Geometric and Spectral Analysis

CRM Workshops on:
- Geometry of Eigenvalues and Eigenfunctions
  June 4–8, 2012
- Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis
  July 2–6, 2012
- Spectral Invariants on Non-compact and Singular Spaces
  July 23–27, 2012

Centre de Recherches Mathématiques, Université de Montréal, Quebec, Canada

Pierre Albin
Dmitry Jakobson
Frédéric Rochon
Editors

American Mathematical Society
Providence, Rhode Island
Centre de Recherches Mathématiques
Montréal, Québec, Canada
Geometric and Spectral Analysis
Geometric and Spectral Analysis

CRM Workshops on:
Geometry of Eigenvalues and Eigenfunctions
June 4–8, 2012

Manifolds of Metrics and Probabilistic Methods
in Geometry and Analysis
July 2–6, 2012

Spectral Invariants on Non-compact and Singular Spaces
July 23–27, 2012
Centre de Recherches Mathématiques,
Université de Montréal, Quebec, Canada

Pierre Albin
Dmitry Jakobson
Frédéric Rochon
Editors

American Mathematical Society
Providence, Rhode Island

Centre de Recherches Mathématiques
Montréal, Québec, Canada
Library of Congress Cataloging-in-Publication Data
Geometric and spectral analysis / Pierre Albin, Dmitry Jakobson, Frédéric Rochon, editors.
   p. cm. – (Contemporary mathematics ; volume 630)
Centre de Recherches Mathématiques Proceedings.
Includes bibliographical references.
   II. Jakobson, Dmitry, 1970– III. Rochon, Frédéric, 1978–.
   QA20.7.S64G46 2014
   516–dc23 2014021478
Contemporary Mathematics ISSN: 0271-4132 (print); ISSN: 1098-3627 (online)
DOI: http://dx.doi.org/10.1090/conm/630

Copying and reprinting. Individual readers of this publication, and nonprofit libraries
acting for them, are permitted to make fair use of the material, such as to copy select pages for
use in teaching or research. Permission is granted to quote brief passages from this publication in
reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication
is permitted only under license from the American Mathematical Society. Permissions to reuse
portions of AMS publication content are handled by Copyright Clearance Center’s RightsLink®
service. For more information, please visit: http://www.ams.org/rightslink.

Send requests for translation rights and licensed reprints to reprint-permission@ams.org.

Excluded from these provisions is material for which the author holds copyright. In such cases,
requests for permission to reuse or reprint material should be addressed directly to the author(s).
Copyright ownership is indicated on the copyright page, or on the lower right-hand corner of the
first page of each article within proceedings volumes.

© 2014 by the American Mathematical Society. All rights reserved.
The American Mathematical Society retains all rights
except those granted to the United States Government.
Copyright of individual articles may revert to the public domain 28 years
after publication. Contact the AMS for copyright status of individual articles.
Printed in the United States of America.

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/
Contents

Preface vii

A New Proof of a Bismut-Zhang Formula for Some Class of Representations
MAXIM BRAVERMAN and BORIS VERTMAN 1

Tunneling, the Quillen Metric and Analytic Torsion for High Powers
of a Holomorphic Line Bundle
ROBERT J. BERMAN 15

Simple Spectrum and Rayleigh Quotients
VICTOR GUILLEMIN, EVELINE LEGENDRE, and ROSA SENA-DIAS 33

Smooth and Singular Kähler–Einstein Metrics
YANIR A. RUBINSTEIN 45

Complex b-Manifolds
GERARDO A. MENDOZA 139

Iterative Structures on Singular Manifolds
BERT-WOLFGANG SCHULZE 173

The Fundamental Gap and One-Dimensional Collapse
ZHIQIN LU and JULIE ROWLETT 223

The Logarithmic Singularities of the Green Functions of the Conformal
Powers of the Laplacian
RAPHAËL PONGE 247

A Symbolic Calculus for Fourier Integral Operators
YURI SAFAROV 275

Strichartz Estimates on Exterior Polygonal Domains
DEAN BASKIN, JEREMY L. MARZUOLA, and JARED WUNSCH 291

Probabilistic Sobolev Embeddings, Applications to Eigenfunctions Estimates
NICOLAS BURQ and GILLES LEBEAU 307

Random Smooth Functions on Compact Riemannian Manifolds
LIVIU I. NICOLAEŞCU 319
Preface

In 2012, thematic programs on Geometric Analysis and Spectral Theory and on Moduli spaces, Extremality and Global Invariants were held at the CRM. There were several workshops on cutting-edge topics in geometric analysis and spectral theory, notably on the Geometry of Eigenvalues and Eigenfunctions, on Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis and on Spectral invariants on non-compact and singular spaces. This was the occasion to learn about the most recent developments in various interconnected fields and also the starting point for new investigations and fruitful collaborations.

The 2012 Spring Semester on Geometric Analysis and Spectral Theory at the CRM brought together researchers working in these fields interpreted in a broad sense, so as to include applications to fundamental problems in geometry, PDE, dynamical systems and mathematical physics. The present CRM Proceedings includes articles written by participants of the workshops in Geometry of Eigenvalues and Eigenfunctions and on Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis.

