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On Lipschitz inversion of nonlinear redundant representations

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Abstract. In this note we show that reconstruction from magnitudes of frame coefficients (the so called "phase retrieval problem") can be performed using Lipschitz continuous maps. Specifically we show that when the nonlinear analysis map $\alpha : H \to \mathbb{R}^m$ is injective, with $(\alpha(x))_k = |\langle x, f_k\rangle|^2$, where $\{f_1, \cdots, f_m\}$ is a frame for the Hilbert space H, then there exists a left inverse map $\omega : \mathbb{R}^m \to H$ that is Lipschitz continuous. Additionally we obtain that the Lipschitz constant of this inverse map is at most 12 divided by the lower Lipschitz constant of α.

1. Introduction

Let H be an n-dimensional Hilbert space and $\mathcal{F} = \{f_1, f_2, \cdots, f_m\}$ be a spanning set for H. Since H has finite dimension, $\mathcal{F}$ forms a frame for H, that is, there exist two positive constants A and B such that

$$A \|x\|^2 \leq \sum_{k=1}^{m} |\langle x, f_k\rangle|^2 \leq B \|x\|^2 , \quad \forall x \in H \tag{1.1}$$

In this paper, H can be a real or complex Hilbert space and the result applies to both cases. On H we consider the equivalent replation $x \sim y$ if and only if there is a scalar a of magnitude one, $|a| = 1$, so that $y = ax$. Let $\hat{H} = H/\sim$ denote the set of equivalence classes. Note that $\hat{H} \setminus \{0\}$ is equivalent to the cross-product between a real or complex projective space $\mathcal{P}^{n-1}$ of dimension $n-1$ and the positive semiaxis $\mathbb{R}^+ = (0, \infty)$.

Let α denote the nonlinear map

$$\alpha : H \to \mathbb{R}^m \ , \ \alpha(x) = \left(|\langle x, f_k\rangle|^2\right)_{1\leq k\leq m} \tag{1.2}$$

Note that α induces a nonlinear map which is well defined on $\hat{H}$. By abuse of notation we also denote it by α. The *phase retrieval problem* (or the *phaseless reconstruction problem*) refers to analyzing when α is an injective map, and in this

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case to finding "good" left inverses. The frame $\mathcal{F}$ is said to be *phase retrievable* if the nonlinear map α is injective. In this paper we assume α is injective (hence $\mathcal{F}$ is phase retrievable).

A continuous map $f : (X, d_X) \to (Y, d_Y)$, defined between metric spaces X and Y with distances d_X and d_Y respectively, is said to be Lipschitz continuous with Lipschitz constant $Lip(f)$ if

$$Lip(f) := \sup_{x_1 \neq x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} < \infty \tag{1.3}$$

Existing literature (e.g. [**BW**]) establishes that when α is injective, it is also bi-Lipschitz for metric d_1 (the nuclear norm, which is defined in (2.4)) in $\hat{H}$ and Euclidian norm in $\mathbb{R}^m$. As a consequence of these results we obtain that a left inverse of α is Lipschitz when restricted to the image of $\hat{H}$ through α. In this paper we show that this left inverse admits a Lipschitz continuous extension to the entire $\mathbb{R}^m$. Surprisingly we obtain the Lipschitz constant of this extension is just a small factor larger than the minimal Lipschitz constant, a factor that is independent of the dimension n or the number of frame vectors m.

The Lipschitz properties of α is related to the stability of reconstruction. Consider the noisy model for the reconstruction of a signal x with the measurements

$$y = \alpha(x) + \nu \tag{1.4}$$

where $\nu \in \mathbb{R}^m$ is the noise. The stability of specific reconstruction methods is studied in, for instance, [**BCMN**], [**BH**] and [**CSV**]. In general, if we can find (guaranteed by the result of this paper) a Lipschitz continuous map defined on the whole $\mathbb{R}^m$, say $\omega : (\mathbb{R}^m, \|\cdot\|) \to (\hat{H}, d_1)$, such that $\omega(\alpha(x)) = x$ for all $x \in \hat{H}$, then we have a stable reconstruction in the following sense: Let $x_0 \in \hat{H}$ be the original signal and y_1 be the measurement from the noisy model (1.4) with noise ν_1. Let $x_1 = \omega(y_1)$. Then

$$d_1(x_0, x_1) = d_1(\omega(\alpha(x_0)), \omega(y_1)) \leq Lip(\omega) \cdot \|\alpha(x_0) - y_1\| = Lip(\omega) \cdot \|\nu_1\|. \tag{1.5}$$

