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The subgroup structure of finite groups

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The papers in this volume are versions of talks given at a conference in honor of John Conway and the Atlas. The Atlas supplies a lot of information about the small finite simple groups L. For example it gives an (almost complete) list of the maximal subgroups of the almost simple groups G with minimal normal subgroup L. (Recall G is *almost simple* if G has a unique minimal normal subgroup D, and D is a nonabelian simple group.)

Now it turns out that in many problems in finite group theory, one can reduce to a situation where the group G in question is almost simple (or perhaps almost almost simple), and to solve the reduced problem, one needs strong information about the subgroup structure of G. In particular a good description of the maximal subgroups of the almost simple groups is sometimes sufficient.

This article discusses such reductions for permutation groups. It also indicates how one describes the subgroup structure of almost simple groups. Finally the last few sections of the article illustrate the reduction process with an open question coming from universal algebra.

1. Reductions for permutation groups

Suppose we have a question about permutation groups G on finite sets X. Experience suggests we can usually reduce our question to the case G transitive on X. In that case for $x \in X$, the stabilizer G_x of x in G is a subgroup of G, and the representation of G on X is equivalent to its representation by right multiplication on the coset space G/G_x. Moreover a transitive representation of G on a second set Y is equivalent to our first representation if and only if G_x and G_y are conjugate in G. This says that permutation group theory is virtually the same as the study of the subgroup structure of groups.

The next common reduction reduces us to the case where G is *primitive* on X: that is G preserves no nontrivial partition of X. This is equivalent to the condition that G_x is a maximal subgroup of G. Thus already we see that the maximal subgroups of a group are important.

What is the general structure of a primitive group G? It develops that there is a nice answer that is relatively easy to prove.

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THEOREM (Structure Theorem for Primitive Groups). *Let G be a primitive permutation group on a set X of finite order n and $H = G_x$ the stabilizer of a point $x \in X$. Let $D = F^*(G)$. Then $G = HD$ and one of the following holds:*

(1) $|X| = p^e$ is a prime power, $D \cong E_{p^e}$ is regular on X, and H is a complement to D in G that is irreducible on D (regarded as an $\mathbf{F}_pH$-module) via conjugation.

(2) $D = D_1 \times D_2$ where D_1 and D_2 are isomorphic normal subgroups of G, and for $i = 1, 2$, D_i is the direct product of k nonabelian simple groups L permuted transitively by H, D_i is regular on X, D_x is a full diagonal subgroup of D, and $H = N_G(D_x)$. Further $n = |L|^k$.

(3) D is the direct product of the set $\mathcal{L}$ of components of G, H is transitive on $\mathcal{L}$, and choosing $L \in \mathcal{L}$, L is simple and one of the following holds:

(i) D_x is the product of the groups F_x for $F \in \mathcal{L}$, $L_x \neq 1$, and $Aut_H(L)$ is maximal in $Aut_G(L)$. Further $n = |L : L_x|^{|\mathcal{L}|}$.

(ii) There exists a maximal G-invariant partition Σ of $\mathcal{L}$ such that D_x is the direct product of full diagonal subgroups of $\langle\sigma\rangle$, for $\sigma \in \Sigma$. Further $n = |L|^{|\mathcal{L}|-|\Sigma|}$.

(iii) H is a complement to D in G and $Inn(L) \leq Aut_H(L)$. Further $n = |L|^{|\mathcal{L}|}$.

A *full diagonal subgroup* of a direct product $Y = Y_1 \times \cdots \times Y_k$ of isomorphic groups Y_i is a subgroup K such that for each $1 \leq i \leq k$, the projection $K \to Y_i$ is an isomorphism.

Len Scott was the first to state a preliminary version of the Structure Theorem, and to sketch a proof of the theorem in [**26**]. The first complete statement and proof appear in [**10**].

The Structure Theorem is often called the O'Nan-Scott Theorem, but we will reserve that name for the theorem in section 3 on the subgroups of the symmetric group; this latter theorem can be proved using the Structure Theorem.

In case (3i) of the Structure Theorem, if $|\mathcal{L}| = 1$ then G is almost simple: that is D is a nonabelian simple group. In each of the remaining cases, G preserves some fairly natural structure on X. We next describe the structures that arise.

