Contemporary Mathematics
Volume **650**, 2015
http://dx.doi.org/10.1090/conm/650/13042

Tight and random nonorthogonal fusion frames

Jameson Cahill, Peter G. Casazza, Martin Ehler, and Shidong Li

Abstract. Nonorthogonal fusion frames were introduced in **Cahill, Casazza, and Li** [**4**] to provide a general method for constructing sparse and/or tight fusion frames. In this paper we will resolve some of the fundamental questions left open in **Cahill, Casazza, and Li** [**4**]. First we show that tight nonorthogonal fusion frames are surprisingly easy to construct. In order to do this we will establish classifications of when and how to write certain self adjoint operators as a product (or sum) of (nonorthogonal) projection operators. We also discuss random versions of nonorthogonal fusion frames, derive constructions based on orbits of irreducible finite subgroups of the unitary group, and study the fusion frame potential in the nonorthogonal setting.

1. Introduction

Fusion frames were introduced in [**7**] and further developed in [**8**]. Recently there has been much activity around the idea of fusion frames, see [**6**] and references therein or go to the *Fusion Frame* website: www.fusionframes.org. Loosely speaking, a fusion frame is a collection of subspaces $\{W_i\}_{i=1}^m$ all contained in some bigger Hilbert space $\mathcal{H}$ such that any signal $f \in \mathcal{H}$ can be stably reconstructed from the set of orthogonal projections $\{\pi_i f\}_{i=1}^m$, where π_i denotes the orthogonal projection from $\mathcal{H}$ onto W_i. Typically we think of the dimension of each subspace W_i as being much smaller than the dimension of $\mathcal{H}$ so that a high dimensional signal f can be reconstructed from several low dimensional measurements $\{\pi_i f\}_{i=1}^m$.

In [**4**], the authors introduced *nonorthogonal fusion frames* in order to achieve sparsity of the fusion frame operator. The basic observation in [**4**] is that replacing orthogonal projections π_i in the original definition of fusion frames [**8**] by nonorthogonal projections P_i onto the same subspaces W_i can result in a fusion frame operator which is much sparser. This is because, for example, one can always choose the null space of the projection P_i to contain some basis elements $\{e_{i_j}\}_j$ that are complementary to the subspace W_i, and thereby nullify some columns of P_i, which in turn results in sparsity of the (new) fusion frame operator. One further observation which was made in [**4**] but was not explored very thoroughly there is that tight nonorthogonal fusion frames are much more abundant than tight (orthogonal) fusion frames. This is important since it was shown in [**9**] that there

2010 *Mathematics Subject Classification.* Primary 15:42C15.

Key words and phrases. Fusion frame, projection.

The first and seconde authors were supported by NSF 1307685; and NSF ATD 1042701; AFOSR DGE51: FA9550-11-1-0245.

are very few tight fusion frames in general. In this paper, constructions of tight nonorthogonal fusion frames and nonorthogonal fusion frames of a prescribed fusion frame operator are provided.

We now give a formal definition of nonorthogonal fusion frames. Throughout this paper, let $\mathcal{H}_n$ denote an n-dimensional Hilbert space.

Definition 1.1. *An operator $P : \mathcal{H}_n \to \mathcal{H}_n$ is called a projection if $P^2 = P$. If in addition we have $P^* = P$ then P is called an orthogonal projection.*

Definition 1.2. *Let $\{P_i\}_{i=1}^m$ be a collection of projections on $\mathcal{H}$ and $\{v_i\}_{i=1}^m$ a collection of positive real numbers. We say $\{(P_i, v_i)\}_{i=1}^m$ is a nonorthogonal fusion frame for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{i=1}^m v_i^2 \|P_i f\|^2 \le B\|f\|^2$$

for every $f \in \mathcal{H}$. We say it is tight if $A = B$.

Given a nonorthogonal fusion frame we define the nonorthogonal fusion frame operator $S : \mathcal{H}_n \to \mathcal{H}_n$ by

$$Sf = \sum_{i=1}^m v_i^2 P_i^* P_i f. \tag{1}$$

We observe that $\{(v_i, P_i)\}$ is tight if and only if $S = \lambda I$ (where $\lambda = A = B$). Therefore, much of this paper is devoted to studying ways of writing multiples of the identity in the form of the right hand side of equation (1). We will also usually assume that $v_i = 1$ for every $i = 1, \dots, m$

Before leaving the introduction we collect some basic facts about projections and fix some notation that will be used throughout this paper.

Proposition 1.3. *Let P be a projection on $\mathcal{H}$, $W = \text{im}P$, $W^* \equiv (\ker P)^\perp = [(I-P)(\mathcal{H})]^\perp$. Denote by P^* the adjoint of P in $\mathcal{H}$. Then:*

(1) $(P^*)^2 = P^*$, and $\text{im}P^* = W^*$, $\ker P^* = W^\perp$.

(2) $W^* = \text{im}P^* = \text{im}P^*P$.

(3) P is an invertible operator mapping W^* onto W.

(4) $\dim(W) = \dim(W^*)$.

Corollary 1.4. *Given subspaces W, W^* of $\mathcal{H}$ with dim W = dim W^*, there is a projection P onto W with $P^*P(\mathcal{H}) = W^*$ if and only if $(W^*)^\perp \cap W = \{0\}$.*

Notation 1.5. Throughout this paper we will always use the notation of Proposition 1.3; *i.e.*, P will always stand for a (nonorthogonal) projection, W will always be the image of P, and W^* will always be the image of P^*. Furthermore, we will always use the symbol π_W to denote the *orthogonal* projection onto the subspace $W \subseteq \mathcal{H}_n$.

