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Topology and Field Theories

Center for Mathematics at Notre Dame

Summer School and Conference
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May 29–June 8, 2012
University of Notre Dame, Notre Dame, Indiana

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Preface

Finding an appropriate mathematical language for quantum field theories and developing tools for their construction and classification is a major task and challenge for mathematics. One approach to quantum field theories was pioneered by Sir Michael Atiyah, Maxim Kontsevich and Graham Segal in the 1980's. They defined an n-dimensional topological quantum field theory to be a functor from a suitable n-dimensional bordism category to the category of complex vector spaces and linear maps. Unwrapping this definition, an n-TQFT associates to a closed manifold Y of dimension $n-1$ (an object of the bordism category) a vector space, and to a n-dimensional bordism from Y_1 to Y_2 (a morphism) a linear map between the associated vector spaces. Moreover, it is required that the disjoint union of manifolds corresponds to the tensor product of the associated algebraic objects; in technical terms, a field theory is a symmetric monoidal functor.

An n-dimensional field theory determines a numerical invariant for closed n-manifolds by interpreting a closed manifold as a bordism from the empty set to itself and applying the functor to it. This numerical invariant is often referred to as the *partition function* of the field theory. The field theory itself can then be thought of as a refinement or gluing formula for this numerical invariant, allowing a calculation of this number for a closed manifold in terms of the algebraic data associated to a decomposition of this manifold into manifolds with boundary.

There are many examples of topological field theories, for example the n-dimensional Dijkgraf-Witten theories associated to a finite group G and a cohomology class in $\eta \in H^{n+1}(BG;\mathbb{Z})$, whose partition function counts principal G-bundles on closed n-manifolds (with a multiplicity determined by η). These are quantizations of the classical Chern-Simons theory. For $n = 3$, there is a construction of the Chern-Simons TQFT for a compact Lie group G due to Witten and Reshetikhin-Turaev.

There are many variants and refinements of this basic definition of field theory. For example, the bordisms could be equipped with additional structure, like a framing or a conformal structure, leading to the notion of *framed field theory* and *conformal field theory*, respectively (for CFTs, the target category should be a suitable category of topological vector spaces). In the case of 3-dimensional bordisms, the extra structure could be a link in this 3-manifold (required to be transverse to the boundary). The partition function of such field theories then provide invariants of links and knots. Another variant replaces the vector space target category by some other symmetric monoidal category.

A very significant refinement is the notion of an *extended* or *local* field theory. An n-dimensional local field theory associates data not only to manifolds of dimension n and $n-1$, but to manifolds with corners of dimension k for $0 \leq k \leq n$.

It also provides much more general gluing formulas, allowing a decomposition of a k-manifold into k-manifolds with corners of arbitrary codimension. Not surprisingly, it takes a lot of effort to organize this tremendous amount of data and relations in a manageable form. However, in the end, the formulation is surprisingly clean: the manifolds of dimension k, $0 \leq k \leq n$ are organized in a higher category whose objects are 0-manifolds, whose morphisms are 1-manifolds, whose 2-morphisms are 2-manifolds, and so forth for k-morphisms for $k \leq n$. It doesn't stop there: $(n+1)$-morphisms are diffeomorphisms between n-manifolds, $(n+2)$-morphisms are paths of diffeomorphisms, etc. So there are k-morphisms for all $k \geq 0$, but k-morphisms for $k > n$ are invertible. Such a higher category is called an (∞, n)-category. Given a symmetric monoidal (∞, n)-category $\mathcal{C}$ a *local n-TQFT with values in* $\mathcal{C}$ is then a symmetric monoidal functor from this bordism (∞, n)-category to $\mathcal{C}$.

Concerning the classification of TQFTs there is the classical folklore result that there is a bijection between 2-dimensional TQFTs and commutative Frobenius algebras. It is given by mapping a field theory to the vectorspace it associates to the circle. A recent breakthrough in the classification of TQFTs was achieved by Jacob Lurie's proof of the Baez-Dolan Cobordism Hypothesis, according to which a local framed n-TQFT with values in an (∞, n)-category $\mathcal{C}$ is determined by the object of $\mathcal{C}$ the field theory associates to the 0-manifold pt consisting of one point. Moreover, an object of $\mathcal{C}$ comes from a field theory if and only if it is *fully dualizable* (the object pt of the framed bordism (∞, n)-category has that property, forcing its image under a field theory to be fully dualizable).

The four papers in this volume are based on talks given during the program on Topology and Field Theories at the Center of Mathematics at Notre Dame. This program was held from May 29 till June 8 of 2012 and consisted of a weeklong summer school followed by a conference. André Henriques, Sergei Gukov, Jacob Lurie and Chris Schommer-Pries were the speakers at the summer school. This volume contains the material of these lectures, as well as their talks at the conference in the case of Gukov, Lurie and Schommer-Pries.