Suppose first that $n = p^e$ is a prime power. Then X admits an addition $+$ which makes X into an abelian group that we can regard as an e-dimensional vector space over the field $\mathbf{F}_p$ of order p. Then this linear structure induces an *affine space structure*

$$R = \{(a, b, c, b + c - a) : a, b, c \in X\}$$

on X. In case (1) of the Structure Theorem, G preserves the affine space structure induced by the linear group D.

Suppose next that $m \geq 5$ and $k > 1$ are integers such that $n = m^k$. A *regular (m, k)-product structure* on X is an equivalence class of identifications of X with the set product of k-copies of an m-set. See Definition 1.5 in [**5**] for a more precise definition of a regular product structure and section 1 of [**5**] for a more detailed discussion of such structures.

In case (2) of the Structure Theorem if $k > 1$ then G preserves an (m, k)-product structure with $m = |L|$. In case (3i) of the Structure Theorem when $k > 1$, G preserves an (m, k)-product structure for $m = |L : L_x|$. In case (3ii) when $k = |\Sigma| > 1$, G preserves an (m, k)-product structure for $m = |L|^{|\sigma|-1}$ and $\sigma \in \Sigma$. Finally in case (3iii), G preserves an (m, k)-product structure for $k = |\mathcal{L}|$ and $m = |L|$. In each case G preserves a product structure by an appropriate application of 1.6 in [**5**].

Suppose that $n = c^{k-1}$ for some integer $k > 1$ and some nonabelian simple group L of order c. Assume $D = D_1 \times \cdots \times D_k$ is a transitive subgroup of $Sym(X)$ such that each member of $\mathcal{D} = \{D_1, \ldots, D_k\}$ is isomorphic to L, and such that $D_x = F$ is a full diagonal subgroup of D with respect to the direct product decomposition. Then $\mathcal{D}$ and F define a *diagonal structure* $\mathfrak{d} = \mathrm{diag}(\mathcal{D}, \mathrm{F})$ on X, whose stabilizer $N_{Sym(X)}(\mathfrak{d})$ we decree to be $D(N_{Sym(X)}(D) \cap N_{Sym(X)}(F))$. In case (2) of the Structure Theorem when D_1 is simple, G preserves $\mathrm{diag}(\mathcal{D}, \mathrm{D_x})$. In case (3ii) when $|\Sigma| = 1$, G preserves $\mathrm{diag}(\mathcal{D}, \mathrm{D_x})$.

REMARK 1.1. We've seen that if G is primitive on X and G is not almost simple, then G preserves a nice structure on X: an affine structure, a product structure, or a diagonal structure. In our reduction process, this structure can often be used to solve our problem in the primitive case. This leaves only the case where G is almost simple.

In short, we can hope to reduce many problems in permutation group theory to the case G almost simple and primitive, where we may be able to solve our problem given enough information about the maximal subgroups of G. (Recall that G_x is maximal in G and we may take $X = G/G_x$.)

How do we go about obtaining such information? We start with the classification of the finite simple groups.

2. The finite simple groups

Classification Theorem. Each finite simple group is isomorphic to one of the following:

(1) A group of prime order.
(2) An alternating group.
(3) A group of Lie type.
(4) One of 26 sporadic groups.

The maximal subgroups of the sporadics are (almost) listed in the Atlas; thus in this paper we will ignore the sporadics. The socle of an almost simple group is nonabelian, and hence not of prime order. So suppose G is an almost simple group with minimal normal subgroup D which is an alternating group or a group of Lie type. How do we describe the maximal subgroups of G? We represent G as essentially the automorphism group of some highly homogeneous object X, and describe the subgroup structure of G in terms of structures on X.

For example one criterion for high homogeneity of X is the presence of the *Witt property* for X: if A and B are subobjects of X and $\alpha : A \to B$ is an isomorphism, then α extends to an automorphism of X. It turns out that the defining objects for the alternating groups and for the classical groups of Lie type satisfy the Witt property. The n-set X is easily seen to satisfy the Witt property. Witt's Lemma (cf. section 20 in [**13**]) says that if X is a linear, symplectic, orthogonal, or unitary space, then X satisfies the Witt property.

