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Algebraic and Analytic Aspects of Integrable Systems and Painlevé Equations

AMS Special Session
Algebraic and Analytic Aspects of Integrable Systems
and Painlevé Equations
January 18, 2014
Baltimore, Maryland

Anton Dzhamay
Kenichi Maruno
Christopher M. Ormerod
Editors

American Mathematical Society

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Preface

The theory of integrable systems has been at the forefront of some of the most important developments in mathematical physics in the last 50 years. The techniques to study such systems have solid foundations in algebraic geometry, differential geometry, and group representation theory. The analytic tools developed to study integrable systems have countless applications in random matrix theory, statistical mechanics and quantum gravity. One of the most exciting developments has been the emergence of good and interesting discrete analogues of the classical integrable differential equations, such as the Painlevé equations and soliton equations. Many algebraic and analytic ideas developed for continuous integrable systems generalize in a beautifully natural manner to discrete integrable systems.

The theory of integrable systems has been enriched by some very powerful tools of a more geometric nature. In particular, these tools have elucidated the central role played by elliptic curves in constructing interesting generalizations of some important classes of integrable systems. Systems whose geometry is described by the most nondegenerate elliptic curves, such as the elliptic Painlevé equation, the discrete and continuous Krichever-Novikov equation, the elliptic solutions to the Yang-Baxter equation, and elliptic hypergeometric functions all play the role of the master class of systems from which many other cases may be derived as degenerations. Understanding and developing these ideas has led to many new current research directions and improved our understanding of known integrable models.

Another active area of research concerns the well-established theme of integrable systems admitting nontrivial symmetries. The symmetries of integrable systems often define examples of representations and realizations of interesting algebraic structures. Generalizations of these structures to quantum and q-deformed versions allow for the construction of new integrable systems. These ideas have been responsible for the continuing fruitful interaction between representation theory and integrable systems.

A related important field of research is the theory of Painlevé equations and special functions. The motivation behind the original definition of Painlevé equations is to obtain new genuinely nonlinear special functions as their general solutions. Such solutions, called the *Painlevé transcendents*, are playing an increasingly important role in a wide range of applications in mathematics and physics. In the opposite direction, Painlevé equations often have important exact solutions expressed in terms of special functions, such as the hypergeometric functions. Currently a lot of research activity has been directed on constructing a discrete analogue of Painlevé equations and their higher dimensional generalization. Yet again, algebraic geometry has been playing a very important role in these developments.

Many powerful techniques for studying nonlinear dynamical systems have been obtained by connecting them, usually in a highly nontrivial way, to some auxiliary linear problems. The paradigm for such techniques is the Lax pair formalism and at present a considerable effort has been devoted to understanding the connections between this linear framework and a geometric framework for integrable systems. But there are also many other interesting applications of the classical theory of linear systems for integrable systems. For example, the Riemann-Hilbert and the inverse scattering techniques continue to be powerful tools in analyzing important problems in statistical mechanics, random matrix theory, and the physical sciences.

This volume arose as a result of the AMS Special Session on *Algebraic and Analytic Aspects of Integrable Systems and Painlevé Equations* at the 2014 Joint Mathematics Meetings in Baltimore. In organizing this special session we aimed at presenting a wide range of new research ideas and directions in the theory of integrable equations, Painlevé equations, and applications. In a similar spirit, the present volume contains a collection of expository and research articles that represent a good cross-section of ideas in these active research areas.

We start our volume with *Padé Interpolation and Hypergeometric Series* by Masatoshi Noumi. This article is representative of the origins of integrable systems through its links with special functions. Of great importance in the theory of many integrable systems is the idea of special function solutions expressible in terms of hypergeometric and basic hypergeometric series and their orthogonal polynomial degenerations. This article delves into the relatively new and very active area of elliptic hypergeometric functions.

Our second article, *A q-analogue of the Drinfeld-Sokolov Hierarchy of Type A and q-Painlevé System* by Takao Suzuki, considers higher order analogues of the q-analogue of the sixth Painlevé equation arising as similarity reductions of the q-analogue of the Drinfeld-Sokolov hierarchy. In this way, this article ties together two interesting discrete analogues of classical integrable differential equations. The article details the construction of a Lax formalism for these systems and presents special solutions in terms of the basic hypergeometric functions of type ${}_n\phi_{n-1}$.

We may also generalize classical integrable systems by assuming the systems lie in some specified noncommutative space such as skew fields over $\mathbb{C}$. Results of this nature are presented in the third article, *Fractional Calculus of Quantum Painlevé Systems of Type $A_l^{(1)}$* by Hajime Nagoya. The author realizes a representation of the affine Weyl group of type $A_l^{(1)}$ on the skew field of Schrödinger operators, which defines a quantum Painlevé system.

The fourth article, *Spectral Curves and Discrete Painlevé Equations* by Christopher Ormerod, is an exploratory piece that considers the spectral curve for the associated linear problems of the discrete Painlevé equations as a way of examining the geometry of Lax pairs and how they are related to the geometry of the discrete Painlevé equations.

The fifth article, *Geometric Analysis of Reductions from Schlesinger Transformations to Difference Painlevé Equations* by Anton Dzhamay and Tomoyuki Takenawa, continues the theme of Lax pairs by presenting two discrete Painlevé equations $\left(d\text{-P}\left(A_2^{(1)*}\right)\right.$ and $\left.d\text{-P}\left(A_1^{(1)*}\right)\right)$ as Schlesinger transformations of Fuchsian differential equations. They thoroughly describe and connect the geometry of both the Lax pairs and the resulting nonlinear difference equations.

The sixth article, *Beta Ensembles, Quantum Painlevé Equations and Isomonodromy Systems* by Igor Rumanov, is a review article of some of the recent developments in random matrix theory and their relations with conformal field theory. The main point is that the integrable structures arising in connection with the quantum Painlevé equations for β-ensembles where $\beta = 2$ may be extended to more arbitrary β values.

The last article, *Inverse Scattering Transform for the Focusing Nonlinear Schrödinger Equation with a One-Sided Non-Zero Boundary* by Barbara Prinari and Federica Vitale, presents results in studies of integrable systems that are of a more classical nature. The authors use inverse scattering to solve the focusing nonlinear Schrödinger equation with a one-sided non-zero boundary condition. The focusing NLS with these boundary conditions is not only interesting from the point of view of integrability; it has tangible applications in the investigation of rogue waves and perturbed soliton solutions in physical media.

We would like to thank Christine Thivierge, AMS Associate Editor for Proceedings, for her valuable help and constant support during the preparation of this volume. We hope that you enjoy the articles presented here.

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This volume contains the proceedings of the AMS Special Session on Algebraic and Analytic Aspects of Integrable Systems and Painlevé Equations, held on January 18, 2014, at the Joint Mathematics Meetings in Baltimore, MD.

The theory of integrable systems has been at the forefront of some of the most important developments in mathematical physics in the last 50 years. The techniques to study such systems have solid foundations in algebraic geometry, differential geometry, and group representation theory.

Many important special solutions of continuous and discrete integrable systems can be written in terms of special functions such as hypergeometric and basic hypergeometric functions. The analytic tools developed to study integrable systems have numerous applications in random matrix theory, statistical mechanics and quantum gravity. One of the most exciting recent developments has been the emergence of good and interesting discrete and quantum analogues of classical integrable differential equations, such as the Painlevé equations and soliton equations. Many algebraic and analytic ideas developed in the continuous case generalize in a beautifully natural manner to discrete integrable systems. The editors have sought to bring together a collection of expository and research articles that represent a good cross section of ideas and methods in these active areas of research within integrable systems and their applications.

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