γ such that X divides Y. Then, denoting by $\mathcal{S}$ the set of neighbors of X different from Y, every vertex of Γ is adjacent to a vertex in $\mathcal{S} \cup \{X, Y\}$. Moreover, the subgraph $\Gamma_{\mathcal{S}}$ of $\Gamma(G)$ induced by $\mathcal{S}$ is connected.*

PROOF. We can clearly assume that Γ is not a complete graph. Therefore, defining $\mathcal{S}' = \mathcal{S} \cup \{X, Y\}$, the set $\mathcal{T}$ of vertices of Γ not lying in $\mathcal{S}'$ is nonempty (note that $\mathcal{S}$ is also the set of neighbors of Y different from X).

The first conclusion of the statement follows at once by Proposition 2.1; in fact, a vertex in $\mathcal{T}$ having no neighbors in $\mathcal{S}'$ (i.e., in $\mathcal{S}$) would have distance at least 3 from X.

As for the second conclusion, let us assume, for a proof by contradiction, that $\Gamma_{\mathcal{S}}$ is disconnected. We claim that in this situation no vertex of the prime graph $\Delta(G)$ is adjacent to all the other vertices of $\Delta(G)$. This, together with Theorem 3.3, will lead to a contradiction.

Let p be a prime dividing some class size of G.

If p divides only one size A, then consider a prime $q \neq p$ dividing some size but not dividing A (such a prime q does exist, otherwise A would be adjacent in Γ to all the other vertices and we are done). Now clearly pq does not divide any class size of G.

Assume that p divides at least two class sizes of G, including X or Y. If p divides only X and Y, then consider any prime q dividing a class size in $\mathcal{T}$; if p divides also a size A in $\mathcal{S}$, take a size B in $\mathcal{S}$ not lying in the same $\Gamma_{\mathcal{S}}$-connected component of A, and take a prime q joining B to a size in $\mathcal{T}$. In all cases, pq does not divide any class size of G.

Assume that p divides at least two class sizes of G, but not Y. if p divides a size A in $\mathcal{S}$ and a size in $\mathcal{T}$, then take a prime q dividing X and a size B in $\mathcal{S}$ but not in the $\Gamma_{\mathcal{S}}$-connected component of A. If p divides only sizes in $\mathcal{T}$, then take a prime q dividing X. Finally, if p divides only sizes in $\mathcal{S}$ (say it joins A and B in $\mathcal{S}$), then take a prime q joining a size in $\mathcal{T}$ with a size C in $\mathcal{S}$ not in the $\Gamma_{\mathcal{S}}$-connected component of A and B. In all cases, pq does not divide any size.

The claim is proved, and the proof is complete. □

THEOREM 4.2. *Let Γ be a graph, and assume that there exists a group G with $\Gamma = \Gamma(G)$. Then, for $k \in \{1, 2, 3, 4, 5\}$, Γ is k-regular if and only if it is a complete graph with $k+1$ vertices.*

PROOF. Clearly it is enough to consider the "only if" part of the statement. In view of Theorem 1.5, we are done if G is an **F**-group; therefore, we can assume there exist two distinct class sizes $X \neq 1$ and Y of G such that X is a divisor of Y. As in the proof of the previous lemma, we denote by $\mathcal{S}$ the set of vertices of Γ that are adjacent to X, but different from Y; we also define $\mathcal{S}' = \mathcal{S} \cup \{X, Y\}$, and $\mathcal{T}$ to be the set of vertices of Γ not lying in $\mathcal{S}'$ (we can clearly assume $\mathcal{T} \neq \emptyset$). The previous lemma shows that every vertex in $\mathcal{T}$ is adjacent to a vertex in $\mathcal{S}$, and that the subgraph $\Gamma_{\mathcal{S}}$ of Γ induced by $\mathcal{S}$ is connected.

Note that the theorem is now obviously true for $k \leq 3$. As for the case when Γ is 4-regular (and $|\mathcal{S}| = 3$), it is easy to check that the total number of adjacencies between vertices in $\mathcal{S}$ and vertices in $\mathcal{T}$ is 2; now, as each element of $\mathcal{T}$ is adjacent to some vertex in $\mathcal{S}$, we get $|\mathcal{T}| \leq 2$. But the elements of $\mathcal{T}$ have at least $|\mathcal{T}|(4 - (|\mathcal{T}| - 1)) = |\mathcal{T}|(5 - |\mathcal{T}|)$ adjacencies in $\mathcal{S}$; thus $|\mathcal{T}|(5 - |\mathcal{T}|) \leq 2$, which yields $|\mathcal{T}| \geq 5$, a contradiction.

It remains to treat the case $k = 5$, so Γ is 5-regular and $|\mathcal{S}| = 4$. The connectedness of $\Gamma_{\mathcal{S}}$ forces the vertices in $\mathcal{S}$ to have altogether an even number, not larger than 6, of adjacencies with vertices in $\mathcal{T}$; therefore, as each element of $\mathcal{T}$ is adjacent to some vertex in $\mathcal{S}$, we get $|\mathcal{T}| \leq 6$. On the other hand, each element of $\mathcal{T}$ has at most $|\mathcal{S}| + |\mathcal{T}| - 1 = |\mathcal{T}| + 3$ neighbors, thus $|\mathcal{T}| \geq 2$. But the elements of $\mathcal{T}$ have at least $|\mathcal{T}|(5 - (|\mathcal{T}| - 1)) = |\mathcal{T}|(6 - |\mathcal{T}|)$ adjacencies in $\mathcal{S}$; thus $|\mathcal{T}|(6 - |\mathcal{T}|) \leq 6$, which in view of $|\mathcal{T}| \geq 2$ yields $|\mathcal{T}| \in \{5, 6\}$.