3. The alternating and symmetric groups

In this section we assume D is an alternating group of degree n. Here we choose the defining object X for D to be the n-set admitting D. Then (unless $n = 6$) if G is an almost simple group with minimal normal subgroup D, then either $G = D$ or $G = Sym(X)$, and in either case G acts on X.

Theorem (O'Nan-Scott Theorem). *Assume X is an n-set for some integer $n \geq 5$ and $G = Sym(X)$. Then if $H \leq G$ then either H preserves one of a number of natural structures on X, or H is almost simple and primitive on X.*

The O'Nan-Scott Theorem was proved independently in unpublished work by Mike O'Nan and by Len Scott in [**26**].

The examples of structures that arise are: proper nonempty subsets of X; regular partitions of X; affine structures on X; regular product structures on X; and diagonal structures on X. We have already discussed affine structures, product structures, and diagonal structures in section 1. A *regular partition* of X is a partition with $k > 1$ blocks, each of the same size $m > 1$.

Observe that the Structure Theorem implies the O'Nan-Scott Theorem. For suppose $H \leq Sym(X)$. If H is intransitive on X then H acts on some proper nonempty subset of X. Therefore we may assume H is transitive. If H is imprimitive, then H preserves some nontrivial partition Σ of X, and as H is transitive on X it is also transitive on Σ, so Σ is regular. Therefore we may assume H is primitive on X, so H is described in one of the cases appearing in the statement of the Structure Theorem. Now from Remark 1.1, either H preserves an affine structure, a product structure, or a diagonal structure on X, or H is almost simple. This completes the proof of the O'Nan-Scott Theorem.

As an immediate consequence of the O'Nan-Scott Theorem, if M is a maximal subgroup of G, then either M is the stabilizer of one of our structures, or M is almost simple and primitive on X. Two questions then arise: When is the stabilizer in G of a natural structure on X maximal in G? If H is an almost simple primitive subgroup of G, when is H maximal in G? Both of these questions are answered in a paper of Liebeck, Praeger, and Saxl [**18**]:

Theorem 3.1 (Liebeck-Praeger-Saxl). *If H is an almost simple primitive subgroup of $G = Alt(X)$ or $Sym(X)$ then, with known exceptions, H is maximal in G. Moreover almost always, and with known exceptions, the stabilizers of structures are maximal.*

The O'Nan-Scott Theorem and Theorem 3.1 give a weak classification of the maximal subgroups of G. The classification is "weak" in the sense that for a given n we don't know the almost simple primitive maximal subgroups of G; while we know which such subgroups are *not* maximal, it is hopeless to attempt to enumerate all the almost simple primitive subgroups H of G unless n is small. For example if H is alternating or symmetric of degree k, to determine the copies of H in G would require a list of the maximal subgroups of H of index n.

While our description of the maximal subgroups of G is weak quantitatively, it is often effective qualitatively. Moreover it can be used to look deeper into the subgroup structure of G. For example in [**6**] we find qualitative statements about the lattice of overgroups of primitive subgroups of G, particularly the almost simple primitive subgroups.

Example 3.2. By Theorem A in [**6**], for almost all almost simple primitive subgroups H of $Sym(X)$, all overgroups of H in $Sym(X)$ are almost simple, and the number of maximal overgroups is very small; indeed with known exceptions there is a unique maximal overgroup of H in $Sym(X)$. Moreover by Theorem E in [**6**] if H is a primitive subgroup of $Sym(X)$ then the overgroup lattice of H in G is not a $D\Delta$-lattice (cf. section 6); this fact will be important later in our discussion in section 6 of the Palfy-Pudlak Question.

4. Groups of Lie type

In this section we assume D is a simple group of Lie type. Here we choose our defining object X for D to be the projective space of a suitable module V for a covering group $\hat{D}$ of D, together with the geometry on that space induced by some $\hat{D}$-invariant polynomial structure on V. For example if we want a uniform treatment of groups of Lie type we might choose V to be the Lie algebra of D. But for the strongest results on the subgroup structure of those almost simple groups G with minimal normal subgroup D in a given family of simple groups of Lie type, it is almost always better to choose X to be the projective space of a minimal dimensional irreducible for $\hat{D}$, together with the geometry induced by the defining forms on V.

EXAMPLE 4.1. If D is classical, choose X to be the projective space of the defining module V for $\hat{D}$ together with a bilinear or unitary form. Thus D is $PSL(V)$, $PSp(V)$, $P\Omega(V)$, or $PSU(V)$.