Two-dimensional conformal field theory is the topic of the paper by Henriques. He extends a construction due to Fuchs, Runkel and Schweigert, whose input consists of a chiral conformal field theory and a Frobenius object in the monoidal category the chiral theory associates to the circle. The output is a conformal field theory. A chiral CFT is an intricate type of conformal field theory which in particular involves a $\mathbb{C}$-linear category associated to 1-manifolds, and continuous linear operators associated to conformal bordisms which are required to depend holomorphically on the bordism. Despite the more intricate definition it is easier to construct examples of chiral CFTs than of CFTs. In particular, there is (or rather is expected to be) a chiral CFT associated to a compact Lie group G and a cohomology class $k \in H^4(BG; \mathbb{Z})$ (the details of this construction are complete only for $G = SU(n)$). For $G = SU(2)$, there is a beautiful ADE-classification of the Frobenius objects in the associated linear category due to Ostrik. The paper by Henriques goes a long way toward producing from a chiral CFT and a Frobenius object a *local* CFT (which he calls 'three-tier CFT') that extends the Fuchs-Runkel-Schweigert CFT.

In their paper Gukov and Saberi give physics interpretations of various polynomials associated to knots and links. For example, they explain the 'A-polynomial' of a knot which is a polynomial in two variables whose zero-set is the curve $\mathcal{C} \subset \mathbb{C}^* \times \mathbb{C}^*$ given by the image of the representation variety consisting of conjugacy classes of homomorphisms $\pi_1(M) \to SL_2(\mathbb{C})$ in the representation variety associated to the fundamental group of $\partial M \cong S^1 \times S^1$, where M is the complement of an open neighborhood of the knot. From a physics perspective, $\mathbb{C}^* \times \mathbb{C}^*$ is the classical phase space of Chern-Simons theory on $S^1 \times S^1 \times \mathbb{R}$, and the curve $\mathcal{C}$ is a Lagrangian submanifold. After a quick introduction to geometric quantization, the (colored) Jones polynomial is interpreted as the state vector associated by quantization to the Lagrangian submanifold $\mathcal{C}$. This is a neat story, discovered some years ago, but told in a way to make it accessible to non-experts. Finally, they talk about knot homology groups as a 'categorification' of the corresponding knot polynomials, or, from a physics perspective, as 4-dimensional topological quantum field theories which upon dimensional reduction give 3-dimensional TQFTs whose partition functions are the corresponding knot polynomials.

In the paper by Heuts and Lurie, they look at the Dijkgraf-Witten TQFT associated to a finite group G through the eyes of homotopy theorists, thinking of maps to the classifying space BG instead of principal G-bundles, and replacing BG by some topological space X. They ask for a sufficient condition on X which guarantees the existence of a Dijkgraf-Witten type TQFT with values in a given category $\mathcal{C}$. They call this property $\mathcal{C}$-ambidextrous. If $\mathcal{C}$ is the category of complex vector spaces the classifying space BG of a finite group is ambidextrous; this leads to the classical Dijkgraf-Witten TQFT. A main result due to Hopkins and Lurie is the statement that every space satisfying a strong finiteness condition on its homotopy groups is $\mathcal{C}$-ambidextrous where $\mathcal{C}$ is the category (or rather $(\infty, 1)$-category) of spectra which are local with respect to the Morava K-theory spectrum $K(n)$. This leads to many new TQFTs of Dijkgraf-Witten flavor with values in the $(\infty, 1)$-category of $K(n)$-local spectra.

The first part of the paper by Schommer-Pries, based on his lectures at the summer school, explores fully-dualizability for objects of (∞, n)-categories $\mathcal{C}$ for $n = 1, 2, 3$ and the action of the orthogonal group $O(n)$ on the full subcategory of $\mathcal{C}$ of fully-dualizable objects. This action is induced by the change-of-framing action of $O(n)$ on the framed bordism (∞, n)-category via Lurie's classification of local framed n-TQFTs. An understanding of this action is essential for determining other types of local n-TQFTs (for example, oriented field theories) which according to another result of Lurie's can be expressed in terms of the homotopy fixed point category of an appropriate subgroup of $O(n)$. The second part of his paper, based on Schommer-Pries' talk at the conference, is an introduction to his joint work with Clark Barwick on the axiomatizatic characterization of (∞, n)-categories.

The program on Topology and Field Theories was funded by the Center of Mathematics at Notre Dame and the NSF. I would like to thank Gregory Crawford, Dean of the College of Science at Notre Dame, for enabling us to establish this center, and Claudia Polini, Director of the Center, for the financial support making it possible to tape all lectures and making them available online. I also would like to thank Sam Evens for helpful organizational advice, and especially Lisa Driver for her efficient organization of the summer school and conference. Finally, I would like to thank all speakers and participants for making this program a lively and successful event.

Stephan Stolz

This book is a collection of expository articles based on four lecture series presented during the 2012 Notre Dame Summer School in Topology and Field Theories.

The four topics covered in this volume are: Construction of a local conformal field theory associated to a compact Lie group, a level and a Frobenius object in the corresponding fusion category; Field theory interpretation of certain polynomial invariants associated to knots and links; Homotopy theoretic construction of far-reaching generalizations of the topological field theories that Dijkgraf and Witten associated to finite groups; and a discussion of the action of the orthogonal group $O(n)$ on the full subcategory of an n-category consisting of the fully dualizable objects.

The expository style of the articles enables non-experts to understand the basic ideas of this wide range of important topics.

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