Moreover, let y_1 and y_2 be two different measurements of $\alpha(x_0)$ from (1.4) with noise ν_1, ν_2, respectively. Then we have

$$d_1(x_1, x_2) = d_1(\omega(y_1), \omega(y_2)) \leq Lip(\omega) \cdot \|y_1 - y_2\| = Lip(\omega) \cdot \|\nu_1 - \nu_2\|. \tag{1.6}$$

Note that in general (1.5) does not imply (1.6).

2. Notations and Statement of Main Results

The nonlinear map α defined by (1.2) naturally induces a linear map between the space $Sym(H) = \{T : H \to H \ , \ T = T^*\}$ of symmetric operators on H and $\mathbb{R}^m$:

$$\mathcal{A} : Sym(H) \to \mathbb{R}^m \ , \ \mathcal{A}(T) = (\langle T f_k, f_k \rangle)_{1 \leq k \leq m} \tag{2.1}$$

Note that $\alpha(x) = \mathcal{A}([\![x, x]\!])$ where

$$[\![x, y]\!] = \frac{1}{2}(\langle \cdot, x \rangle y + \langle \cdot, y \rangle x) \tag{2.2}$$

denotes the symmetric outer product between vectors x and y.

The linear map $\mathcal{A}$ has first been observed in [**BBCE**] and it has been exploited successfully in various papers e.g. [**Ba2, CSV, Ba3**].

Let $S^{p,q}(H)$ denote the set of symmetric operators that have at most p strictly positive eigenvalues and q strictly negative eigenvalues.

In particular $S^{1,0}(H)$ denotes the set of non-negative symmetric operators of rank at most one:

$$S^{1,0}(H) = \{T \in Sym(H)\ s.t.\ \exists x \in H, \forall y \in H\ ,\ T(y) = \langle y, x\rangle x\} \tag{2.3}$$

In [**Ba4**] we studied in more depth geometric and analytic properties of this set. The map α is injective if and only if $\mathcal{A}$ restricted to $S^{1,0}(H)$ is injective. On the space $\hat{H}$ we define the *matrix norm induced metrics* as follows: For every $1 \leq p \leq \infty$ and $x, y \in H$,

$$d_p(\hat{x}, \hat{y}) = \|[\![x,x]\!] - [\![y,y]\!]\|_p = \begin{cases} \left(\sum_{k=1}^n (\sigma_k)^p\right)^{1/p} & for \quad 1 \leq p < \infty \\ \max_{1\leq k\leq n} \sigma_k & for \quad p = \infty \end{cases} \tag{2.4}$$

where $(\sigma_k)_{1\leq k\leq n}$ are the singular values of the matrix $[\![x,x]\!] - [\![y,y]\!]$, which is of rank at most 2. In particular, for $p = 1$, d_1 corresponds to the nuclear norm $\|\cdot\|_1$ in $Sym(H)$ (the sum of singular values); for $p = \infty$, d_∞ corresponds to the operator norm $\|\cdot\|_\infty$ in $Sym(H)$ (the largest singular value). In the following parts, when no subscript is used, $\|\cdot\| = \|\cdot\|_2$.

In previous papers [**Ba4, BW**] we showed a result that is equivalent to the following theorem:

THEOREM 2.1. *If $\mathcal{F}$ is phase retrievable, then there exist constants a_0, $b_0 > 0$ such that for every x, $y \in H$,*

$$\sqrt{a_0} d_1(x, y) \leq \|\alpha(x) - \alpha(y)\| \leq \sqrt{b_0} d_1(x, y) \tag{2.5}$$

i.e. α is bi-Lipschitz between $(\hat{H}, d_1)$ and $(\mathbb{R}^m, \|\cdot\|)$.