REMARK 4.2. Suppose G is almost simple with minimal normal subgroup D. Then $Z(O(V))\hat{D} = F^*(O(V))$, where $O(V)$ is the isometry group of the linear, symplectic, orthogonal, or unitary space V. Usually the representation of $\hat{D}$ lifts to a representation of some group $\hat{G}$ with $\hat{G}$ mapping onto G, so $G \leq P\Gamma(V)$ preserves the structure on X. But in certain instances this need not be the case, namely when D is $L_n(q)$, $Sp_4(q)$ with q even, $D_4(q)$, or $E_6(q)$, and some element of G induces an automorphism of D nontrivial on the Dynkin diagram of D. However, for ease of exposition, we will ignore these cases in this paper.

In treating the classical groups, once again we can consider certain natural structures on X and show that for $H \leq G$, either H preserves one of these structures, or H is almost simple, primitive (in the linear sense), tensor indecomposable, etc. This is accomplished in [**1**]. The structures involved include certain proper nonzero subspaces of V; certain regular direct sum decompositions of V; certain subfield and extension field structures; and certain tensor product structures and regular tensor product structures. The exact definition of the various structures and the statement of the main theorem of [**1**] is rather complicated, so we will not state that theorem here.

As in the case of the alternating groups, the main theorem of [**1**] implies that if M is a maximal subgroup of G then either M is the stabilizer of one of our structures, or M is almost simple, primitive, tensor indecomposable, etc. Then, again as in the case of the alternating groups, we would like to answer two questions: Which structure stabilizers are maximal? Which almost simple irreducible subgroups of G are maximal?

When $\dim(V) \geq 13$, Kleidman and Liebeck determine in [**17**] which structure stabilizers are actually maximal in G. When $\dim(V) \leq 12$ this is accomplished by Bray, Holt, and Roney-Dougal in [**12**], where they also determine the almost simple maximal subgroups of these small rank classical groups.

Unfortunately as yet we don't know quite as much about the final class of almost simple subgroups of large rank classical groups as we do in the case of alternating and symmetric groups. See [**24**] for a discussion of the current state of the art in the description of almost simple irreducible subgroups of classical groups.

5. Exceptional groups of Lie type

In this section we consider the case where D is an exceptional group of Lie type. In this case there is a theory similar to that of the classical groups. On the one hand this theory is less satisfactory than that for the classical groups in that the defining module, together with its polynomial geometry, is less homogeneous, but on the other hand the theory is more satisfactory in the sense that the modules are of small bounded dimension and there are only a small finite number of families of exceptional groups.

For purposes of this discussion, the exceptional groups can be divided into two classes: the twisted groups of small Lie rank, and the untwisted group G_2 together with the exceptional groups of Lie rank at least 4.

The first class consists of the groups ${}^2B_2(2^e)$, ${}^2G_2(3^e)$, ${}^2F_4(2^e)$, and ${}^3D_4(q)$, with e odd. The groups in the first two subclasses are of Lie rank 1, and have extremely sparse subgroup structure. Those in the remaining two subclasses are of Lie rank 2; their subgroup structure is only slightly more complicated. For references to papers determining the maximal subgroups of such groups, see Table 4 in [**16**]; there is also a treatment of such subgroups in [**12**], as the subgroups can be retrieved in part from the embedding of the group in its defining untwisted overgroup.

The second class consists of the groups G_2, F_4, ${}^2E_6$, E_6, E_7, and E_8. The minimum dimension of a nontrivial module for the group is (generically) 7, 26, 27, 27, 56, and 248, respectively. In the case of G_2, F_4, and E_6, the maximal subgroups are described in terms of this module in [**2**], [**23**], and [**9**], although in the last two cases the existence and conjugacy of certain potential small maximal subgroups is left open. Also Magaard only treats F_4 in characteristic distinct from 2 and 3. In unpublished work, Kleidman determines the maximal subgroups of ${}^2E_6$ using its embedding in E_6, again modulo the existence and uniqueness of some small subgroups.

