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Transcendental numbers and special values of Dirichlet series

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To the memory of F. Momose, with respect and admiration

Abstract. We give a short survey of results and conjectures regarding special values of certain Dirichlet series

Contents

1. Introduction
2. The discovery of transcendental numbers
3. An overview of problems and results
4. Euler's theorem revisited
5. Special values of Dirichlet L-series
6. Summation of infinite series of rational functions
7. Multiple zeta values
8. The Chowla-Milnor conjecture
9. The Riemann zeta function at odd arguments
10. Hecke's conjecture and the Siegel-Klingen theorem
11. Artin L-series
12. Schanuel's conjecture and special values at $s = 1$.
13. The Chowla and Erdös conjectures
14. Concluding remarks
Acknowledgments
References

1. Introduction

In this paper, we are concerned with special values of Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

assuming that the series is convergent. Most of the time, these Dirichlet series will be zeta and L-functions that arise out of number theory. Sometimes, they

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will be general series arising from other considerations (as we will see below as in the case of the Chowla and Erdös problems or the Chowla-Milnor conjectures). Several different perspectives are available to study the former case of zeta functions. The cohomological approach of Beilinson and his predictions regarding the special values in terms of generalized regulators is an active area of current research and we refer the reader to several good expositions such as Soulé [**51**] and Ramakrishnan [**44**]. What we want to highlight in this exposition is a more analytic and classical approach.

Beginning with Euler's work on the explicit evaluation of the Riemann zeta function at even arguments and the role the cotangent function plays in this evaluation, we amplify the role of special functions that emerge in our understanding of these special values. The logarithm function, the dilogarithm function and the polylogarithm functions assume a central place in this study, as well as the gamma function, the digamma function and the polygamma functions. Special values of modular forms are also closely connected to special values of certain Dirichlet series and L-functions. Other zeta functions, such as the Hurwitz zeta function, the Lerch zeta function, the multiple zeta functions and the multiple Hurwitz zeta functions also make an appearance.

Often the evaluation of these Dirichlet series and the determination of their algebraic or transcendental nature require the marriage of several disparate branches of number theory. The former is an analytic-arithmetic viewpoint, and the latter being transcendental number theory. It is hoped that this survey will serve to highlight the beauty of all these viewpoints and propose some new questions for further research. The reader may find additional exposition in the monograph [**36**].

2. The discovery of transcendental numbers

An *algebraic number* is a complex number which is a root of a non-trivial polynomial with integer coefficients. It is a beautiful theorem of algebra that the totality of algebraic numbers forms a field. If a complex number is not algebraic, we call it *transcendental*. The notion of a transcendental number may be traced back to Euler but the first use of the term "transcendental" occurs in a 1682 paper of Leibnitz where he showed that $\sin x$ is not an algebraic function of x. In 1844, using the theory of continued fractions, Joseph Liouville proved that transcendental numbers exist. It was not until 1851 that he realized that there was a simple way to construct some examples. His construction was based on the following elementary idea which had a profound impact on the development of transcendental number theory.

Liouville observed that if α is algebraic then we may consider a polynomial $f(x) \in \mathbb{Z}[x]$ of minimal degree for which it is a root. This polynomial is unique up to an integral factor. The *degree of* α is defined to be the degree of f. Thus, the algebraic numbers which have degree 1 are precisely the rational numbers. If α has degree ≥ 2, and $f(x)$ is an irreducible polynomial with integer coefficients such that $f(\alpha) = 0$, Liouville proved that there is a positive constant C (depending on f) such that for any rational number p/q, we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^n}.$$

To prove this, let us note that if $|\alpha - p/q| > 1$, we are done since $1 \geq 1/q^n$ and we can choose $C = 1$. So, let us suppose that $|\alpha - p/q| \leq 1$. If $\alpha = \alpha_1, ..., \alpha_n$ are the roots of f, and A is the leading coefficient of $f(x)$, then,

$$A(\alpha - p/q)(\alpha_2 - p/q) \cdots (\alpha_n - p/q) = f(p/q).$$

Observe that for any rational number p/q,

$$|f(\alpha) - f(p/q)| = |f(p/q)| \geq 1/q^n$$

because the numerator is a non-zero integer. Since

$$\left|\alpha_i - \frac{p}{q}\right| \leq |\alpha_i - \alpha_1| + \left|\alpha_1 - \frac{p}{q}\right| \leq |\alpha_i - \alpha_1| + 1,$$

we see immediately on choosing

$$M = |A| \prod_{i=2}^{n} (|\alpha_i - \alpha| + 1),$$

that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{Mq^n}.$$

Setting $C = \max(1, 1/M)$ gives the result.

What Liouville's theorem says is that algebraic numbers are not too well approximable by rational numbers. Consequently, if a number is too well approximable by rational numbers, in the above sense, it must be transcendental. Applying his theorem to numbers of the form

$$\sum_{n=0}^{\infty} \frac{1}{2^{n!}},$$

Liouville deduced that these must be transcendental numbers since partial sums of these numbers are rational numbers that approximate the sum too well for the sum to be algebraic. These were perhaps the first class of infinite series shown to be transcendental.

Liouville's numbers were exotic constructions. It wasn't clear at that time whether numbers like e or π were transcendental. This had already been conjectured by Johann Heinrich Lambert in his 1761 paper where he proved that π is irrational. In 1873, Charles Hermite proved that e is transcendental. A year later, in 1874, Cantor gave his famous diagonal argument to show that transcendental numbers are uncountable. It was as late as 1882 when Ferdinand von Lindemann proved that π is transcendental using methods initiated by Hermite. In his paper, Lindemann stated many results without proof. For example, he stated that if α is algebraic and non-zero, then e^α is transcendental. Since $e^{\pi i} = -1$, we deduce that π is transcendental. This more general result was later proved rigorously by Hermite. Lindemann also stated that if $\alpha_1, ..., \alpha_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, ..., e^{\alpha_n}$ are algebraically independent. The proof of this statement was given by Weierstrass. (For some reason, Lindemann did not give much attention to the line of research he initiated. Instead, he turned his gaze to Fermat's Last Theorem and published a book on it with a general "proof", which was unfortunately wrong.)

Perhaps the oldest explicit evaluation of an infinite series was first carried out by Madhava in 14th century India, when he proved that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

This evaluation is often attributed to Leibnitz, but there is now well-documented evidence to the contrary. Recent research has uncovered the contributions of the Kerala school of mathematicians, led by Madhava. Their writings show that much of what we would now call "pre-calculus" was well developed in 14th century India by the Kerala school and we refer the reader to [**26**] for an account of this fascinating history.