The workshop on Geometry of Eigenvalues and Eigenfunctions was held at the CRM on June 4–8, 2012. It was organized by D. Jakobson (McGill) and I. Polterovich (Montréal). The workshop brought together the leading researchers and young mathematicians working in various areas of geometric spectral theory. Many problems in the field are motivated by questions originating in the study of real-life phenomena: quantum-mechanical effects, vibration of membranes and plates, oscillations of fluids, etc. One of the main highlights of the meeting was a series of Aisenstadt lectures on Quantum Unique Ergodicity by Elon Lindenstrauss. Quantum Ergodicity and properties of eigenfunctions in the semiclassical limit were among the central themes of the workshop. In recent work, Jakobson, Safarov and Strohmaier analyzed the semiclassical limit of spectral theory on manifolds whose metrics have jump-like discontinuities; the corresponding semiclassical limit does not relate to a classical flow but rather to branching (ray-splitting) billiard dynamics. A Quantum Ergodicity theorem was established for discontinous systems, and a new notion of ergodicity for the ray-splitting dynamics was introduced, and a class of examples of such ergodic systems was provided by Yves Colin de Verdière. Very important technical results in that work were established in the paper by Y. Safarov appearing in these proceedings. His paper discusses Fourier integral operators (FIOs) associated with canonical transformations. It is well known that the composition of two such operators is a FIO associated with the composition of the corresponding transformations. It is also known that the principal symbol of a FIO is a half-density with values in a linear bundle over a Lagrangian manifold. The latter makes it impossible to write down a simple formula for the principal symbol...
of the composition: such a formula must involve the product of principal symbols but it is not a half-density. It will be shown that, under a slightly different definition of the principal symbol, this problem can be avoided, and then the formulae for principal symbols of compositions and adjoint operators become almost as simple as for pseudodifferential operators.

Another topic considered at the conference was the behavior of eigenfunctions of the Laplacian on Riemannian manifolds and Euclidean domains. In their paper appearing in these proceedings, B. Nguyen, A. Delytisn, and D. Grebenkov discuss the eigenvalue problem for the Laplace operator in a planar domain which can be decomposed into a bounded domain of arbitrary shape and elongated “branches” of variable cross-sectional profiles. They show that when the eigenvalue is smaller than a prescribed threshold, the corresponding eigenfunction decays exponentially along each branch.

Two papers appearing in these proceedings concern the behavior of eigenvalues of differential operators. In the paper by V. Guillemin, E. Legendre, and R. Sena-Dias, the authors calculate the variation of the Rayleigh quotient under conformal variations and under variations within a Kähler class. They use the conformal formula to reprove Uhlenbeck’s result: a generic metric can be perturbed by a conformal factor into a metric whose eigenspaces for the Laplace operator have dimension 1. In the case of a Kähler manifold, they write an explicit variational formula for the Rayleigh quotient under Kähler deformations and discuss this relationship with the following question: Is the spectrum of a generic Kähler metric simple?

G. Poliquin’s paper considers lower bounds for the principal frequency of the $p$-Laplacian on $n$-dimensional Euclidean domains. In particular, he extends the classical results involving the inner radius of a domain and the first eigenvalue of the Laplace operator to the case $p \neq 2$. As a by-product, he obtains a lower bound on the size of the nodal set of an eigenfunction of the $p$-Laplacian on planar domains.

The workshop on Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis was held at CRM on July 2–6, 2012. It was organized by D. Jakobson (McGill), S. Klevtsov (ULB), and S. Zelditch (Northwestern). The workshop brought together mathematicians and physicists working on questions related to random geometry in a broad sense.

The study of geometry of the infinite-dimensional space of Kähler metrics was the subject of many talks at the conference, as well as two papers in these proceedings. The main result in the paper of R. Berman provides a new formula for the distribution of the exponentially small eigenvalues of Dolbeault Laplacians associated to high tensor powers of a line bundle over a compact complex manifold, which in physics terminology is a measure of “tunneling” of the Dolbeault complex. Along the way a new proof of the asymptotics of the induced Quillen metric on the corresponding determinant line is obtained. A brief comparison with the tunneling effect for Witten Laplacians and large deviation principles for fermions is also made.

A long paper by Y. Rubinstein provides a survey on Kähler-Einstein metrics in the smooth and singular (edge) setting, presented from a unified point of view. The paper takes the reader from the very beginning of Kähler geometry and the Einstein equations, through the edge calculus for Kähler-Einstein edge metrics as developed in the work of Rubinstein, Jeffres and Mazzeo, over the geometry of the space of Kähler metrics and its associated functionals, to the very recent work of
Rubinstein-Cheltsov on asymptotic log Fano varieties. The survey is concluded
with a conjectural picture concerning a generalization of the Calabi conjecture in
the asymptotic log setting.