There exists no treatment for E_7 in terms of its 56-dimensional module, and the 248-dimensional module for E_8 is its Lie algebra. In a series of papers [**19**]-[**22**], Liebeck and Seitz give a treatment of the subgroup structure of all the groups in the second class in terms of their Lie algebras. Once again the existence and conjugacy of certain small potential maximal subgroups is left open.

6. The Palfy-Pudlak Question

Where do we go from here? We have a fairly good qualitative description of the maximal subgroups of the almost simple groups. There are many open questions, but they are difficult and often quite technical. So perhaps it is time to look more deeply into the lattice of subgroups of finite groups in search of new insights.

Let G be a finite group and $H \leq G$. Define $\mathcal{O}_G(H) = \{K \leq G : H \leq K\}$. The members of $\mathcal{O}_G(H)$ are the *overgroups* of H in G. We can study $\mathcal{O}_G(H)$ from many points of view. For example for suitable subgroups H, it is possible to determine explicitly the maximal overgroups of H. We can also regard $\mathcal{O}_G(H)$ as a lattice, and this is the point of view we will focus on here. What can we say about this lattice?

Theorem 6.1 (Palfy-Pudlak, 1980 [**25**]). *The following are equivalent:*
(1) Each finite lattice is the lattice of congruences of some finite algebra.
(2) Each finite lattice is an interval in the subgroup lattice of some finite group.

An *interval* in a lattice Λ is a sublattice $[u, v] = \{x \in \Lambda : u \leq x \leq v\}$ for some $u \leq v$ in Λ.

Note. If Λ is the lattice of subgroups of a group G and $H \leq K \leq G$ then the interval $[H, K]$ is $\mathcal{O}_K(H)$. So condition (2) in the Palfy-Pudlak Theorem says that each finite lattice is an overgroup lattice. This leads to the **Palfy-Pudlak Question**.

Palfy-Pudlak Question. Is each finite lattice of the form $\mathcal{O}_G(H)$ for some finite group G and subgroup H of G?

Now (cf. [**15**]) it is known that the algebraic lattices are the lattices realizable as the lattice of congruences of an algebra, and that the finite lattices are algebraic. However it is not known if each finite lattice can be realized as a lattice of congruences of a *finite* algebra.

Universal algebraists seem to think the answer to the Palfy-Pudlak Question is probably yes, but most finite group theorists seem to think the answer is no. In the latter direction we have:

Conjecture. (Shareshian, [**27**]) Let G be a finite group and $H \leq G$. Then the order complex of the poset $\mathcal{O}_G(H) - \{H, G\}$ has the homotopy type of a wedge of spheres.

If Shareshian's conjecture is correct, the answer to the Palfy-Pudlak Question is a resounding no. Further Shareshian has suggested a family of potential counter examples.

Given a positive integer m, write $\Delta(m)$ for the lattice of all subsets of an m-set. Call such lattices *simplices*. Define a $D\Delta$-lattice to be a lattice built from $k \geq 2$ simplices via the following process: Write ∞ for the greatest element of a finite lattice Λ and 0 for the least element of Λ. Set $\Lambda' = \Lambda - \{0, \infty\}$. Given lattices Γ and Σ write $\Lambda = \Gamma * \Sigma$ if Γ and Σ are sublattices of Λ such that $0, \infty \in \Gamma \cap \Sigma$, Λ' is the disjoint union of Γ' and Σ', and no member of Γ' is comparable to a member of Σ'. Then Λ is $D\Delta$-*lattice* if $\Lambda = \Lambda_1 * \cdots * \Lambda_k$ with $k \geq 2$ and for each $1 \leq i \leq k$, $\Lambda_i \cong \Delta(m_i)$ for some $m_i \geq 3$.

If Shareshian's conjecture is correct then no $D\Delta$-lattice is an overgroup lattice. We want to prove this subconjecture, or at least show that *most* $D\Delta$-lattices are not overgroup lattices, as this would suffice to show the Palfy-Pudlak Question has a negative answer. The idea is to reduce to the almost simple case and use our knowledge of the subgroup structure of such groups. The hope is that, in attacking this somewhat different kind of problem, we will be led to new insights about the subgroup structure of finite groups.

Theorem 6.2 (Reduction Theorem [**4**]). *Let Λ be a $D\Delta$-lattice and G, H minimal subject to $\Lambda \cong \mathcal{O}_G(H)$. Then either G is almost simple or*

(1) G has a unique minimal normal subgroup D.