Several centuries later, the famous Basel problem asking for an explicit evaluation of

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

was solved by Euler in 1735 almost a century after Pietro Mengoli proposed the problem in 1644. Euler went on to show, after a decade of work, that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

is a rational multiple of π^{2k}. The underlying reason for the explicit evaluation of both the Madhava-Leibnitz series and Euler's determination of the special values of the Riemann zeta function at even arguments is due to the properties of the cotangent function, though this is not overtly clear from the study of their works.

The Madhava-Leibnitz formula is an explicit evaluation of a Dirichlet L-series. In fact, it is $L(1, \chi)$ with χ being the non-trivial Dirichlet character (mod 4). From this, together with Lindemann's result that π is transcendental and Euler's explicit evaluation of special values of the Riemann zeta function, there emerges a new theme of determining the nature of special values of general zeta and L-functions and more generally special values of Dirichlet series. This determination requires a two-fold (perhaps three-fold) development of number theory. On the one hand, one needs a general method to evaluate these series and this is often difficult. It involves an analytic and arithmetic study of special functions. One then needs results from transcendental number theory regarding the precise nature of the special values. These special values often factor as a product of an algebraic number and a transcendental number (a "period" in modern parlance), and the algebraic number is often pregnant with arithmetic meaning. These determinations stimulate a three-fold development of number theory. At the moment, some of these strands are developing faster than others. The slowest seems to be transcendental number theory where it is difficult to determine whether a given number (or "period") is transcendental. Such results require tremendous advances in our understanding the nature of special functions. This is best highlighted by relating the story of some of the Hilbert problems.

In his famous list of problems at the International Congress of Mathematicians held in 1900, Hilbert asked: if α is algebraic $\neq 0, 1$ and β is any irrational algebraic number, then is α^{β} transcendental? If true, it would imply that e^{π} is transcendental since $e^{\pi} = (-1)^{-i}$. Apparently, Hilbert considered this problem to be notoriously difficult. He predicted that the Riemann hypothesis and Fermat's Last Theorem would be solved before this problem (see p. 84 of [**49**]). History proved otherwise! It is dangerous to make predictions!

In 1934, Gelfond and Schneider independently resolved this problem. This represented a major advance in the field. Thus, not only are e and π transcendental numbers but so is e^π by the Gelfond-Schneider theorem. But what about $e+\pi$ or πe? Are these transcendental? (It is easy to see that at least one of the two is transcendental.) Is there an algebraic relation between e and π? The answers to these questions are unknown at the moment. However, in the 1960's, Schanuel (as cited in Lang's monograph on Transcendental Numbers) conjectured that if $x_1, ..., x_n$ are linearly independent over $\mathbb{Q}$, then

$$\operatorname{tr} \deg \mathbb{Q}(x_1, ..., x_n, e^{x_1}, ..., e^{x_n}) \geq n.$$

The Lindemann- Weierstrass theorem states that if $x_1, ..., x_n$ are linearly independent *algebraic* numbers, then this conjecture is true. For $n = 1$, Schanuel's conjecture follows from the Hermite - Lindemann theorem. Already for $n = 2$, we do not know the truth of the conjecture. However, the general case is still a major unsolved problem and many interesting consequences emerge from it. For example, we immediately deduce that e and π are algebraically independent. To see this, consider $x_1 = 1, x_2 = \pi i$. These are linearly independent over $\mathbb{Q}$ and so the

$$\operatorname{tr} \deg \mathbb{Q}(\pi, e) = \operatorname{tr} \deg \mathbb{Q}(1, \pi i, e, e^{\pi i})$$

is at least 2. Thus, e and π are algebraically independent. Therefore Schanuel implies that both $e+\pi$ and $e\pi$ are transcendental and algebraically independent. Also, $1, \log \pi$ are linearly independent over $\mathbb{Q}$ for otherwise, we have $a + b \log \pi = 0$, for some integers a, b from which we get $e^a \pi^b = 1$, contradicting the algebraic independence of e and π. To deduce that $\log \pi$ is transcendental, we consider $\pi i, \log \pi$ which are linearly independent over $\mathbb{Q}$ since $\pi \neq \pm 1$. Thus, π and $\log \pi$ are algebraically independent. In particular, $\log \pi$ is transcendental (modulo Schanuel).

There has been some progress on Schanuel's conjecture but not much. A special case of the conjecture is the following conjecture of Gelfond and Schneider: if α is an algebraic number and $\alpha \neq 0, 1$, and if β is an irrational algebraic number of degree d, then the $d-1$ numbers

$$\alpha^{\beta}, \alpha^{\beta^2}, ..., \alpha^{\beta^{d-1}}$$

are algebraically independent. We call this the Gelfond-Schneider conjecture. What is know is a result of Diaz [**14**] which states that the transcendence degree of the field generated by the $d-1$ numbers above is at least $[(d+1)/2]$.

Schanuel's conjecture will help us later to determine the transcendental nature of special values of many Dirichlet series (more specifically Artin L-series at $s = 1$).

3. An overview of problems and results

The prototypical example of an L-series is the Riemann zeta function, denoted by $\zeta(s)$ and defined by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$ and then extended to the entire complex plane via a classical method of Riemann [**46**]. Related to this, perhaps the most celebrated and most beautiful of results on special values of L-series is Euler's theorem [**15**] that

$$\zeta(2k) = \frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

for every positive integer k. Here, B_k denotes the k-th Bernoulli number, given via the generating function:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}.$$

By the functional equation for $\zeta(s)$, this can be written equivalently as $\zeta(1-k) = -B_k/k$. It is easy to see that $B_k = 0$ for k odd and greater than 1. Thus, the Riemann zeta function vanishes at $s = -2, -4, ...$ and these are referred to as the trivial zeros. Since the Bernoulli numbers are rational numbers, we see that $\zeta(2k) \in \pi^{2k}\mathbb{Q}^*$, and hence the special value is a transcendental number by a famous theorem of Lindemann [**30**]. It is conjectured that all of the special values $\zeta(2k+1)$ with k a natural number are transcendental and that in fact, $1, \pi, \zeta(3), \zeta(5), \zeta(7), ...$ are algebraically independent (see [**8**]). Though this conjecture is still open, some spectacular progress has been made in the recent past. Beginning with the work of Apéry [**1**] in 1978 that $\zeta(3)$ is irrational, we have the theorem of Rivoal [**47**] that infinitely many of the values $\zeta(2k+1)$ are irrational. There are even some stronger results giving a lower bound for the dimension of the $\mathbb{Q}$-vector space spanned by the values $\zeta(2k+1)$ with $k \leq a$.