Remark 1.6. We will be working with finite dimensional Hilbert spaces. Some of the results here can be generalized to infinite dimensions but we have chosen not to cover this case.

2. Classification of self adjoint operators via projections

Let $T : \mathcal{H}_n \to \mathcal{H}_n$ be a positive, self adjoint, linear operator. The main result of this section is to classify the set

$$\Omega(T) = \{P : P^2 = P, P^*P = T\}.$$

Armed with this result and its consequences, in the next section we will show how to construct large families of sparse and tight non-orthogonal fusion frames. The spectral theorem tells us that $T = \sum_{j=1}^{n} \lambda_j \pi_j$ where the λ_j's are the eigenvalues of T and π_j is the orthogonal projection onto the one dimensional span of the jth eigenvector of T. Therefore $P \in \Omega(T)$ if and only if P^*P has the same eigenvalues and eigenvectors as T. Also note that if $P \in \Omega(T)$ then $\ker(P) = \operatorname{im}(T)^{\perp}$, and since a projection is uniquely determined by its kernel and its image we have a natural bijection between $\Omega(T)$ and the set

$$\tilde{\Omega}(T) := \{W \subseteq \mathcal{H}_n : \operatorname{im}(P) = W \text{ for some } P \in \Omega(T)\}.$$

given by

$$\Omega(T) \ni P \mapsto \operatorname{im}(P) \in \tilde{\Omega}(T).$$

We start with two elementary lemmas.

LEMMA 2.1. *Let P be a projection and let $\{e_j\}_{j=1}^{k}$ be an orthonormal basis of W^* consisting of eigenvectors of P^*P with corresponding nonzero eigenvalues $\{\lambda_j\}$. Then $\{Pe_j\}_{j=1}^{k}$ is an orthogonal basis for W and $\|Pe_j\| = \sqrt{\lambda_j}$.*

PROOF. Just observe that $\langle Pe_j, Pe_\ell\rangle = \langle P^*Pe_j, e_\ell\rangle = \lambda_j\langle e_j, e_\ell\rangle$. ☐

LEMMA 2.2. *Let P be a projection and suppose λ is an eigenvalue of P^*P, $\lambda \neq 0$. Then $\lambda \geq 1$. Moreover, $\lambda = 1$ if and only if the corresponding eigenvector is in $W \cap W^*$.*

PROOF. Note that $W^* = \operatorname{im} P^*P$, so all eigenvectors of P^*P corresponding to nonzero eigenvalues are in W^*. Let $x \in W^*$ and write $Px = x + (P - I)x$. Since $x \perp (I - P)x$,

$$\|Px\|^2 = \|x\|^2 + \|(P - I)x\|^2 \geq \|x\|^2. \tag{2}$$

By the same argument on P^* we get $\|P^*Px\| \geq \|Px\| \geq \|x\|$ for all $x \in W^*$. Therefore, if $P^*Px = \lambda x$ we have that $\lambda \geq 1$.

Finally, by equation (2), $\lambda = 1$ if and only if $(I - P)x = 0$, or $x = Px \in W$. Hence $x \in W \cap W^*$. ☐

The next proposition allows us reduce our problem to the case when $\operatorname{rank}(T) \leq n/2$.

PROPOSITION 2.3. *Let P be a projection, then we can write*

$$P = P' + \pi_{W\cap W^*}$$

where $\pi_{W\cap W^*}$ is the orthogonal projection onto $W \cap W^*$, and P' is a projection such that all nonzero eigenvalues of P'^*P' are strictly greater than 1.

PROOF. First note that Lemma 2.2 says that $W \cap W^* = \{x : P^*Px = x\}$. Now let W' be the orthogonal complement of $W \cap W^*$ in W and let P' be the projection onto W' along $\ker(P) + W \cap W^*$. Then $P'\pi_{W\cap W^*} = \pi_{W\cap W^*}P' = 0$, so $(P' + \pi_{W\cap W^*})^2 = P'^2 + \pi_{W\cap W^*}^2 = P' + \pi_{W\cap W^*}$. It is clear that $\operatorname{im}(P' + \pi_{W\cap W^*}) =$

W. Since $\ker P = W^{*\perp} \subseteq (W \cap W^*)^\perp$ it follows that $\ker(P) \subseteq \ker(P' + \pi_{W\cap W^*})$ so we must have $\ker(P) = \ker(P' + \pi_{W\cap W^*})$. Therefore $P = P' + \pi_{W\cap W^*}$, and the nonzero eigenvalues of P'^*P' are precisely the nonzero eigenvalues of P^*P which are greater than 1. $\square$

We can now state the main theorem of this section:

THEOREM 2.4. *Let $T : \mathcal{H}_n \to \mathcal{H}_n$ be a positive, self-adjoint operator of rank $k \leq \frac{n}{2}$. Let $\{\lambda_j\}_{j=1}^k$ be the nonzero eigenvalues of T and suppose $\lambda_j \geq 1$ for $i = 1, \ldots, k$ and let $\{e_j\}_{j=1}^k$ be an orthonormal basis of im(T) consisting of eigenvectors of T. Then*

$$\tilde{\Omega}(T) = \{span\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}\} : \{e_j\}_{j=1}^{2k} \text{ is orthonormal}\}.$$