(2) D is the direct product of a set $\mathcal{L}$ of nonabelian simple groups permuted transitively by H.

(3) H is a complement to D in G and for $L \in \mathcal{L}$, $Inn(L) \leq Aut_H(L)$.

(4) Λ is a lower signalizer lattice in the lattice of subgroups of some almost simple group.

The notation and terminology in part (4) of the Reduction Theorem and in Problem 6.2 below will be explained in section 7.

The Reduction Theorem reduces Shareshian's subconjecture to two problems on the subgroup lattice of almost simple groups G:

PROBLEM 6.1. Let G be an almost simple finite group and $H \leq G$. Prove $\mathcal{O}_G(H)$ is not a $D\Delta$-lattice.

PROBLEM 6.2. Let G be an almost simple finite group. Prove for each nonabelian simple group L there is no $\tau = (G, N, I) \in \mathcal{T}(L)$ such that the lower signalizer lattice $\Xi(\tau)$ is a $D\Delta$-lattice.

In [**10**] and [**7**], Shareshian and I have solved both problems in the case where G is alternating or symmetric.

If we only wish to show that *most* $D\Delta$-lattices are not overgroup lattices, then, in treating Problems 6.1 and 6.2, we can ignore any finite collection of almost simple groups. In particular we need not consider the case where D is sporadic. This leaves only the case where D is of Lie type.

A different approach to showing that the Palfy-Pudlak Question has a negative answer is pursued in [**14**]. Define an *M-lattice* to be a lattice Λ such that Λ' consists of incomparable members. It seems possible that there are M-lattices which are not overgroup lattices, and indeed [**14**] puts forward a candidate for the class of such M-lattices, and reduces the problem of showing such lattices are not overgroup lattices to the verification of a certain set of properties of the finite simple groups.

7. Lower signalizer lattices.

In this section we define the notion of a "lower signalizer lattice" and give some idea of how such lattices arise in the Reduction Theorem.

Let L be a nonabelian finite simple group. Define $\mathcal{T}(L)$ to be the set of triples $\tau = (H, N_H, I_H)$ such that:

(T1) H is a finite group and $N_H \leq H$, and

(T2) $I_H \trianglelefteq N_H$ and $F^*(N_H/I_H) \cong L$.

The set of *signalizers* of N_H in H is the set $\mathcal{W} = \mathcal{W}(\tau)$ of N_H-invariant subgroups W of H such that $W \cap N_H = I_H$. Define

$$\mathcal{W}_1 = \mathcal{W}_1(\tau) = \{W \in \mathcal{W} : W \leq F^*(H)I_H\},$$

and order $\mathcal{W}_1$ by inclusion. Let $\Xi(\tau)$ be the poset obtained by adjoining a greatest member ∞ to $\mathcal{W}_1$. It turns out (cf. 2.11 in [**4**]) that $\Xi(\tau)$ is a lattice, which we call the *lower signalizer lattice* of τ. There is also a *signalizer lattice* $\Lambda(\tau)$ (defined in [**3**]) associated to τ, whose definition is more complicated, and hence is omitted here.

Now assume the hypotheses of the Reduction Theorem with G *not* almost simple. Then by Theorem 3 in [**3**], parts (1)-(3) of the Reduction Theorem hold, and in addition, for $L \in \mathcal{L}$ we have $\tau = (H, N_H(L), C_H(L)) \in \mathcal{T}(L)$ and $\Lambda \cong \Lambda(\tau)$. Indeed from the proof of Theorem 3 in [**3**], each maximal overgroup of H in G appears in case (3ii) of the Structure Theorem. Then from the proof of the Reduction Theorem at the end of section 7 in [**4**], we can pick $\gamma = (\tilde{G}, \tilde{N}, \tilde{I}) \in \mathcal{T}(L)$ such that $\tilde{G}$ is almost simple and $\Lambda \cong \Lambda(\gamma) \cong \Xi(\gamma)$ is a lower signalizer lattice.

8. A question and a theorem

Here is a question whose positive answer could be used (I believe) to give a reasonably nice proof that (large) $D\Delta$-lattices are not realized (as in Problem 6.1) in any almost simple group whose minimal normal subgroup is an exceptional group:

QUESTION 8.1. Let G be a finite group and H a subgroup of G such that $\mathcal{O}_G(H) \cong \Delta(2)$. Let M_1 and M_2 be the maximal overgroups of H in G. Is it true that M_1 and M_2 are not conjugate in G?