There are several different directions in which these results can be extended. Firstly, Euler's theorem on the special values of the Riemann zeta function was first extended by Hecke [**24**] to the case of a real quadratic field. He showed that if F is a real quadratic field and $\zeta_F(s)$ is the Dedekind zeta function of F, then $\zeta_F(2k)$ is an algebraic multiple of π^{4k}. This led him to conjecture that if F is a totally real algebraic number field, then $\zeta_F(2k)$ is an algebraic multiple of π^{2dk} where $d = [F : \mathbb{Q}]$. This conjecture was later proved by Siegel and Klingen [**28**]. These results raise further questions. What happens if F is not totally real? If F is totally real, what about $\zeta_F(2k+1)$? Are these transcendental numbers? The answer is most likely "yes" but we are far from knowing this. We discuss what is known and unknown in the larger context of Artin L-series, in a later section. In this connection, it is worth noting that Euler's theorem was first extended to Dirichlet L-series by Hecke as late as 1940, though the ideas needed for this work were already there at the time of Euler in the 18th century. We give the details of this development in a section below.

Another direction worthy of study is to fix a value of s and study special values of a family of L-series at s. The most celebrated example of this phenomenon is Dirichlet's class number formula. This formula states that if

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is the Dirichlet L-series attached to a Dirichlet character $\chi \pmod q$, then for χ a non-trivial quadratic character,

$$L(1, \chi) = \begin{cases} \frac{2\pi h}{w\sqrt{|d|}}, & \chi(-1) = -1 \\ \frac{h \log \epsilon}{\sqrt{|d|}} & \chi(-1) = 1, \end{cases}$$

where h denotes the class number of the quadratic field cut out by χ and ϵ is the fundamental unit in the real quadratic case, and w is the number of roots of unity in the associated quadratic field.

If χ is not a quadratic character, one can also write down a precise formula for $L(1,\chi)$. In all cases, Dirichlet's celebrated result is that $L(1,\chi) \neq 0$ and this is equivalent to the infinitude of primes in a given arithmetic progression (mod q). Inspired by this formula, Chowla [**12**] proposed the following problem. Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ and consider the Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Under what general conditions can we assert that $L(1,f) \neq 0$? Is it possible to evaluate $L(1,f)$? If f is algebraic valued, can we say that if $L(1,f) \neq 0$, then $L(1,f)$ is transcendental? More generally, what can we say about $L(k,f)$? These questions led Chowla and his daughter, Paromita Chowla, [**13**] to a variety of conjectures. These conjectures were generalized by Milnor [**31**]. One can ask for analogues of Chowla's question to number fields. But this investigation is still in its infancy. V.K. Murty and M.R. Murty [**34**] initiated this study by considering first imaginary quadratic fields. We describe this below.

Yet another direction of study is via multiple zeta values. These are defined as:

$$\zeta(k_1,...,k_r) = \sum_{n_1>n_2>...>n_r} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r},$$

with $k_1 \geq 2$, and $k_2,...,k_{r-1} \geq 1$ and the n_i run over all positive integers.. The *weight* of the multiple zeta value $\zeta(k_1,...,k_r)$ is defined as the sum $k_1 + \cdots + k_r$ and its *depth* as r. A recent remarkable theorem of Brown [**7**] is that **all** multiple zeta values of weight n are $\mathbb{Q}$-linear combinations of

$$\{\zeta(a_1,...,a_r) : \text{where} \quad a_i = 2 \ \text{ or } \ 3, \ \text{ and } \ a_1 + \cdots + a_r = n\}.$$

Using this theorem, we can see that the dimension of the $\mathbb{Q}$-vector space V_n spanned by multiple zeta values of weight n is bounded by d_n where d_n satisfies the recurrence relation $d_n = d_{n-2} + d_{n-3}$, with $d_0 = 1$, $d_1 = 0$, and $d_2 = 1$. It is conjectured that the dimension of V_n is exactly d_n but this has not yet been proved. The recursion shows that d_n grows exponentially as a function of n and yet, not a single value of n is known for which dim V_n is at least 2. Gun, Murty and Rath [**21**] showed that if the Chowla-Milnor conjecture is true, then there are infinitely many values of n for which the dimension is at least 2. This goes to indicate that the Chowla-Milnor conjecture is quite difficult.

A fourth direction of study is the special values of L-series attached to modular forms and more generally automorphic forms. Already, in the modular forms case, there are quite a number of results and conjectures. The full extent of these conjectures is beyond the scope of this survey. We relegate this to a future occasion. For the time being, we refer the reader to the excellent survey by Raghuram and Shahidi [**43**] where more references on this theme can be found.

4. Euler's theorem revisited

The analytic continuation of the Riemann zeta function to the entire complex plane was first proved by Riemann [**46**]. He also established the functional equation for the zeta function:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

If it is only the analytic continuation we desire and not the functional equation, there is an elementary way to derive it. This was noted in the author's paper with Reece [**37**]. For this purpose, it is useful to consider the Hurwitz zeta function:

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

for $0 < x \le 1$. Thus, $\zeta(s,1) = \zeta(s)$. This series converges absolutely for $\Re(s) > 1$ and it is surprising that one can derive an analytic continuation of the Hurwitz zeta function by a simple induction argument as follows. Let us observe that

$$-\frac{1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{n=1}^{\infty}\left\{\frac{1}{(n+x)^s} - \frac{1}{n^s}\right\}.$$

Writing the summand as

$$\frac{1}{n^s}\left(\left(1+\frac{x}{n}\right)^{-s} - 1\right),$$

we can apply the binomial theorem for $0 < x < 1$ and get

$$-\frac{1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{r=1}^{\infty}\binom{-s}{r}\zeta(s+r)x^r. \tag{4.1}$$

Several observations can now be made. First, if $x = 1/2$, we observe that $\zeta(s,1/2) = (2^s-1)\zeta(s)$ so that

$$-2^s + (2^s-2)\zeta(s) = \sum_{r=1}^{\infty}\zeta(s+r)2^{-r},$$

which serves to provide a meromorphic continuation of the Riemann zeta function to the entire complex plane by a simple induction argument. This argument shows that $\zeta(s)$ extends analytically to the entire complex plane except possibly at those s for which $2^s = 2$. Indeed, let us first consider the region $\Re(s) > 0$. In this region, the only poles are at

$$s = \frac{2\pi i m}{\log 2}, \qquad m \in \mathbb{Z}.$$

For any natural number $q > 1$, we also have the identity

$$(q^s - q)\zeta(s) = \sum_{a=1}^{q}\left(\zeta(s,\frac{a}{q}) - \zeta(s)\right)$$

and the right hand side is analytic for $\Re(s) > 0$ by (4.1). This identity serves to imply that if $\zeta(s)$ has any poles in this region, they occur at

$$s = 1 + \frac{2\pi i m}{\log q}, \qquad m \in \mathbb{Z},$$

and they are all simple. Taking $q = 3$ say, and noting that $s = 1$ is the only element in the intersection of

$$\left\{1 + \frac{2\pi i m}{\log 2} : m \in \mathbb{Z}\right\} \cap \left\{1 + \frac{2\pi i n}{\log 3} : n \in \mathbb{Z}\right\},$$

we deduce that $\zeta(s)$ extends to $\Re(s) > 0$, except for a simple pole at $s = 1$. Proceeding inductively, we deduce the analytic continuation of $\zeta(s)$ to the entire complex plane. This then serves to give the analytic continuation of $\zeta(s, x)$ as well.