PROOF. First suppose $W \in \tilde{\Omega}(T)$ and let P be the projection onto W along $\mathrm{im}(T)^\perp$. By Lemma 2.1 we know that $\{\frac{Pe_j}{\|Pe_j\|}\}_{j=1}^k$ is an orthonormal basis for W. We also know that $\|Pe_j\| = \sqrt{\lambda_j}$ for $j = 1, 2, \ldots, k$ and that e_i is in the range of P^* which is orthogonal to $ker\ P = Im\ (I - P)$. So

$$\langle e_i, (P - I)e_i\rangle = \langle (P - I)^* e_i, e_i\rangle \langle 0, e_i\rangle = 0.$$

It follows that

$$\lambda_j = \|e_j\|^2 + \|(P - I)e_j\|^2 = 1 + \|(P - I)e_j\|^2$$

which means

$$\|(P - I)e_j\| = \sqrt{\lambda_j - 1}$$

so if we set for $1 \leq j \leq k$,

$$e_{j+k} = \frac{(P - I)e_j}{\sqrt{1 - \lambda_j}},$$

then $\{e_j\}_{j=1}^{2k}$ is an orthonormal set and

$$\frac{Pe_j}{\|Pe_j\|} = \frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}.$$

Conversely suppose $W = \mathrm{span}\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}\}_{j=1}^k$ with $\{e_j\}_{j=1}^{2k}$ orthonormal. Let P be the projection onto W along $\mathrm{im}(T)^\perp$. Notice that for $1 \leq j \leq k$, $e_j = e_j + \sqrt{\lambda_j - 1}e_{j+k} - \sqrt{\lambda_j - 1}e_{j+k}$ with $e_j + \sqrt{\lambda_j - 1}e_{j+k} \in W$ and $-\sqrt{\lambda_j - 1}e_{j+k} \in \mathrm{im}(T)^\perp$, so $Pe_j = e_j + \sqrt{\lambda_j - 1}e_{j+k}$ for $j = 1, \ldots, k$. Similarly $e_j + \sqrt{\lambda_j - 1}e_{j+k} = \lambda_j e_j + (1 - \lambda_j)e_j + \sqrt{\lambda_j - 1}e_{j+k}$ with $\lambda_j e_j \in W^* = \mathrm{im}P^*$ and $(1 - \lambda_j)e_j + \sqrt{\lambda_j - 1}e_{j+k} \in W^\perp = \ker(P^*)$, so $P^*Pe_j = \lambda_j e_j$ for $j = 1, \ldots, k$. Therefore, P^*P has the same eigenvectors and corresponding eigenvalues as T, so $P^*P = T$, and $W \in \tilde{\Omega}(T)$. $\square$

Before proceeding we remark that Theorem 2.4 is independent of our choice of eigenbasis for T. To see this let $\{e'_j\}_{j=1}^k$ be any other eigenbasis for T and let $W = \mathrm{span}\{\frac{1}{\sqrt{\lambda_j}}e'_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e'_{j+k}\}_{j=1}^k$ with $\{e'_j\}_{j=1}^{2k}$ orthonormal. By the second part of the proof of Theorem 2.4 we have that $W \in \tilde{\Omega}(T)$, and so by the first part of the proof we have that in fact $W = \text{span}\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j-1}{\lambda_j}}e_{j+k}\}_{j=1}^{k}$ with $\{e_j\}_{j=1}^{2k}$ orthonormal.

We now state several consequences of Theorem 2.4. The first corollary appeared first in [**12**].

COROLLARY 2.5. *If T is a positive self-adjoint operator of rank $\leq \frac{n}{2}$ with all nonzero eigenvalues ≥ 1, then there is a projection P so that $T = P^*P$.*

COROLLARY 2.6. *If T is a positive self-adjoint operator of rank $\leq \frac{n}{2}$, then there is a projection P and a weight $v > 0$ so that $T = v^2P^*P$.*

PROOF. Let λ_k be the smallest non-zero eigenvalue of T. So all nonzero eigenvalues of $\frac{1}{\lambda_k}T$ are greater than or equal to 1 and by Corollary 2.5 there is a projection P so that $P^*P = \frac{1}{\lambda_k}T$. Let $v = \sqrt{\lambda_k}$ to finish the proof. □

In the rest of this section we will analyze the case where $\text{rank}(T) > n/2$. Our first proposition can be found in [**12**]. But we feel our proof is much more instructive and visual, so we include it here.

PROPOSITION 2.7. *Let T be a positive self-adjoint operator of rank $k > \frac{n}{2}$ with eigenvectors $\{e_j\}_{j=1}^n$ and respective eigenvalues $\{\lambda_j\}_{j=1}^n$. The following are equivalent:*

(1) There is a projection P so that $T = P^*P$.

(2) The nonzero eigenvalues of T are greater than or equal to 1 and we have

$$|\{j : \lambda_j > 1\}| \leq |\{j : \lambda_j = 0\}|.$$

In particular,

$$|\{j : \lambda_j = 1\}| \geq k - \left\lfloor \tfrac{n}{2} \right\rfloor .$$

PROOF. Let $A_1 = \{j : \lambda_j > 1\}$, $A_2 = \{j : \lambda_j = 0\}$, and $A_3 = \{j : \lambda_j = 1\}$, and let π_i be the orthogonal projection onto $\text{span}\{e_j : j \in A_i\}$ for $i = 1, 2, 3$.