A few years ago I posed Question 8.1 and wrote down a sketch of what I believed to be a proof that the question reduces to the case G almost simple. However more recently when I began to fill in the details of that sketch, I found a gap near the end of the argument, and then went on to produce examples of pairs G, H where M_1 and M_2 *are* conjugate in G. However in these examples, G is *not* almost simple. This suggests a modified version of Question 8.1:

QUESTION 8.2. Let G be an almost simple finite group and H a subgroup of G such that $\mathcal{O}_G(H) \cong \Delta(2)$. Let M_1 and M_2 be the maximal overgroups of H in G. Is it true that M_1 and M_2 are not conjugate in G?

If the condition in Question 8.2 can be verified whenever $F^*(G) = D$ is exceptional and M_1 is large, then one should be able to show there is no subgroup H of such a group G such that $\Lambda = \mathcal{O}_G(H)$ is a large $D\Delta$-lattice, where the two notions of "large" are independent of the choice of the exceptional group. For then maximal overgroups of H in the same connected component Λ_i' of Λ' are not conjugate in G. However as D is exceptional of type $\Phi(q)$, the number of classes of large maximal subgroups of G, other than subfield subgroups, is bounded independent of Φ and q. Hence many maximal members of Λ_i' are subfield subgroups, and it should be possible to use this fact to obtain a contradiction.

In short, we can hope to give a relatively attractive proof that Problem 6.1 has a positive solution when $F^*(G)$ is an exceptional group of Lie type.

This leaves the most difficult case of Problem 6.1: the case where $F^*(G)$ is classical.

We close our discussion with a theorem that is very useful in approaching Problem 6.1 when $F^*(G)$ is a group of Lie type.

THEOREM 8.3. *Assume G is an almost simple finite group such that $F^*(G)$ is of Lie type and $\mathcal{O}_G(H)$ is a $D\Delta$-lattice for some $H \leq G$. Then H is not contained in any proper parabolic subgroup of G.*

The proof appears in [**8**]. It uses a result of Timmesfeld [**28**] which says that if R is the unipotent radical of a proper parabolic then the maximal overgroups of R are parabolics. It also uses a theorem in [**8**] on the maximal subgroups of maximal parabolics M which do not contain the radical of M.

Here is a corollary of Theorem 8.3.

COROLLARY 8.4. *Assume G is almost simple with $F^*(G) = PSL(V)$ for some vector space V and $H \leq G \leq P\Gamma(V)$ such that $\mathcal{O}_G(H)$ is a $D\Delta$-lattice. Then H is irreducible on V.*

PROOF. If not then H acts on some proper nonzero subspace U of V. But $N_G(U)$ is a proper parabolic subgroup of G, contrary to Theorem 8.3. □

References

[1] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Invent. Math. **76** (1984), no. 3, 469–514, DOI 10.1007/BF01388470. MR746539

[2] Michael Aschbacher, *Chevalley groups of type G_2 as the group of a trilinear form*, J. Algebra **109** (1987), no. 1, 193–259, DOI 10.1016/0021-8693(87)90173-6. MR898346

[3] Michael Aschbacher, *On intervals in subgroup lattices of finite groups*, J. Amer. Math. Soc. **21** (2008), no. 3, 809–830, DOI 10.1090/S0894-0347-08-00602-4. MR2393428

[4] Michael Aschbacher, *Signalizer lattices in finite groups*, Michigan Math. J. **58** (2009), no. 1, 79–103, DOI 10.1307/mmj/1242071684. MR2526079

[5] Michael Aschbacher, *Overgroups of primitive groups*, J. Aust. Math. Soc. **87** (2009), no. 1, 37–82, DOI 10.1017/S1446788708000785. MR2538638

[6] Michael Aschbacher, *Overgroups of primitive groups. II*, J. Algebra **322** (2009), no. 5, 1586–1626, DOI 10.1016/j.jalgebra.2009.04.044. MR2543625

[7] Michael Aschbacher, *Lower signalizer lattices in alternating and symmetric groups*, J. Group Theory **15** (2012), no. 2, 151–225, DOI 10.1515/jgt-2011-0112. MR2900223

[8] Michael Aschbacher, *Overgroup lattices in finite groups of Lie type containing a parabolic*, J. Algebra **382** (2013), 71–99, DOI 10.1016/j.jalgebra.2013.01.034. MR3034474

[9] M. Aschbacher *The maximal subgroups of E_6*, preprint.