What is impressive about (4.1) is that it shows that $\zeta(1-k, x)$ is a polynomial in x of degree $k-1$ for any positive value of k. To see this, one need only note that for $s = 1-k$, the infinite series becomes a finite series because the binomial coefficients vanish for $r \geq k$. One can also derive from this that $\zeta(1-k) = -B_k/k$, by an elementary induction argument as follows. The recursion above allows us to deduce that for any positive integer m,

$$m\zeta(1-m) = \frac{(-1)^m}{m+1} - m\sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}\frac{\zeta(1-m+r)}{r+1}.$$

Recall that the generating function for the Bernoulli numbers is

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty}\frac{B_k t^k}{k!}.$$

From this we deduce that the B_k are rational numbers and the following recurrence for them:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}}{k+1} = 0.$$

Moreover,

$$\frac{t}{e^t-1} + \frac{t}{2}$$

is an even function of t so that the Bernoulli numbers for odd subscripts ≥ 3 vanish. We can then prove by induction the formula

$$\zeta(1-k) = (-1)^{k-1}\frac{B_k}{k}$$

using (4.1). One can now use the functional equation to deduce the explicit value of $\zeta(2k)$.

Of course, Euler's approach was completely different. Since this approach will be useful later in our study of multiple zeta values, we indicate briefly his point of view.

In 1735, Euler discovered experimentally that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{4.2}$$

He gave a "rigorous" proof much later, in 1742. Here is a sketch of Euler's proof. The polynomial

$$\left(1-\frac{x}{r_1}\right)\left(1-\frac{x}{r_2}\right)\ldots\left(1-\frac{x}{r_n}\right)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coefficient of x is

$$-\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}\right).$$

Using this observation, Euler proceeded "by analogy." Supposing that $\sin \pi x$ "behaves" like a polynomial and noting that its roots are at $x = 0, \pm 1, \pm 2, ...,$ Euler puts

$$f(x) = \frac{\sin \pi x}{\pi x}.$$

By l'Hôpital's rule, $f(0) = 1$. Now $f(x)$ has roots at $x = \pm 1, \pm 2, ...$ and so

$$f(x) = (1-x)(1+x)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{3}\right)\left(1+\frac{x}{3}\right)\cdots.$$

That is,

$$f(x) = (1-x^2)\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\cdots.$$

The coefficient of x^2 on the right hand side is

$$-\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\right).$$

By Taylor's expansion,

$$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + \cdots$$

so that comparing the coefficients gives us formula (4.2).

The main question is whether all of this can be justified. Euler certainly didn't have a completely rigorous proof of his argument. To make the above discussion rigorous, one needs either the theory of Weierstrass products discovered in 1876 (see page 79 of [**45**]) or Hadamard's theory of factorization of entire funtions, a theory developed much later in 1892, in Jacques Hadamard's doctoral thesis. Still, we credit Euler for the discovery of this result since the basic idea is sound.

The next question is whether Euler's result can be generalized. For example, can we evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Euler had difficulty with the first question but managed to show, using a similar argument, that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and more generally that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

It is not hard to see that Euler's proof can be modified to deduce the above results. Indeed, if $i = \sqrt{-1}$, then observing that

$$f(ix) = (1+x^2)\left(1+\frac{x^2}{4}\right)\cdots$$

we see that

$$f(x)f(ix) = (1-x^4)\left(1-\frac{x^4}{2^4}\right)\left(1-\frac{x^4}{3^4}\right)\cdots$$

But the Taylor expansion of $f(x)f(ix)$ is

$$\left(1-\frac{\pi^2x^2}{3!}+\frac{\pi^4x^4}{5!}-\cdots\right)\left(1+\frac{\pi^2x^2}{3!}+\frac{\pi^4x^4}{5!}+\cdots\right).$$

Computing the coefficient of x^4 yields

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Continuing in this way, it is not difficult to see how Euler arrived at the assertion that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

Euler's viewpoint is useful in the study of multiple zeta values. He seemed to suggest that for $\zeta(2k+1)$ the value is a product of π^{2k+1} and "a function of $\log 2$" (see the last line of p. 1078 of [**2**]). This conjecture is probably wrong but no definite disproof has yet been found.

5. Special values of Dirichlet L-series

The logarithmic derivative of the Γ-function is called the *digamma function*. Higher derivatives of the digamma function give rise to the *polygamma functions*. More precisely, the digamma function is defined by

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right), \quad z \neq 0, \pm 1, \pm 2, ...$$

so that the polygamma functions $\psi_k(z)$ are given by

$$\psi_k(z) = (-1)^{k-1} k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}.$$

It is easily seen from the series expansion for $\psi_k(z)$ that

$$\psi_k(z+1) = \psi_k(z) + \frac{(-1)^k k!}{z^{k+1}}.$$

This allows us to deduce that for every integer $k \geq 0$,

$$-\frac{d^k}{dz^k}(\pi \cot \pi z) = \psi_k(z) + (-1)^{k+1}\psi_k(-z) + (-1)^k \frac{k!}{z^{k+1}}.$$

Indeed, from the partial fraction expansion of the cotangent function, we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

The result is now easily deduced by taking successive derivatives. This allows us to relate the cotangent function to the polygamma functions. Indeed, it is readily seen that

$$-\frac{d^k}{dz^k}(\pi \cot \pi z) = \psi_k(z) + (-1)^{k+1}\psi_k(1-z). \tag{5.1}$$

These identities are at the heart of Hecke's 1940 generalization of Euler's explicit determination of (see Ayoub [**2**] and [**15**]) $\zeta(2k)$ of 1749 and it is surprising that it took almost two centuries to write them down. This highlights the importance of a survey, when we can look back and see what has been done and what is yet to be done. Here is a brief description of Hecke's theorem.