(1) ⇒ (2): By Proposition 2.3, we can write

$$P = P' + \pi_{W \cap W^*},$$

where $\pi_{W\cap W^*}$ is the orthogonal projection onto $W \cap W^*$, and P' is the projection onto the orthogonal complement W' of $W \cap W^*$ in W along $ker\ P + W \cap W^*$. Define $W'^* \equiv \text{im}\, P'^*$. Then P' is an invertible operator from W'^* onto W', $W'^* \perp W \cap W^*$ and $W' \perp W \cap W^*$, and $W' \cap W'^* = \{0\}$. Hence,

$$\begin{aligned} 2 \dim W'^* &= \dim W' + \dim W'^* \\ &= \dim(W' + W'^*) \\ &\leq \dim W'^* + \dim \text{span}\{e_j : j \in A_2\}. \end{aligned}$$

Since $W'^* = span\ \{e_j : j \in A_1\}$, it follows that $|A_1| \leq |A_2|$.

(2) ⇒ (1): Let $T_1 = T(\pi_1 + \pi_2)$, so $T = T_1 + \pi_3$. By our assumption

$$\text{rank } T_1 \leq \frac{n}{2},$$

and all non-zero eigenvalues of T_1 are strictly greater than 1. By Theorem 2.4 there is a projection P' so that $P'^*P' = T_1$. Let $P = P' + \pi_3$. Then $P'\pi_3 = \pi_3 P' = 0$. Hence, $P = P^2$ is a projection and

$$P^*P = P'^*P' + \pi_3 = T_1 + \pi_3 = T.$$

□

We now will see that in general it may happen that $\Omega(T) = \emptyset$.

Corollary 2.8. *If rank$(T) = k > \frac{n}{2}$ and T does not have 1 as an eigenvalue with multiplicity at least $k - \lfloor \frac{n}{2} \rfloor$, then $\Omega(T) = \emptyset$.*

Remark 2.9. Similar to the proof of Corollary 2.6, if T is a positive self-adjoint operator of rank $> \frac{n}{2}$ with eigenvalues $\{\lambda_1 \geq \cdots \geq \lambda_k > 0 = \lambda_{k+1} = \cdots \lambda_n\}$, then there is a projection P and a weight $v = \sqrt{\lambda_k}$ so that $T = v^2 P^* P$ if and only if

$$|\{j : \lambda_j > \lambda_k\}| \leq |\{j : \lambda_j = 0\}|.$$

Proposition 2.10. *Let $T : \mathcal{H}_n \to \mathcal{H}_n$ be a positive, self adjoint operator of rank $k > \frac{n}{2}$ with eigenvalues $\{\lambda_j\}_{j=1}^n$ and whose nonzero eigenvalues are all greater than or equal to 1. If either*

(1) n is even, or

(2) n is odd and T has at least one eigenvalue in the set $\{0, 1, 2\}$

then there are two projections P_1 and P_2 such that $T = P_1^* P_1 + P_2^* P_2$.

Proof. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of $\mathcal{H}_n$ consisting of eigenvectors of T with respective eigenvalues $\{\lambda_j\}_{j=1}^n$, in decreasing order.

Case 1: n is even.

Let $V = \text{span}\{e_j\}_{j \in I}$, $|I| = \frac{n}{2}$. Note that $T = T\pi_V + T\pi_{V^\perp}$. Also, since T, π_V, and $\pi_{V^\perp}$ are all diagonal with respect to $\{e_j\}_{j=1}^n$ it follows that T commutes with both π_V and $\pi_{V^\perp}$. Therefore $(T\pi_V)^* = \pi_V^* T^* = \pi_V T = T\pi_V$, so by Theorem 2.4 there is a projection P_1 such that $T\pi_V = P_1^* P_1$. Similarly we can find a projection P_2 such that $T\pi_{V^\perp} = P_2^* P_2$.

Case 2: n is odd and T has an eigenvalue in the set $\{0, 1, 2\}$.

We will look at the case for each eigenvalue separately.

Subcase 1: $\lambda_n = 0$.

Let $\mathcal{H}_1 = \text{span}\{e_j : 1 \leq j \leq n-1\}$. Then $\dim(\mathcal{H}_1)$ is even so we can apply the same argument as above to $\mathcal{H}_1$.

Subcase 2: $\lambda_n = 1$.

Define T_1, T_2 by

$$T_1 e_j = \begin{cases} Te_j & \text{if } j = 1, 2, \ldots, \frac{n-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$T_2 e_j = \begin{cases} Te_j & \text{if } j = \frac{n-1}{2} + 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\text{rank}(T_1) = \text{rank}(T_2) = \frac{n-1}{2} < \frac{n}{2}$ so by Corollary 2.5, we can write

$$T_i = P_i^* P_i, \quad i = 1, 2.$$

Let π be the orthogonal projection of $\mathcal{H}_n$ onto $\text{span}\{e_n\}$ and let

$$Q = P_2 + \pi,$$

which is clearly a projection. Then we have

$$T = P_1^* P_1 + Q^* Q.$$

Subcase 3: $\lambda_j = 2$ for some j.

Without loss of generality, re-index $\{\lambda_j\}_{j=1}^n$ so that $\lambda_n = 2$. Define T_1, T_2, and π as above. As in the previous case, define two projections $\{P_i\}_{i=1}^2$ so that

$$T_i = P_i^* P_i.$$

Now let $Q_i = P_i + \pi$, $i = 1, 2$. Then

$$T = Q_1^* Q_1 + Q_2^* Q_2. \qquad \square$$

COROLLARY 2.11. *Let $T : \mathcal{H}_n \to \mathcal{H}_n$ be a positive, self adjoint operator of rank $k > \frac{n}{2}$. There is a weight v and projections P_1 and P_2 so that*

$$T = v^2 P_1^* P_1 + v^2 P_2^* P_2.$$

PROOF. Apply Proposition 2.10 to $\frac{1}{\lambda_k} T$ and set $v = \sqrt{\lambda_k}$. $\square$

It is important to note that, without weighting, we can always write every positive self-adjoint T as the sum of $P_i^* P_i$ with three projections.