Consequently, the inverse map defined on the range of α from metric space $(\alpha(\hat{H}), \|\cdot\|)$ to $(\hat{H}, d_1)$:

$$\tilde{\omega} : \alpha(\hat{H}) \subset \mathbb{R}^m \to \hat{H}\ ,\ \tilde{\omega}(c) = x \text{ if } \alpha(x) = c \tag{2.6}$$

is Lipschitz and its Lipschitz constant is bounded by $\frac{1}{\sqrt{a_0}}$.

Now we state the main result of this paper:

THEOREM 2.2. *Let $\mathcal{F} = \{f_1, \cdots, f_m\}$ be a phase retrievable frame for the n-dimensional Hilbert space H, and let $\alpha : \hat{H} \to \mathbb{R}^m$ denote the injective nonlinear analysis map $\alpha(x) = (|\langle x, f_k\rangle|^2)_{1\leq k\leq m}$. Then there exists a Lipschitz continuous function $\omega : \mathbb{R}^m \to \hat{H}$ such that $\omega(\alpha(x)) = x$ for all $x \in \hat{H}$. ω has a Lipschitz constant $Lip(\omega)$ between $(\mathbb{R}^m, \|\cdot\|_2)$ and $(\hat{H}, d_1)$ bounded by*

$$Lip(\omega) \leq \frac{12}{\sqrt{a_0}} \tag{2.7}$$

The proof of Theorem 2.2, presented in the next section, requires construction of a special Lipschitz map. We believe this particular result is interesting in itself and may be used in other constructions. Due to its importance we state it here:

LEMMA 2.3. *Consider the spectral decomposition of any self-adjoint operator in $Sym(H)$, $A = \sum_{k=1}^d \lambda_{m(k)} P_k$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the n eigenvalues including multiplicities, and P_1,...,P_d are the orthogonal projections associated to* *the d distinct eigenvalues. Additionally, $m(1) = 1$ and $m(k+1) = m(k) + r(k)$, where $r(k) = rank(P_k)$ is the multiplicity of eigenvalue $\lambda_{m(k)}$. Then the map*

$$\pi : Sym(H) \to S^{1,0}(H)\ ,\ \pi(A) = (\lambda_1 - \lambda_2)P_1 \tag{2.8}$$

satisfies the following two properties:

(1) *π is Lipschitz continuous from $(Sym(H), \|\cdot\|_\infty)$ to $(S^{1,0}(H), \|\cdot\|_\infty)$ with Lipschitz constant less than or equal to 6;*
(2) *$\pi(A) = A$ for all $A \in S^{1,0}(H)$.*

The estimates of Theorem 2.2 and Lemma 2.3 are not optimal. In a separate publication [**BZ**] we improve it and extend the estimates to other metrics.

3. Proof of Results

The proof of Theorem 2.2 requires the Kirszbraun Theorem (see, e.g. [**WW**], Ch.10-11). Kirszbraun Theorem applies when two metric spaces have the following property:

Definition 3.1 (Kirszbraun Property (K)). Let X and Y be two metric spaces with metric d_x and d_y respectively. (X, Y) is said to have Property (K) if for any pair of families of closed balls $\{B(x_i, r_i) : i \in I\}$, $\{B(y_i, r_i) : i \in I\}$, such that $d_y(y_i, y_j) \leq d_x(x_i, x_j)$ for each $i, j \in I$, it holds that $\bigcap_{i\in I} B(x_i, r_i) \neq \emptyset \Rightarrow \bigcap_{i\in I} B(y_i, r_i) \neq \emptyset$.

Kirszbraun Theorem states the following:

Theorem 3.2 (Kirszbraun Theorem). *Let X and Y be two metric spaces and (X, Y) has Property (K). Suppose U is a subset of X and $f : U \to Y$ is a Lipschitz map. Then there exists a Lipschitz map $F : X \to Y$ which extends f to X and $Lip(F) = Lip(f)$. In particular, (X, Y) has Property (K) if X and Y are Hilbert spaces and d_X, d_Y are the correspondingly induced metrics.*

Note that we cannot use the Kirszbraun Theorem directly to extend $\tilde{\omega}$. Specifically, our pair of spaces $(\mathbb{R}^m, \hat{H})$ does not satisfy the Kirszbraun Property. We give the following counterexample.