Let q be a natural number and let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ be a Dirichlet character and define the L-series $L(s,\chi)$ by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Then

$$(k-1)!L(k,\chi) = (-q)^{-k}\sum_{a=1}^{q}\chi(a)\psi_{k-1}\left(\frac{a}{q}\right).$$

Here k is a positive integer and if $k=1$, we assume χ is non-trivial. Indeed, we may write

$$L(k,\chi) = \sum_{a=1}^{q}\chi(a)\sum_{n\equiv a(\operatorname{mod} q)} n^{-k} = \sum_{a=1}^{q}\chi(a)\sum_{j=0}^{\infty}(qj+a)^{-k}$$

from which the desired result is apparent.

If χ is an even character (that is, $\chi(-1)=1$) and k is even, then

$$-2(k-1)!L(k,\chi) = (-q)^{-k}\sum_{a=1}^{q}\chi(a)\frac{d^k}{dz^k}(\pi\cot\pi z)\Big|_{z=a/q}.$$

Then, $L(k,\chi)$ is an algebraic multiple of π^k.

To see this, we can write

$$2(k-1)!L(k,\chi) = (-q)^{-k}\sum_{a=1}^{q}\chi(a)\psi_{k-1}\left(\frac{a}{q}\right)+\chi(q-a)\psi_{k-1}\left(1-\frac{a}{q}\right)$$

which is

$$= (-q)^{-k}\sum_{a=1}^{q}\chi(a)\left(\psi_{k-1}\left(\frac{a}{q}\right)+\psi_{k-1}\left(1-\frac{a}{q}\right)\right).$$

By our earlier observation (5.1), the result follows. The last assertion is immediate upon noting that

$$\frac{d^k}{dz^k}(\pi\cot\pi z)\Big|_{z=a/q}$$

is an algebraic multiple of π^k. It is also important to note that this calculation allows us to determine the algebraic number precisely. An analogous calculation can be made for odd characters.

If χ is an odd character (that is, $\chi(-1)=-1$) and k is odd, then

$$-2(k-1)!L(k,\chi) = (-q)^{-k}\sum_{a=1}^{q}\chi(a)\frac{d^k}{dz^k}(\pi\cot\pi z)\Big|_{z=a/q}.$$

and $L(k,\chi)$ is an algebraic multiple of π^k.

As before we can write

$$2(k-1)!L(k,\chi) = (-q)^{-k}\sum_{a=1}^{q}\chi(a)\psi_{k-1}\left(\frac{a}{q}\right)+\chi(q-a)\psi_{k-1}\left(1-\frac{a}{q}\right)$$

which is now equal to (since χ is odd)

$$= (-q)^{-k}\sum_{a=1}^{q}\chi(a)\left(\psi_{k-1}\left(\frac{a}{q}\right)-\psi_{k-1}\left(1-\frac{a}{q}\right)\right).$$

As before, the result follows by noting the relation of the polygamma function to the cotangent function. The last assertion is also immediate upon noting (as before) that

$$\frac{d^k}{dz^k}(\pi \cot \pi z)\Big|_{z=a/q}$$

is an algebraic multiple of π^k.

When k and χ have opposite parity, the situation is as difficult as the determination of the transcendental nature of the Riemann zeta function at odd arguments. The simplest unknown case concerns the famous Catalan constant:

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$$

We do not know if this number is irrational.

6. Summation of infinite series of rational functions

The essential success of explicit evaluations of the special values of Riemann's zeta function and Dirichlet's L-series discussed in the earlier sections is mainly due to our understanding of the cotangent function. More precisely, the identity

$$\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{n+z}, \quad z \notin \mathbb{Z}$$

and its derivatives allow us a precise knowledge of

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^k}.$$

This idea also allows us to explicitly evaluate infinite series of the form

$$\sum_{n \in \mathbb{Z}} \frac{A(n)}{B(n)},$$

where $A(x), B(x)$ are polynomials and we assume that the sum converges and that $B(x)$ has no integral roots. We may further allow for $B(x)$ to have integer roots and then restrict the sum so that we exclude these (finite number of) roots in the summation. In any case, one can apply partial fraction expansions and derive very beautiful explicit formulas for these sums. In the case $A(x)$ and $B(x)$ have algebraic coefficients, the transcendental nature of these sums can sometimes be determined thanks to the work of Nesterenko [**42**]. All of these investigations have been carried out in the paper by Murty and Weatherby [**40**]. Among other results proved in [**40**], here is a representative one. If the Gelfond-Schneider conjecture is true, then for any $A(x), B(x) \in \overline{\mathbb{Q}}[x]$, with degree of A less than the degree of B and $B(n) \neq 0$ for any $n \in \mathbb{Z}$, the sum

$$\sum_{n \in \mathbb{Z}} \frac{A(n)}{B(n)}$$

is either zero or transcendental. Perhaps the most striking of formulas derived through these investigations is a superb generalization of Euler's theorem. The sum

$$\sum_{n \in \mathbb{Z}} \frac{1}{An^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}/A} - 1}{e^{2\pi\sqrt{D}/A} - 2(\cos(\pi B/A))e^{\pi\sqrt{D}/A} + 1} \right)$$

is transcendental if $A, B, C \in \mathbb{Z}$ and $-D = B^2 - 4AC < 0$. The explicit evaluation is simply an application of the cotangent expansion. The transcendence is derived by applying a result of Nesterenko which states that π and $e^{\pi\sqrt{D}}$ are algebraically independent. Viewing the right hand side as a function of C and applying successive differentiation with respect to C, we deduce an explicit formula for

$$\sum_{n\in\mathbb{Z}} \frac{1}{(An^2+Bn+C)^k},$$

and this is exposed in [**41**]. More can be done in this direction and one can study sums of the form

$$\sum_{n\in\mathbb{Z}} \frac{f(n)}{(An^2+Bn+C)^k},$$

with f a periodic function (mod q). More generally, we can study

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)} f(n).$$

Some partial results have been obtained in [**54**]. However, the full extent of our knowledge of these special values has not yet been determined.

7. Multiple zeta values

To understand the arithmetic nature of special values of the Riemann zeta function, it has become increasingly clear that multiple zeta values (MZV's for short) must be studied. These are defined as follows:

$$\zeta(k_1,...,k_r) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{1}{n_1^{k_1} n_2^{k_2}\cdots n_r^{k_r}},$$

where $k_1, k_2, ..., k_r$ are positive integers with the proviso that $k_1 \geq 2$. The last condition is imposed to ensure convergence of the series.