COROLLARY 2.12. *If $T : \mathcal{H}_n \to \mathcal{H}_n$ is a positive, self adjoint operator of rank $k > \frac{n}{2}$ whose nonzero eigenvalues are all greater than or equal to 1, then there are projections $\{P_i\}_{i=1}^3$ so that*

$$T = P_1^* P_1 + P_2^* P_2 + P_3^* P_3.$$

PROOF. If n is even, we can write T as the sum of two projections. Suppose n is odd and let $\{e_j\}_{j=1}^n$ be an eigenbasis of T. Suppose $J_1 \cup J_2 \cup J_3 = \{1, \ldots, n\}$ with $|J_i| < \frac{n}{2}$ and let π_i be the orthogonal projection onto $\text{span}\{e_j : j \in J_i\}$ for $i = 1, 2, 3$. Then $T = T(\pi_1 + \pi_2 + \pi_3)$ and $T\pi_i$ satisfies Corollary 2.5 for each i. $\square$

3. Tight nonorthogonal fusion frames

As applications of the results of the previous section, in this section we address some issues regarding tight nonorthogonal fusion frames. The first theorem shows which sets of dimensions allow the existence of a tight nonorthogonal fusion frame. The corresponding problem for fusion frames has received considerable attention proven to be quite difficult, see [**20**], [**9**], and [**3**].

THEOREM 3.1. *Suppose $n_1 + \cdots + n_m \geq n$, $n_i \leq \frac{n}{2}$. Then there exists a tight nonorthogonal fusion frame $\{P_i\}_{i=1}^m$ ($v_i = 1$ for every i) for $\mathcal{H}_n$ such that rank$(P_i) = n_i$ for $i = 1, \ldots, m$.*

PROOF. Choose an orthonormal basis $\{e_j\}_{j=1}^n$ for $\mathcal{H}_n$ and choose a collection of subspaces $\{W_i\}_{i=1}^m$ such that:
1) $W_i = \text{span}\{e_j\}_{j \in J_i}$ with $|J_i| = n_i$ for each $i = 1, \ldots, m$, and
2) $W_1 + \cdots + W_m = \mathcal{H}_n$.
Let π_i be the orthogonal projection onto W_i and let $S = \sum_{i=1}^m \pi_i$. Observe that $I = S^{-1}S = \sum_{i=1}^m S^{-1}\pi_i$. Since each π_i is diagonal with respect to $\{e_j\}_{j=1}^n$ it follows that S^{-1} commutes with π_i, so $S^{-1}\pi_i$ is positive and self adjoint for every $i = 1, \ldots, m$. Let γ be the smallest nonzero eigenvalue of any $S^{-1}\pi_i$, then $\frac{1}{\gamma} S^{-1}\pi_i$

satisfies the hypotheses of Corollary 2.5 so there is a projection P_i so that $P_i^* P_i = \frac{1}{\gamma} S^{-1} \pi_i$, and we have

$$\sum_{i=1}^{m} P_i^* P_i = \frac{1}{\gamma} I.$$

□

Theorem 3.1 should be compared with Theorem 3.2.2 in [**20**]. Also note that the proof of Theorem 3.1 is constructive, cf. [**9**]. It was shown in [**9**] that this theorem fails for orthogonal projections for almost all choices of dimensions for the subspaces. The next theorem deals with adding projections to a given nonorthogonal fusion frame it order to get a tight nonorthogonal fusion frame. Somewhat surprisingly, this can always be achieved with only two projections.

THEOREM 3.2. *Let $\{P_i\}_{i=1}^m$ be projections on $\mathcal{H}_n$, $n \geq 2$. Then there are two projections $\{P_i\}_{i=m+1}^{m+2}$ and a λ so that*

$$\sum_{i=1}^{m+2} P_i^* P_i = \lambda I.$$

PROOF. Let

$$S = \sum_{i=1}^{m} P_i^* P_i,$$

and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of S. Let $\lambda = \lambda_1 + 1$ and let

$$T = \lambda I - S.$$

Then T is a positive self-adjoint operator with all of its eigenvalues ≥ 1 and at least one eigenvalue equal to one. By Proposition 2.10, we can find projections $\{P_i\}_{i=m+1}^{m+2}$ so that

$$T = P_{m+1}^* P_{m+1} + P_{m+2}^* P_{m+2}.$$

Thus,

$$\lambda I = S + T = \sum_{i=1}^{m+2} P_i^* P_i.$$

□

No such theorem exists for frames or regular (orthogonal) fusion frames. In general we need to add $n-1$ vectors to a frame in $\mathcal{H}_n$ in order to get a tight frame (see Proposition 2.1 in [**10**]). However, in this context Theorem 3.2 may be misleading, as the ranks of the projections we need to add could be quite large. The next result tells us how to deal with the case where we want small rank projections.