Example 3.3. Let $X = \mathbb{R}^m$ for any $m \in \mathbb{N}$ and $Y = \hat{H}$ with $H = \mathbb{C}^2$. We want to show that (X, Y) does not have Property (K). Let $\tilde{y}_1 = (1, 0)$ and $\tilde{y}_2 = (0, \sqrt{3})$ be representitives of y_1, $y_2 \in Y$, respectively. Then $d_1(y_1, y_2) = 4$. Pick any two points x_1, x_2 in X with $\|x_1 - x_2\| = 4$. Then $B(x_1, 2)$ and $B(x_2, 2)$ intersect at $x_3 = (x_1 + x_2)/2 \in X$. It suffices to show that the closed balls $B(y_1, 2)$ and $B(y_2, 2)$ have no intersection in H. Assume on the contrary that the two balls intersect at y_3, then pick a representive of y_3, say $\tilde{y}_3 = (a, b)$ where $a, b \in \mathbb{C}$. It can be computed that

$$d_1(y_1, y_3) = |a|^4 + |b|^4 - 2|a|^2 + 2|b|^2 + 2|a|^2|b|^2 + 1 \tag{3.1}$$

and

$$d_1(y_2, y_3) = |a|^4 + |b|^4 + 6|a|^2 - 6|b|^2 + 2|a|^2|b|^2 + 9 \tag{3.2}$$

Set $d_1(y_1, y_3) = d_1(y_2, y_3) = 2$. Take the difference of the right hand side of (3.1) and (3.2), we have $|b|^2 - |a|^2 = 1$ and thus $|b|^2 \geq 1$. However, the right hand side of (3.1) can be rewritten as $(|a|^2 + |b|^2 - 1)^2 + 4|b|^2$, so $d_1(y_1, y_3) = 2$ would imply that $|b|^2 \leq 1/2$. This is a contradiction.

We start with the proof of Lemma 2.3.

PROOF OF LEMMA 2.3. We prove (1) only. (2) follows directly from the expression of π.

Let A, $B \in Sym(H)$ where $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$ is the spectral decomposition as stated in the lemma and $B = \sum_{k'=1}^{d'} \mu_{m(k')} Q_{k'}$ is a decomposition in the same manner. We now show that

$$\|\pi(A) - \pi(B)\|_\infty \leq 6\,\|A - B\|_\infty \tag{3.3}$$

Assume $\lambda_1 - \lambda_2 \leq \mu_1 - \mu_2$. Otherwise switch the notations for A and B. If $\mu_1 - \mu_2 = 0$ then $\pi(A) = \pi(B) = 0$ and the inequality (3.3) is satisfied. Assume now $\mu_1 - \mu_2 > 0$. Thus Q_1 is of rank 1 and therefore $\|Q_1\|_\infty = 1$.

First note thats

$$\begin{aligned} \pi(A) - \pi(B) &= (\lambda_1 - \lambda_2)P_1 - (\mu_1 - \mu_2)Q_1 \\ &= (\lambda_1 - \lambda_2)(P_1 - Q_1) + (\lambda_1 - \mu_1 - (\lambda_2 - \mu_2))Q_1 \end{aligned} \tag{3.4}$$