There are several advantages to introducing these multiple zeta functions. First, they have an algebraic structure which we describe. It is easy to see that

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1,s_2) + \zeta(s_2,s_1) + \zeta(s_1+s_2).$$

Indeed, the left hand side can be decomposed as

$$\sum_{n_1,n_2} \frac{1}{n_1^{s_1}n_2^{s_2}} = \sum_{n_1>n_2} \frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1} \frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_1=n_2} \frac{1}{n_1^{s_1}n_2^{s_2}}$$

from which the identity becomes evident. In a similar way, one can show that

$$\zeta(s_1)\zeta(s_2,...,s_t)$$

is again an integral linear combination of multiple zeta values. More generally, the product of any two MZV's is an integral linear combination of MZV's. These identities lead to new relations, like

$$\zeta(2,1) = \zeta(3),$$

an identity which appears in Apéry's proof [**1**] of the irrationality of $\zeta(3)$.

If we let V_r be the $\mathbb{Q}$-vector space spanned by

$$\zeta(s_1,s_2,...,s_k)$$

with $s_1+s_2+\cdots+s_k = r$, then the product formula for MZV's shows that

$$V_rV_s \subseteq V_{r+s}.$$

In this way, we obtain a graded algebra of MZV's. Let d_r be the dimension of V_r as a vector space over $\mathbb{Q}$. For convenience, we set $d_0 = 1$ and $d_1 = 0$. Clearly, $d_2 = 1$ since V_2 is spanned by $\pi^2/6$. Zagier [**55**] has made the following conjecture:

$$d_r = d_{r-2} + d_{r-3},$$

for $r \geq 3$. In other words, d_r satisfies a Fibonacci-type recurrence relation. Consequently, d_r is expected to have exponential growth. Given this prediction, it is rather remarkable that not a single value of r is known for which $d_r \geq 2$! We relate this to the Chowla-Milnor conjecture in a later section.

In view of the identity, $\zeta(2,1) = \zeta(3)$, we see that $d_3 = 1$. What about d_4? V_4 is spanned by $\zeta(4), \zeta(3,1), \zeta(2,2), \zeta(2,1,1)$. What are these numbers? Zagier's conjecture predicts that $d_4 = d_2 + d_1 = 1 + 0 = 1$. Is this true? Let us see.

We can adapt Euler's technique to evaluate $\zeta(2,2)$. As noted earlier

$$\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_n}\right)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coeffcient of x is

$$-\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}\right).$$

The coefficient of x^2 is

$$\sum_{i<j} \frac{1}{r_i r_j}.$$

With this observation, we see from the product expansion

$$f(x) = \frac{\sin \pi x}{\pi x} = (1 - x^2)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right)\cdots$$

that the coefficient of x^4 is precisely $\zeta(2,2)$. An easy computation shows that

$$\zeta(2,2) = \frac{\pi^4}{5!}.$$

It is now clear that this method can be used to evaluate $\zeta(2,2,...,2) = \zeta(\{2\}^m)$ (say). By comparing the coefficient of x^{2m} in our expansion of $f(x)$, we obtain that

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!},$$

which can also be viewed as another generalization of Euler's result. We could have also evaluated $\zeta(2,2)$ using the identity

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4),$$

but we opted for the method above to indicate its generalization which allows us to also evaluate $\zeta(2,2,...,2)$.

What about $\zeta(3,1)$? What can we say about it? This is a bit more difficult and will not come out of our earlier results. In 1998, Borwein, Bradley, Broadhurst and Lisonek [**6**] showed that $\zeta(3,1) = 2\pi^4/6!$.

What about $\zeta(2,1,1)$? Can we relate it to a known constant? With some work, one can show that this is equal to $\zeta(4)$. Thus, we conclude that $d_4 = 1$ as predicted by Zagier.

What about d_5? With more work, we can show that

$$\zeta(2,1,1,1) = \zeta(5); \quad \zeta(3,1,1) = \zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3).$$

$$\zeta(2,1,1) = \zeta(2,3) = 9\zeta(5)/2 - 2\zeta(2)\zeta(3).$$

$$\zeta(2,2,1) = \zeta(3,2) = 3\zeta(2)\zeta(3) - 11\zeta(5)/2.$$

This proves that $d_5 \leq 2$. Zagier conjectures that $d_5 = 2$. In other words, $d_5 = 2$ if and only if $\zeta(2)\zeta(3)/\zeta(5)$ is irrational. Of course, this is not yet known.

Can we prove Zagier's conjecture? To this date, not a single example is known for which $d_n \geq 2$. In the next section, we will show its relation to another conjecture due to Chowla and Milnor. If we write

$$(1 - x^2 - x^3)^{-1} = \sum_{n=1}^{\infty} D_n x^n,$$

then it is easy to see that Zagier's conjecture is equivalent to the assertion that $d_n = D_n$ for all $n \geq 1$. Deligne and Goncharov [**17**] and (independently) Terasoma [**53**] showed that $d_n \leq D_n$. As noted earlier, Brown [**7**] showed that the MZV's are generated by

$$\zeta(k_1, ..., k_r)$$

with the k_i equal to 2 or 3. The number of such values is clearly seen to be D_n since we are looking for the number of ways of writing n as a sum of 2's and 3's. The conjecture therefore is that this collection is a basis for all MZV's.

It is a fruitful line of research to study multiple Hurwitz zeta functions and this was initiated in [**39**]. These are defined as:

$$\zeta(k_1, ..., k_r; x_1, ..., x_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{(n_1 + x_1)^{k_1} \cdots (n_r + x_r)^{k_r}}.$$

However, many of the elegant evaluations of MZV's have not yet been extended to MHZV's.

8. The Chowla-Milnor conjecture

In the sums considered in the previous section, if one restricts the summation to only positive integers, we are led to study

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)}. \tag{8.1}$$

Again, we restrict the sum over those positive n which avoid the zeros of $B(x)$. In these situations, a partial fraction expansion leads us to write the value of the sum in terms of the digamma function and special values of the Hurwitz zeta function. Recall that the digamma function $\psi(x)$ is the logarithmic derivative of the Γ-function and it appears as the constant term in the Laurent expansion of the Hurwitz zeta function:

$$\zeta(s, x) = \frac{1}{s-1} - \psi(x) + \cdots$$

Indeed, the partial fraction expansions lead to sums of the form

$$\sum_{n=1}^{\infty} \frac{1}{(n+\alpha)^k}.$$

If $k \geq 2$, then the sum is the special value of the Hurwitz zeta function $\zeta(k, \alpha)$. If we assume that the series (8.1) converges, the terms corresponding to $k = 1$ disappear. The nature of the emerging sums has not been investigated since there is only meager knowledge about the special values of Hurwitz zeta functions. In fact, there are very few conjectures on what we may expect. Foremost among the sparse set of conjectures is the one due to Paromita and Sarvadaman Chowla [**13**] and its generalization due to Milnor [**31**]. Their conjecture is that the special values for a fixed natural number $k \geq 2$ and $q \geq 1$

$$\zeta\left(k, \frac{a}{q}\right), \qquad (a, q) = 1,$$

are linearly independent over the rational numbers. In particular, this conjecture would imply that if $f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{Q}$ is a rational-valued function and supported on the coprime residue classes (mod q), then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^k} \neq 0.$$