[10] M. Aschbacher and L. Scott, *Maximal subgroups of finite groups*, J. Algebra **92** (1985), no. 1, 44–80, DOI 10.1016/0021-8693(85)90145-0. MR772471

[11] Michael Aschbacher and John Shareshian, *Restrictions on the structure of subgroup lattices of finite alternating and symmetric groups*, J. Algebra **322** (2009), no. 7, 2449–2463, DOI 10.1016/j.jalgebra.2009.05.042. MR2553689

[12] John N. Bray, Derek F. Holt, and Colva M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, London Mathematical Society Lecture Note Series, vol. 407, Cambridge University Press, Cambridge, 2013. With a foreword by Martin Liebeck. MR3098485

[13] Michael Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 1986. MR895134

[14] Robert Baddeley and Andrea Lucchini, *On representing finite lattices as intervals in subgroup lattices of finite groups*, J. Algebra **196** (1997), no. 1, 1–100, DOI 10.1006/jabr.1997.7069. MR1474164

[15] George Grätzer, *Two problems that shaped a century of lattice theory*, Notices Amer. Math. Soc. **54** (2007), no. 6, 696–707. MR2327971

[16] Peter B. Kleidman and Martin W. Liebeck, *A survey of the maximal subgroups of the finite simple groups*, Geom. Dedicata **25** (1988), no. 1-3, 375–389, DOI 10.1007/BF00191933. Geometries and groups (Noordwijkerhout, 1986). MR925843

[17] Peter Kleidman and Martin Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990. MR1057341

[18] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, *A classification of the maximal subgroups of the finite alternating and symmetric groups*, J. Algebra **111** (1987), no. 2, 365–383, DOI 10.1016/0021-8693(87)90223-7. MR916173

[19] Martin W. Liebeck and Gary M. Seitz, *Maximal subgroups of exceptional groups of Lie type, finite and algebraic*, Geom. Dedicata **35** (1990), no. 1-3, 353–387, DOI 10.1007/BF00147353. MR1066572

[20] Martin W. Liebeck and Gary M. Seitz, *On finite subgroups of exceptional algebraic groups*, J. Reine Angew. Math. **515** (1999), 25–72, DOI 10.1515/crll.1999.078. MR1717629

[21] Martin W. Liebeck and Gary M. Seitz, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc. **169** (2004), no. 802, vi+227, DOI 10.1090/memo/0802. MR2044850

[22] Martin W. Liebeck and Gary M. Seitz, *Maximal subgroups of large rank in exceptional groups of Lie type*, J. London Math. Soc. (2) **71** (2005), no. 2, 345–361, DOI 10.1112/S0024610704006179. MR2122433

[23] K. Magaard, *The maximal subgroups of the Chevalley groups $F_4(F)$ where F is a finite or algebraically closed field of characteristic*, $\neq 2, 3$, Caltech thesis (1980).

[24] K. Magaard, *Some remarks on maximal subgroups of finite classical groups*, Finite Simple Groups: Thirty Years of the Atlas and Beyond, Contemp. Math., vol. 694, Amer. Math. Soc., Providence, RI, 2017, pp. 123–137.

[25] Péter Pál Pálfy and Pavel Pudlák, *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups*, Algebra Universalis **11** (1980), no. 1, 22–27, DOI 10.1007/BF02483080. MR593011

[26] Leonard L. Scott, *Representations in characteristic p*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R.I., 1980, pp. 319–331. MR604599

[27] John Shareshian, *Topology of order complexes of intervals in subgroup lattices*, J. Algebra **268** (2003), no. 2, 677–686, DOI 10.1016/S0021-8693(03)00274-6. MR2009327

[28] F. G. Timmesfeld, *Subgroups of Lie type groups containing a unipotent radical*, J. Algebra **323** (2010), no. 5, 1408–1431, DOI 10.1016/j.jalgebra.2009.12.006. MR2584962

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