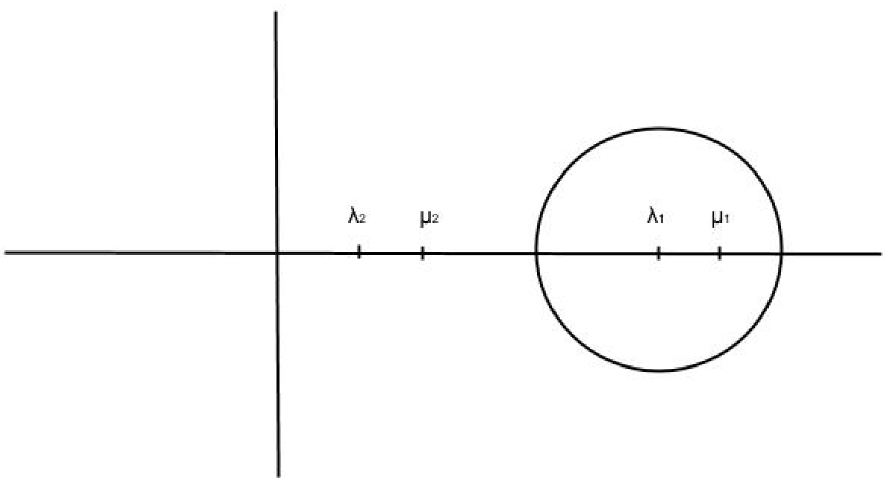
Here $\|P_1\|_\infty = \|Q_1\|_\infty = 1$. Therefore we have $\|P_1 - Q_1\|_\infty \leq 1$ since P_1, Q_1 are both positive semidefinite.

Also, by Weyl's inequality (see [**Bh**] III.2) we have $|\lambda_i - \mu_i| \leq \|A - B\|_\infty$ for each i. Apply this to $i = 1, 2$ we get $|\lambda_1 - \mu_1 - (\lambda_2 - \mu_2)| \leq |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2| \leq 2\,\|A - B\|_\infty$. Thus $|\lambda_1 - \mu_1| + |\lambda_2 - \mu_2| \leq 2\,\|A - B\|_\infty$.

Let $g := \lambda_1 - \lambda_2$, $\delta := \|A - B\|_\infty$, then apply the above inequality to (3.4) we get

$$\|\pi(A) - \pi(B)\|_\infty \leq g\,\|P_1 - Q_1\|_\infty + 2\delta \leq g + 2\delta \tag{3.5}$$

If $0 \leq g \leq 4\delta$, then $\|\pi(A) - \pi(B)\|_\infty \leq 6\delta$ and we are done. Now we consider the case where $g > 4\delta$. In the complex plane, let $\gamma = \gamma(t)$ be the (directed) circle centered at λ_1 with radius $g/2$. Since $\delta < g/4$ we have $|\lambda_1 - \mu_1| < g/4$ and $|\lambda_2 - \mu_2| < g/4$. Therefore the contour encloses μ_1 but not μ_2.



Using holomorphic calculus, we can put

$$P_1 = -\frac{1}{2\pi i} \oint_\gamma R_A \, dz \tag{3.6}$$

and

$$Q_1 = -\frac{1}{2\pi i}\oint_\gamma R_B\ dz \tag{3.7}$$

where $R_A = (A - zI)^{-1}$ and $R_B = (B - zI)^{-1}$.

Now we have

$$\begin{aligned} P_1 - Q_1 &= \frac{1}{2\pi i}\oint_\gamma (R_B - R_A)\ dz \\ &= \frac{1}{2\pi i}\oint_\gamma R_A\,(B-A)\,R_B\ dz \end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned} \|P_1 - Q_1\|_\infty &\le \frac{1}{2\pi}\cdot 2\pi\cdot\frac{g}{2}\cdot\max_z \|A - zI\|_\infty \|B-A\|_\infty \|B - zI\|_\infty \\ &= \frac{g\delta}{2}\cdot\max_z \max\left\{\frac{1}{|\lambda_1 - z|},\frac{1}{|\lambda_2 - z|}\right\}\cdot \\ &\qquad \max_z \max\left\{\frac{1}{|\mu_1 - z|},\frac{1}{|\mu_2 - z|}\right\} \\ &= \frac{g\delta}{2}\cdot\frac{2}{g}\cdot\frac{4}{g} \\ &= \frac{4\delta}{g} \end{aligned} \tag{3.9}$$

Thus by the first inequality in (3.5) we have

$$\|\pi(A) - \pi(B)\|_\infty \le 4\delta + 2\delta = 6\delta \tag{3.10}$$

Therefore, we have proved (3.3).

□

REMARK 3.4. Using the integration contour from [**ZB**], one can derive a slightly stronger bound. We plan to present this result in [**BZ**].

REMARK 3.5. Numerical experiments seem to suggest that the optimal Lipschitz constant in Lemma 2.3 is 2.