This trivial remark is sufficient to conclude that $\Gamma_{\mathcal{S}}$ cannot be a square, as otherwise $\mathcal{S}$ would have precisely 4 adjacencies with $\mathcal{T}$, which would imply $|\mathcal{T}| \leq 4$. For the same reason, denoting by A, B, C and D the vertices in $\mathcal{S}$, if one of them (say A) has valency 3 in $\Gamma_{\mathcal{S}}$, then B, C and D are pairwise nonadjacent. In other words we are left with two situations: either the edges of $\Gamma_{\mathcal{S}}$ are $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, or they are $\{A, B\}$, $\{A, C\}$ and $\{C, D\}$ (up to isomorphism). In any case, the number of adjacencies between elements in $\mathcal{S}$ and elements in $\mathcal{T}$ is 6.

Now the case $|\mathcal{T}| = 5$ can be easily ruled out. In fact, if $|\mathcal{T}| = 5$ and the subgraph $\Gamma_{\mathcal{T}}$ of Γ induced by $\mathcal{T}$ is not complete, then the vertices in $\mathcal{T}$ must have altogether at least 7 adjacencies in $\mathcal{S}$, whereas if $\Gamma_{\mathcal{T}}$ is complete, then the adjacencies between $\mathcal{S}$ and $\mathcal{T}$ would be 5.

The conclusion so far is that $|\mathcal{T}|$ must be 6. Now, both the possible configurations of $\Gamma_{\mathcal{S}}$ yield that $\Gamma_{\mathcal{T}}$ is 4-regular.

Focussing on the case when A has valency 3, recall that the edges involving only vertices in $\mathcal{S}'$ are between each of A, X, Y and every vertex in $\mathcal{S}'$ other than itself, and that A, X, Y are not adjacent to any vertex in $\mathcal{T}$. Setting $\mathcal{T} = \{T_i \mid 1 \leq i \leq 6\}$, we can assume that $\{T_1, B\}$, $\{T_2, B\}$, $\{T_3, C\}$, $\{T_4, C\}$, $\{T_5, D\}$, $\{T_6, D\}$ are edges of Γ. We claim that in this case no vertex of $\Delta = \Delta(G)$ is adjacent to all the other vertices of Δ (therefore, an application of Theorem 3.3 will yield a contradiction). Indeed, let p be a vertex of Δ.

(i) Suppose that p divides at least one of X, Y and A. We may assume, say, that $p \nmid C$. Let q be a prime divisor both of T_3 and of C. Then p does not divide C and any of the T_i, whereas q does not divide any element of $\{A, X, Y, B, D\}$, so p and q are nonadjacent in Δ.

(ii) Suppose that the previous situation does not hold. If, say, $p \mid B$, then p does not divide any element of $\{A, X, Y, C, D, T_3, T_4, T_5, T_6\}$. Now let q be a prime divisor of T_3 and C; then q does not divide any element in $\{B, T_1, T_2\}$, and p is nonadjacent to q in Δ.

(iii) Suppose, finally, that p does not divide any element of $\mathcal{S}'$, and let q be a prime divisor of X. Then q does not divide any vertex in $\mathcal{T}$, and again p is nonadjacent to q in Δ.

Finally, we consider the case when $|\mathcal{T}| = 6$ and no vertex of $\Gamma_{\mathcal{S}}$ has valency 3. Setting as above $\mathcal{T} = \{T_i \mid 1 \leq i \leq 6\}$, we can assume that $\{T_1, A\}$, $\{T_2, B\}$, $\{T_3, B\}$, $\{T_4, C\}$, $\{T_5, D\}$ and $\{T_6, D\}$ are edges of Γ. We claim that no vertex of Δ is adjacent to all the other vertices of Δ, and this will be the final contradiction. Let p be a vertex of Δ.

(i) Suppose that p divides at least one among X, Y. Then p does not divide any of the T_i, and some $U \in S$. Let T_U be a vertex in $\mathcal{T}$ that is adjacent to U via a prime q; then q does not divide any vertex in $\mathcal{S}' \setminus U$. Thus p and q are nonadjacent in Δ.

(ii) Suppose that the previous situation does not hold. If p divides some $U \in \mathcal{S}$, then there exists $V \in \mathcal{S}$ that is not divisible by p; therefore, p does not divide X, Y, V, and all the elements in $\mathcal{T}$ that are nonadjacent with U. Now, let q be a prime divisor of V and of a vertex in $\mathcal{T}$; such a q does not divide any element in $\mathcal{S} \setminus \{V\}$, and of course does not divide any element in $\mathcal{T}$ which is nonadjacent to V. It is easy to see that pq does not divide any class size of G.

(iii) Suppose, finally, that p does not divide any element in $\mathcal{S}'$. If $q \mid X$, Then q does not divide any element in $\mathcal{T}$, and pq does not divide any class size of G.

The proof is now complete. □

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