In [**20**], Gun, Murty and Rath proved that the Chowla-Milnor conjecture implies that

$$\left(\frac{\zeta(2k+1)}{\pi^{2k+1}}\right)^2 \notin \mathbb{Q}$$

for any $k \geq 1$. They also showed that the Chowla-Milnor conjecture for the single modulus $q = 4$ is equivalent to the irrationality of

$$\frac{\zeta(2k+1)}{\pi^{2k+1}}$$

for all $k \geq 1$ (see Proposition 4 in [**20**]). At the end of their paper, the authors formulate a stronger form of the conjecture, namely that,

$$1, \zeta(k, a/q), \qquad (a, q) = 1, \qquad 1 \leq a < q,$$

are linearly independent over the rationals. If this stronger conjecture is true either for $q = 3$ or $q = 4$, then it would imply that $\zeta(2k+1)$ is irrational for every value of k. It would also imply that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^k}$$

is irrational whenever f is rational-valued. From these implications, we see how difficult the Chowla-Milnor conjecture is.

9. The Riemann zeta function at odd arguments

In the notebooks of Ramanujan published by the Tata Institute in 1957, we find some elegant formulas for the special values of the Riemann zeta function at odd arguments. In particular, on page 171 of Volume 2, we see

(9.1)
$$\alpha^{-k}\left\{\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty} \frac{n^{-2k-1}}{e^{2\alpha n} - 1}\right\} = (-\beta)^{-k}\left\{\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty} \frac{n^{-2k-1}}{e^{2\beta n} - 1}\right\}$$

$$(9.2) \qquad - 2^{2k}\sum_{j=0}^{k+1}(-1)^j \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}\alpha^{k+1-j}\beta^j,$$

where $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, k is any nonzero integer and B_j is the jth Bernoulli number. If we put $\alpha = \beta = \pi$ and k is odd, we deduce

$$(9.3) \quad \zeta(2k+1)+2\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2\pi n}-1)} = \pi^{2k+1}2^{2k}\sum_{j=0}^{k+1}(-1)^{j+1}\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!},$$

a formula apparently due to Lerch [**29**] and published in an obscure journal (see also[**5**]). It seems that Grosswald [**18**] rediscovered this formula and published its proof in 1970 only to learn later that it was discovered earlier by Ramanujan and even earlier by Lerch.

Since the left hand side of this equation is non-zero (both the terms being positive), the right hand side is a non-zero rational multiple of π^{2k+1}. Consequently, at least one of

$$\zeta(4k+3), \quad \sum_{n=1}^{\infty}\frac{1}{n^{4k+3}(e^{2\pi n}-1)}$$

is transcendental for every integer $k \geq 0$.

Motivated by these identities, the authors of [**22**] introduced the function:

$$F_k(z) = \sum_{n=1}^{\infty}\sigma_{-k}(n)e^{2\pi i n z}.$$

and proved the following theorem:

THEOREM 9.1. *Let k be a non-negative integer and set $\delta = 0, 1, 2, 3$ according as the* gcd$(k, 6)$ *equals 1,2,3 or 6. With at most $2k + 2 + \delta$ exceptions, the number*

$$F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha)$$

is transcendental for every algebraic $\alpha \in \mathfrak{H}$. In particular, there are at most $2k+2+\delta$ algebraic numbers $\alpha \in \mathfrak{H}$ such that $F_{2k+1}(\alpha)$ and $F_{2k+1}(-1/\alpha)$ are both algebraic.

Notice the similarity between $F_k(z)$ and the classical Eisenstein series $E_{k+1}(z)$ with k odd. It is therefore not unreasonable to study special values of modular forms and quasi-modular forms at algebraic arguments and this is investigated in [**23**]. The interest in this theorem as it relates to $\zeta(2k + 1)$ is highlighted by the following observation. There exist algebraic numbers α in the upper half plane $\mathfrak{H}$ for which the numbers

$$F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha)$$

are non-zero algebraic multiples of $\zeta(2k + 1)$ for all $k \geq 4$. Thus, Theorem 9.1 comes very close to showing transcendence of $\zeta(2k + 1)$. The function $F_k(z)$ is closely related to the classical Eisenstein series $E_{k+1}(z)$ and is really an example of an Eichler integral. Clearly, the study of the special values of these Eichler integrals will shed new light on the nature of the Riemann zeta function at odd arguments.

Much of the analysis used to derive (9.2) can be carried out for Dirichlet L-functions $L(2k + 1, \chi)$ for χ even (and similarly for $L(2k, \chi)$ for χ odd). But this has not been done in a systematic manner and offers a good program for further research. There are some related papers on this topic by Katayama [**27**].

10. Hecke's conjecture and the Siegel-Klingen theorem

The Dedekind zeta function of an algebraic number field F is defined as

$$\zeta_F(s) = \sum_{\mathfrak{a} \neq 0} N(\mathfrak{a})^{-s},$$

where the summation is over non-zero ideals $\mathfrak{a}$ of the ring of integers of F. The analytic continuation and functional equation of $\zeta_F(s)$ was proved by Hecke [**25**] in 1918. From this functional equation, we can see that the Dedekind zeta function has trivial zeroes at $s = -1, -2, ...$ unless F is totally real, in which case it has trivial zeroes only at $s = -2, -4, ...$ just like the Riemann zeta function. It is for this reason that Hecke was able to surmise and Siegel and Klingen were able to extend Euler's theorem describing the special values of $\zeta_F(2k)$ as algebraic multiples of π^{2kd} with $d = [F : \mathbb{Q}]$ when F is totally real.

The gist of the results of an earlier section is that the value of $L(k, \chi)$ when k and χ have the same parity, is a non-zero algebraic multiple of π^k. As mentioned earlier, this fact was first proved by Hecke in 1940 and he noted that this implies an interesting result for real quadratic fields. Namely, if F is a real quadratic field, then $\zeta_F(2m)$ is an algebraic multiple of π^{4m}. This motivated him to ask if such a result holds generally for any totally real field F. That is, if F is totally real of degree d over the rationals, then is it true that $\zeta_F(2m)$ is an algebraic multiple of π^{2dm}? Hecke's calculation also answers his question in another case, namely the case of the cyclotomic subfield $\mathbb{Q}(\zeta + \zeta^{-1})$ where $\zeta = e^{2\pi i/q}$ is a primitive q-th root of unity. This field is also totally real and its Dedekind zeta function is the product of Dirichlet L-functions $L(s, \chi)$ with χ an even character (mod q). Thus, Hecke's simple extension of Euler's theorem allowed him to prove his conjecture in two important cases.