Now we are ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. We construct a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|) \to (\hat{H}, d_1)$ such that $\omega(\alpha(x)) = x$ for all $x \in \hat{H}$ and $Lip(\omega) \le 12/\sqrt{a_0}$.

Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$. By hypothesis, there is a map $\tilde{\omega}_1 : M \to \hat{H}$ that is Lipschitz continuous and satisfies $\tilde{\omega}_1(\alpha(x)) = x$ for all $x \in \hat{H}$. Additionally, the Lipschitz bound between $(M, \|\cdot\|)$ (that is, M with Euclidian distance) and $(\hat{H}, d_1)$ is given by $1/\sqrt{a_0}$.

First we change the metric on $\hat{H}$ from d_1 to d_2 and embed isometrically $\hat{H}$ into $Sym(H)$ with Frobenius norm (i.e. Euclidian metric):

$$(M, \|\cdot\|) \xrightarrow{\tilde{\omega}_1} (\hat{H}, d_1) \xrightarrow{i_{1,2}} (\hat{H}, d_2) \xrightarrow{\kappa} (Sym(H), \|\cdot\|_{Fr}) \tag{3.11}$$

where $i_{1,2}(x) = x$ is the identity of $\hat{H}$ and κ is the isometry given by

$$\kappa : \hat{H} \to S^{1,0}(H), \quad x \mapsto [\![x, x]\!] \tag{3.12}$$

Obviously we have $Lip(i_{1,2}) = 1$ and $Lip(\kappa) = 1$. Thus we obtain a map $\tilde{\omega}_2 : (M, \|\cdot\|) \to (Sym(H), \|\cdot\|_{Fr})$ of Lipschitz constant

$$Lip(\tilde{\omega}_2) \leq Lip(\tilde{\omega}_1)Lip(i_{1,2})Lip(\kappa) = \frac{1}{\sqrt{a_0}} \tag{3.13}$$

Kirszbraun Theorem (Theorem 3.2) extends isometrically $\tilde{\omega}_2$ from M to the entire $\mathbb{R}^m$ with Euclidian metric $\|\cdot\|$. Thus we obtain a Lipschitz map $\omega_2 : (\mathbb{R}^m, \|\cdot\|) \to (Sym(H), \|\cdot\|_{Fr})$ of Lipschitz constant $Lip(\omega_2) = Lip(\tilde{\omega}_2) \leq 1/\sqrt{a_0}$ such that $\omega_2(\alpha(x)) = [\![x, x]\!]$ for all $x \in \hat{H}$.

Now we consider the following maps:

$$\begin{aligned}
(\mathbb{R}^m, \|\cdot\|) &\xrightarrow{\omega_2} (Sym(H), \|\cdot\|_{Fr}) \\
&\xrightarrow{I_{2,\infty}} (Sym(H), \|\cdot\|_{\infty}) \\
&\xrightarrow{\pi} (S^{1,0}(H), \|\cdot\|_{\infty}) \\
&\xrightarrow{\kappa^{-1}} (\hat{H}, d_{\infty}) \\
&\xrightarrow{i_{\infty,1}} (\hat{H}, d_1)
\end{aligned} \tag{3.14}$$

where $I_{2,\infty}$ and $i_{\infty,1}$ are identity maps that change the metrics. The map ω is defined by

$$\omega : (\mathbb{R}^m, \|\cdot\|) \to (\hat{H}, d_1), \quad \omega = i_{\infty,1} \cdot \kappa^{-1} \cdot \pi \cdot I_{2,\infty} \cdot \omega_2 \tag{3.15}$$

The Lipschitz constant is bounded by

$$\begin{aligned}
Lip(\omega) &\leq Lip(\omega_2)Lip(I_{2,\infty})Lip(\pi)Lip(\kappa^{-1})Lip(i_{2,1}) \\
&\leq \frac{1}{\sqrt{a_0}} \cdot 1 \cdot 6 \cdot 1 \cdot 2 \\
&= \frac{12}{\sqrt{a_0}}
\end{aligned} \tag{3.16}$$

Hence we obtained (2.7).

□

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