PROPOSITION 3.3. *If $\{P_i\}_{i=1}^m$ are projections on $\mathcal{H}_n$ and $k \leq \frac{n}{2}$, there are projections $\{Q_i\}_{i=1}^L$ with $L = \lceil \frac{n}{k} \rceil$ and rank$(Q_i) \leq k$, and a λ so that*

$$\sum_{i=1}^{m} P_i^* P_i + \sum_{j=1}^{L} Q_i^* Q_i = \lambda I.$$

PROOF. Let $S = \sum_{i=1}^{m} P_i^* P_i$ and assume S has eigenvectors $\{e_j\}_{j=1}^n$ with respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Partition the set $\{1, \ldots, n\}$ into sets $J_1, \ldots, J_L$ with $|J_\ell| \leq k$ for every $\ell = 1, \ldots, L$. Let π_ℓ denote the orthogonal projection onto $\mathrm{span}\{e_j\}_{j \in J_\ell}$. Set $\lambda = \lambda_1 + 1$ and let $T_\ell = (\lambda I - S)\pi_\ell$. Then each T_ℓ satisfies the hypotheses of Corollary 2.5 so choose any projection $Q_\ell \in \Omega(T_\ell)$. Now we have that

$$\begin{aligned} \sum_{i=1}^{M} P_i^* P_i + \sum_{\ell=1}^{L} Q_\ell^* Q_\ell &= S + \sum_{\ell=1}^{L} T_\ell \\ &= S + \lambda I - S = \lambda I. \end{aligned}$$

□

3.1. 2 projections. As an application of the results of the previous section we will give a complete description of when there are two projections $P_i : \mathcal{H}_n \to \mathcal{H}_n$, $i = 1, 2$ such that

$$P_1^* P_1 + P_2^* P_2 = \lambda I. \tag{3}$$

Let $W_1 = \mathrm{im}(P_1), W_1^* = \mathrm{im}(P_1^*), W_2 = \mathrm{im}(P_2), W_2^* = \mathrm{im}(P_2^*)$. We will examine this in several cases but first we make some general remarks. Note that if $x \in W_1^*$ such that $P_1^* P_1 x = \alpha x$ (for $\alpha \in \mathbb{R}$) then $P_2^* P_2 x = (\lambda - \alpha)x$, so there is an orthonormal bases $\{e_j\}_{j=1}^n$ consisting of eigenvectors of both $P_1^* P_1$ and $P_2^* P_2$. Furthermore, if $P_1^* P_1 x = 0$ then $P_2^* P_2 x = \lambda x$, so $\ker P_1 = W_1^{*\perp} \subseteq W_2^*$, and similarly $W_2^{*\perp} \subseteq W_1^*$.

It follows from (3) that $\mathrm{rank}(P_1) + \mathrm{rank}(P_2) \geq n$. We will examine the cases of equality and strict inequality separately.

PROPOSITION 3.4. *Suppose P_1 and P_2 are projections on $\mathcal{H}_n$ such that $P_1^* P_1 + P_2^* P_2 = \lambda I$ and that rank(P_1) + rank$(P_2) = n$. Then either rank$(P_1) \neq$ rank(P_2) and $\lambda = 1$ or rank(P_1) = rank$(P_2) = \frac{n}{2}$ and $\lambda \geq 1$.*

PROOF. First suppose without loss of generality that $\mathrm{rank}(P_1) = k > \mathrm{rank}(P_2)$. In this case we have that $k > \frac{n}{2}$, so $\dim(W_1 \cap W_1^*) \geq 2k - n > 0$. Then by Proposition 2.3 we have that $P_1 = P_1' + \pi_{W_1 \cap W_1^*}$ and $P_1^* P_1 + P_2^* P_2 = P_1'^* P_1' + \pi_{W_1 \cap W_1^*} + P_2^* P_2$. Let $x \in W_1 \cap W_1^*$, then $P'x = 0$, and since $x \notin W_2^*$ it follows that $P_2 x = 0$. Therefore $(P_1^* P_1 + P_2^* P_2)x = x$ which means $\lambda = 1$, both P_1 and P_2 are orthogonal projections, and $W_j^* = W_j$ $j = 1, 2$.

Now suppose that n is even, and $\dim(W_1) = \dim(W_2) = \frac{n}{2}$. In this case we have that $W_1^* = W_2^{*\perp}$, so it follows immediately that $P_1^* P_1 = \lambda \pi_{W_1^*}$ and $P_2^* P_2 = \lambda \pi_{W_2^*}$. □

PROPOSITION 3.5. *Suppose P_1 and P_2 are projections on $\mathcal{H}_n$ such that $P_1^* P_1 + P_2^* P_2 = \lambda I$ and that rank(P_1) + rank$(P_2) > n$. Then rank(P_1) = rank(P_2), $\lambda = 2$, and $W_1^* \cap W_1 = W_2^* \cap W_2$.*

PROOF. First suppose $\dim(W_1) = k > \ell = \dim(W_2)$. Note that $k > \frac{n}{2}$. By the remarks above we know that 0 must be an eigenvalue of $P_1^* P_1$ with multiplicity $n - k$, λ must be an eigenvalue of $P_1^* P_1$ with multiplicity $n - \ell$, and 1 must be an eigenvalue of $P_1^* P_1$ with multiplicity $\dim(W_1^* \cap W_2^*) \geq 2k - n$. Adding up these multiplicities we get $(n - k) + (n - \ell) + (2k - n) = n + k - \ell = n$ which contradicts the fact that $k > \ell$. Therefore, we may assume that $\dim(W_1) = \dim(W_2)$.