Several sessions at the workshop were devoted to the lectures on Liouville 2d
quantum gravity and the physical definition of random metrics. L. Chen out-
lined a construction (jointly with D. Jakobson) of random measures in dimension
four, inspired by the construction of Duplantier and Sheffield in dimension two,
and described an approach to derive a KPZ-type relation for spherical averages of
those measures. The construction used a conformally covariant differential operator
(Paneitz operator) in dimension four, in place of the Laplacian used in dimension
two. R. Ponge gave an overview of Fefferman’s program, conformally covariant
differential operators (including the Yamabe and the Paneitz operators). In his
paper appearing in these proceedings, Ponge studied the logarithmic singularities
of the Green functions of conformally covariant operators, including the Yamabe
and Paneitz operators, as well as the conformal fractional powers of the Laplacian
arising from scattering theory for Poincaré-Einstein metrics. He explains how to
compute explicitly the logarithmic singularities of the Green functions of the confor-
mal powers of the Laplacian. The results are formulated in terms of Weyl conformal
invariants arising from the ambient metric of Fefferman-Graham. As applications
we obtain characterizations in terms of Green functions of locally conformally flat
manifolds and a spectral theoretic characterization of the conformal class of the
round sphere.

Several talks at the conference concerned the study of random functions. L.
Nicolaescu’s talk described his results on critical values of random functions on a
given compact Riemannian manifold. In his paper appearing in these proceedings,
Nicolaescu surveys his work on the distribution of critical points and critical val-
ues of those functions, defined as a Fourier-like eigenseries whose coefficients are
independent Gaussian random variables.

Nicolas Burq gave a talk on probabilistic Sobolev embeddings, showing that,
from a PDE point of view, randomly chosen functions may behave much more nicely
than what the deterministic theory would predict. In the paper by N. Burq and
G. Lebeau appearing in these proceedings, the authors present some probabilistic
versions of Sobolev embeddings and an application to the growth rate of the $L^p$
norms of spherical harmonics, as well as eigenfunctions on manifolds with transitive
isometry group actions.

Shortly after these workshops, the CRM thematic year on Moduli spaces, ex-
 tremality and global invariant started with its first workshop on Spectral invariants
on non-compact and singular spaces. It was held on July 23–27 at the CRM and was
organized by P. Albin (UIUC) and F. Rochon (UQAM). Since it involved spectral
theory and geometric analysis, as well as global invariants and moduli spaces (which
are a good source of examples of singular spaces), this workshop was a natural tran-
sition between the two thematic programs. A good part of it was concerned with
spectral invariants arising in index theory, like the eta invariant, analytic torsion or
the determinant of Laplacian. The latter also arises naturally in scattering theory
and more generally in spectral theory, which was the other important theme of the
workshop.

The workshop was the starting point of an interesting collaboration between
Dean Baskin, Jeremy Marzuola, and Jared Wunsch, who obtained global Strichartz
estimates for exterior polygonal domains as well as some smoothing estimates. These results have natural applications to the study of some non-linear Schrödinger equations on exterior polygonal domains. Their results appear in the present proceedings as one of the five articles related to this workshop.

There is also the collaboration of Zhiqin Lu and Julie Rowlett, which provides a careful analysis of the gap between the first two positive eigenvalues of the Laplacian of a convex domain in $\mathbb{R}^n$ that degenerates to a domain in $\mathbb{R}^{n-1}$. One of the important conclusions is that this gap detects the geometry of a one-dimensional collapse. In fact, depending on the geometry of the collapse, the gap can diverge or remain bounded.

In another direction, Maxim Braverman and Boris Vertman give a new proof of the Bismut-Zhang formula for the ratio of the analytic torsion and the Reidemeister torsion. Their proof works when the representation of the fundamental group lies in a connected component of a unitary representation and has the advantage of being significantly simpler. The idea is to use the Cheeger-Müller theorem for the unitary representations and extend the result to the whole connected component using analytic continuation.

Gerardo A. Mendoza contributes to these proceedings with a nice paper on $b$-complex manifolds, which are complex manifolds arising from a complex structure of the $b$-tangent bundle of Melrose. This is a subject he has developed over the years. In the paper published in these proceedings, Mendoza studies the rich structure of the cohomology of the indicial complex obtained by restricting the Dolbeault complex to the boundary.

Finally, B.-W. Schulze gives a valuable and useful overview of the iterative approach, an approach he and his collaborators developed to construct and study a pseudodifferential calculus suitable for doing analysis on stratified spaces.

Pierre Albin
Dmitry Jakobson
Frédéric Rochon
In 2012, the Centre de Recherches Mathématiques (CRM) was at the center of many interesting developments in geometric and spectral analysis, with a thematic program on Geometric Analysis and Spectral Theory followed by a thematic year on Moduli Spaces, Extremality and Global Invariants.

This volume contains original contributions as well as useful survey articles of recent developments by participants from three of the workshops organized during these programs: Geometry of Eigenvalues and Eigenfunctions, held from June 4–8, 2012; Manifolds of Metrics and Probabilistic Methods in Geometry and Analysis, held from July 2–6, 2012; and Spectral Invariants on Non-compact and Singular Spaces, held from July 23–27, 2012.

The topics covered in this volume include Fourier integral operators, eigenfunctions, probability and analysis on singular spaces, complex geometry, Kähler-Einstein metrics, analytic torsion, and Strichartz estimates.