By the remarks above we can choose an orthonormal basis $\{e_j\}_{j=1}^n$ of $\mathcal{H}_n$ so that

$$\begin{aligned} P_1^* P_1 e_j &= \lambda e_j \text{ and } P_2^* P_2 e_j = 0 \text{ for } j = 1, \ldots, n-k, \\ P_1^* P_1 e_j &= 0 \text{ and } P_2^* P_2 e_j = \lambda e_i \text{ for } j = k+1, \ldots, n. \end{aligned}$$

Since $\dim(W_1 \cap W_1^*), \dim *(W_2 \cap W_2^*) \geq 2k - n$ it follows that

$$P_1^* P_1 e_j = e_j = P_2^* P_2 e_j \text{ for } j = n-k+1, \ldots, k.$$

Therefore $\lambda = 2$ and $W_1 \cap W_1^* = W_2 \cap W_2^*$. □

Ideally we would like analogous theorems for any number of projections, but this has proven to be a quite difficult problem.

4. Random nonorthogonal fusion frames

Probabilistic versions of frames have been introduced in [**13–15**]. Here, we extend the concept to nonorthogonal fusion frames.

Let Ω be a locally compact Hausdorff space and $\mathcal{B}(\Omega)$ be the Borel-sigma algebra on Ω endowed with a probability measure μ. We denote the collection of projections on $\mathcal{H}_n$ by $\mathcal{P}_n$, endowed with the induced Borel sigma algebra. We say that a random projector $P : \Omega \to \mathcal{P}_n$, is a random nonorthogonal fusion frame if there are nonnegative constants A and B such that

$$A\|x\|^2 \leq \int_\Omega \|P(w)x\|^2 d\mu(\omega) \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}_n.$$

The random projector P is called tight if we can choose $A = B$. The random analysis operator F is defined by

$$F : \mathcal{H}_n \to L_2(\Omega, \mathcal{H}_n, \mu), \quad x \mapsto (\omega \mapsto P(\omega)x).$$

Its adjoint operator T^* is the random synthesis operator

$$F^* : L_2(\Omega, \mathcal{H}_n, \mu) \to \mathcal{H}_n, \quad f \mapsto \int_\Omega P(\omega)^* f(\omega) d\mu(\omega).$$

The random nonorthogonal fusion frame operator $S = F^*F$ then is

$$S : \mathcal{H}_n \to \mathcal{H}_n, \quad x \mapsto \int_\Omega P(\omega)^* P(\omega) x d\mu(\omega).$$

Thus, $S = \int_\Omega P(\omega)^* P\omega d\mu(\omega)$ has a matrix representation $\left(\sum_{i=1}^m \langle P_i^* P_i e_k, e_l \rangle\right)_{i,j}$, where $\{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathcal{H}_n$. Moreover, we obtain

$$An = \sum_{j=1}^n A\|e_j\|^2 = \sum_{j=1}^n \int_\Omega \|P(\omega)e_j\|^2 d\mu(\omega) = \int_\Omega \sum_{j=1}^n \langle P(\omega)e_j, P(\omega)e_j \rangle d\mu(\omega).$$

Thus, if μ is a tight random nonorthogonal fusion frame for $\mathcal{H}_n$, then the frame bound A satisfies

$$A = \frac{1}{n} \int_\Omega \operatorname{trace}(P(\omega)^* P(\omega)) d\mu(\omega).$$

Next, we present a construction of tight (random) nonorthogonal fusion frames that is based on finite groups. Recall that a finite subgroup G of the unitary operators $O(\mathcal{H}_n)$ is called *irreducible* if each orbit Gx, for $0 \neq x \in \mathcal{H}_n$, spans $\mathcal{H}_n$. In other words, any G-invariant subspace is trivial, i.e., either $\mathcal{H}_n$ or $\{0\}$. The following result is a generalization of Theorem 6.3 in [**22**], where finite frames were considered:

THEOREM 4.1. *If P is a nontrivial random projection and G is an irreducible finite subgroup of $O(\mathcal{H}_n)$, then $\frac{1}{|G|}\sum_{g\in G} g^*Pg$ is a tight random nonorthogonal fusion frame.*

PROOF. One can directly check that the fusion frame operator $S : \mathcal{H}_n \to \mathcal{H}_n$,

$$x \mapsto \frac{1}{|G|}\sum_{g\in G}\int_\Omega g^*P(\omega)^*gg^*P(\omega)gx d\mu(\omega) = \frac{1}{|G|}\sum_{g\in G}\int_\Omega g^*P(\omega)^*P(\omega)gx d\mu(\omega)$$

is self-adjoint and positive semi-definite. Since the identity is an element in G and P is not the trivial random projection, S cannot be the zero mapping, so that it has a positive eigenvalue λ. One checks that each $g \in G$ commutes with S. Thus, the λ-eigenspace is a G-invariant subspace. The irreducibility implies that the eigenspace is the full space $\mathcal{H}_n$, so that S is a multiple of the identity. □

For the sake of completeness, we also formulate Theorem 4.1 in terms of finite nonorthogonal fusion frames:

COROLLARY 4.2. *If P is a projection and G an irreducible finite subgroup of $O(\mathcal{H}_n)$, then $\{g^*Pg\}_{g\in G}$ is a tight nonorthogonal fusion frame.*

Next, we shall discuss the fusion frame potential, cf. . [**5**, **14**, **20**], in the setting of nonorthogonal random projectors. If P is a random projector, then we call

$$\mathcal{R}(P) = \int_\Omega\int_\Omega \langle P(\omega)^*P(\omega), P(\omega')^*P(\omega')\rangle d\mu(\omega)d\mu(\omega')$$

its *random nonorthogonal fusion frame potential*.

PROPOSITION 4.3. *If P is a nontrivial random projection, then*

$$\mathcal{R}(P) \geq \frac{M^2}{n}, \quad \text{where } M = \int_\Omega \|P(\omega)\|^2_{HS} d\mu(\omega), \tag{4}$$

and equality holds if and only if P is tight.

Note that $\mathcal{R}(P) = \text{trace}(S^2)$, where S is the random nonorthogonal fusion frame operator of P. This way we see that Proposition 4.3 can be proven by following the lines of the analogues results for orthogonal projectors in [**1**].

If P is a projection, then $\|P\|^2_{HS} \geq \text{rank}(P)$, and equality holds if and only if P is an orthogonal projection. We therefore have the following:

COROLLARY 4.4. *If P is a nontrivial random projection, then*

$$\mathcal{R}(P) \geq \frac{M^2}{n}, \quad \text{where } M = \int_\Omega \text{rank}(P(\omega)) d\mu(\omega), \tag{5}$$

and equality holds if and only if P is tight and an orthogonal projection almost everywhere.

We can expect that the sample of a tight random nonorthogonal fusion frame approximates a tight nonorthogonal fusion frame when the sample size increases. The following theorem generalizes results in [**13**]:

PROPOSITION 4.5. *Let $\{P_i\}_{i=1}^m$ be a collection of independent tight random nonorthogonal fusion frames with frame bounds $\{A_i\}_{i=1}^m$, respectively, such that,*

$$M := \frac{1}{m}\sum_{i=1}^m \int_\Omega \|P_i^*(\omega)P_i(\omega)\|_{HS}^2 d\mu(\omega) < \infty.$$

If $S(\omega) = \sum_{i=1}^m P_i(\omega)^* P_i(\omega)$ denotes the nonorthogonal fusion frame operator associated to $\{P_i(\omega)\}_{i=1}^m$, then

$$\mathbb{E}(\|\frac{1}{m}S - AI\|_{HS}^2) = \frac{1}{m}(M - n\tilde{A}),$$

where $A = \frac{1}{m}\sum_{i=1}^m A_i$ and $\tilde{A} = \frac{1}{m}\sum_{i=1}^m A_i^2$.

PROOF. The (k,l)-th entry of the matrix S is given by $S_{k,l} = \sum_{i=1}^m \langle P_i^* P_i e_k, e_l\rangle$, and we observe that $\mathbb{E}(\langle P_i^* P_i e_k, e_l\rangle)) = A_i\delta_{k,l}$. We derive

$$\begin{aligned}\mathbb{E}(\|\frac{1}{m}S - AI_d\|_{HS}^2) &= \mathbb{E}(\sum_{k,l}\big(\frac{1}{m}S_{k,l} - A\delta_{k,l}\big)^2)\\ &= \mathbb{E}(\sum_{k,l}\frac{1}{m^2}(S_{k,l})^2) - \mathbb{E}(\sum_k \frac{2A}{m}S_{k,k}) + nA^2\\ &= \mathbb{E}(\sum_{k,l}\frac{1}{m^2}(S_{k,l})^2) - nA^2\end{aligned}$$

since $\mathbb{E}(\sum_k \frac{1}{m}S_{k,k}) = nA$. We split the occurring double sum of $(S_{k,l})^2$ into its diagonal and nondiagonal parts so that the independence of $\{P_i\}_{i=1}^m$ yields

$$\begin{aligned}\mathbb{E}(\|\frac{1}{m}S - AI\|_{HS}^2) &= \frac{1}{m}M + \sum_{k,l}\frac{1}{m^2}\sum_{i\neq j}\mathbb{E}(\langle P_j^* P_j e_k, e_l\rangle)E(\langle P_i^* P_i e_k, e_l\rangle) - nA^2\\ &= \frac{1}{m}M + \sum_{k,l}\frac{1}{m^2}\sum_{i\neq j}A_j\delta_{k,l}A_i\delta_{k,l} - nA^2\\ &= \frac{1}{m}M + \frac{n}{m^2}\sum_{i\neq j}A_jA_i - nA^2 = \frac{1}{m}M - \frac{n}{m}\tilde{A}.\end{aligned}$$

□

The Matrix Rosenthal inequality as stated in [**19**] enables the following variation of Proposition 4.5:

THEOREM 4.6. *Let $\{P_i\}_{i=1}^m$ be a collection of independent random tight nonorthogonal fusion frames with frame bound A, such that, $\mathbb{E}\|P_i^*P_i - AI\|_{4p}^{4p} < \infty$. Let S be as in Proposition 4.5. If $p = 1$ or $p \geq 3/2$, then*

$$\mathbb{E}\|\frac{1}{m}S - AI\|_{4p}^{4p} \leq (\frac{4p-1}{m^2})^{2p}\|\big(\sum_{i=1}^m \mathbb{E}(P_i^*P_i - AI)^2\big)^{1/2}\|_{4p}^{4p} + (\frac{4p-1}{m})^{4p}\sum_{i=1}^m \mathbb{E}\|P_i^*P_i - AI\|_{4p}^{4p}.$$

In order to replace the expectation used in Proposition 4.5 and Theorem 4.6 with a proper estimate of the norm of the difference, one can apply large deviation bounds, for which we refer, for instance, to [**23**].

Acknowledgements

M.E. has been funded by the Vienna Science and Technology Fund (WWTF) through project VRG12-009.

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Mathematics Department, Duke University, Durham, North Carolina 27708
E-mail address: jcahill@math.duke.edu

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
E-mail address: casazzap@missouri.edu

Department of Mathematics, University of Vienna, Vienna, Austria
E-mail address: martin.ehler@univie.ac.at

Department of Mathematics, San Francisco State University, San Francisco, California 94132
E-mail address: shidong@sfsu.edu