Hecke's question was anwered in the affirmative by Siegel and Klingen and is now known as the Siegel-Klingen theorem. The proof of the Siegel-Klingen theorem makes use of the theory of classical modular forms and Hilbert modular forms. Since a detailed explanation of the proof is beyond the scope of this survey, we content ourselves with a brief outline and refer the reader to Garrett [**16**] as well as Siegel's exposition [**50**] or the authors [**33**] for further details.

Let F be a totally real number field of degree r and discriminant D. Let $x \mapsto x^{(i)}$ be an indexing of real embeddings of F and let $\mathcal{O}_F$ be the ring of integers of $\mathcal{O}_F$ and $\mathfrak{h}$ denote the upper half-plane. The group $SL_2(\mathcal{O}_F)$ is called the Hilbert modular group and it acts on $\mathfrak{h}^r$ via the map:

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F),$$

$$g \cdot (z_1, .., z_r) = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, ..., \frac{a^{(r)} z_r + b^{(r)}}{c^{(r)} z_r + d^{(r)}} \right).$$

For $a, b \in F$ and $z \in \mathfrak{h}^r$, it is convenient to use the notation

$$N(az + b) := \prod_{i=1}^{r} (a^{(i)} z_i + b^{(i)}).$$

Let k be a natural number (which is even if K has a unit of norm -1). For an ideal $\mathfrak{a}$, we define the analogue of the Eisenstein series as

$$F_k(\mathfrak{a}, z) = N(\mathfrak{a})^k \sum_{(\lambda,\mu)\neq(0,0)} N(\lambda z + \mu)^{-k},$$

where the sum runs over a complete system of pairs of numbers in the ideal $\mathfrak{a}$ different from $(0,0)$ and not differing from one another by a factor which is a unit (that is, not associated). One can show that the series converges absolutely for $k > 2$ and for $k = 2$, one can apply a limit process (Hecke's trick). The function $F_k(\mathfrak{a}, z)$ is an example of a Hilbert modular form in the sense that

$$F_k(\mathfrak{a}, gz) = N(cz+d)^k F_k(\mathfrak{a}, z) \qquad \forall \quad g \in SL_2(\mathcal{O}_F).$$

As $\mathcal{O}_F$ is a lattice and its dual is the inverse *different* $\mathfrak{d}^{-1}$ of $\mathcal{O}_F$, one has a Fourier expansion of $F_k(\mathfrak{a}, z)$ of the form

$$\zeta(\mathfrak{a}, k) + \left(\frac{(2\pi i)^k}{(k-1)!}\right)^r D^{1/2-k} \sum_{\nu\in\mathfrak{d}^{-1},\nu\gg 0} \sigma_{k-1}(\mathfrak{a}, \nu) e^{2\pi i Tr(z\nu)},$$

where ν runs over totally positive numbers in $\mathfrak{d}^{-1}$ and

$$\zeta(\mathfrak{a}, k) = N(\mathfrak{a})^k \sum_{\mathfrak{a}|(\mu)} N(\mu)^{-k},$$

$$\sigma_{k-1}(\mathfrak{a}, \nu) = \sum_{\mathfrak{d}^{-1}|(\alpha)\mathfrak{a}|\nu} \operatorname{sign}(N(\alpha)^k) N((\alpha)\mathfrak{a}\mathfrak{d})^{k-1}.$$

The summation is over principal ideals $(\mu), (\alpha)$ under the conditions given. If we set all the variables $z_1, ..., z_r$ to z, then $F_k(\mathfrak{a}, z)$ becomes a classical modular form of weight rk for the full modular group. The final result is then deduced using the classical theory of modular forms. Indeed, let E_k denote the usual normalized Eisenstein series of weight k, Δ denote Ramanujan's normalized cusp form and j the modular invariant. If $M_k(SL_2(\mathbb{Z}))$ is the space of modular forms of weight k for the full modular group, let $t = \dim M_k(SL_2(\mathbb{Z}))$. Put

$$F_k := E_{12r-k+2}\Delta^{-t}.$$

Then F_k has q-expansion

$$C_{kr}q^{-t} + \cdots + C_{k1}q^{-1} + C_{k0} + \cdots$$

with $C_{kr} = 1$ and $C_{k\ell} \in \mathbb{Z}$ for all ℓ. We need to make one further observation. Then for any integer $m \geq 0$,

$$j^m \frac{dj}{dz}$$

has a q-expansion without a constant term. Using these facts, one shows that if f is a modular form of weight k for the full modular group with q-expansion

$$f(z) = a_0 + a_1 q + a_2 q^2 + \cdots$$

and $C_{k\ell}$ are as above, then

$$C_{k0}a_0 + C_{k1}a_1 + \cdots + C_{kr}a_r = 0.$$

The key observation is then that $C_{k0} \neq 0$, so that the constant term a_0 is a rational linear combination of $a_1, ..., a_r$. Applying this to our modular form $F_k(\mathfrak{a}, z)$, we deduce that $\zeta(\mathfrak{a}, k)$ is a an algebraic multiple of π^{kr}. Since the Dedekind zeta function $\zeta_F(k)$ can be written as a rational linear combination of the values $\zeta(\mathfrak{a}, k)$, the Siegel-Klingen theorem follows from this. The reader can find further details in [**50**] or [**33**].

11. Artin L-series

The Riemann zeta function and Dirichlet L-functions, as well as the Dedekind zeta function of a number field F are all special cases of Artin L-series. Let K/F be a finite Galois extension of algebraic number fields with Galois group G. Let (ρ, V) be a complex linear representation of G. For each prime ideal $\mathfrak{p}$ of F, and prime ideal $\wp$ of K lying above $\mathfrak{p}$, we have the usual inertia group $I_\wp$ which is a subgroup of the Galois group G. We then define the Artin L-series attached to ρ as

$$L(s, \rho, K/F) = \prod_{\mathfrak{p}} \det(1 - \rho(\sigma_\wp) N(\mathfrak{p})^{-s} | V^{I_\wp})^{-1},$$

where the product is over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$. One can show that the product is well-defined and converges absolutely for $\Re(s) > 1$. If ρ is the trivial representation, then the corresponding Artin L-series is the Dedekind zeta function of F. If F is $\mathbb{Q}$ and K is the cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is primitive q-th root of unity, then the Galois group is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^*$ and the Artin L-series attached to this Galois group coincide with the classical Dirichlet L-series.

The first question that arises about special values of these Artin L-series is if there is an analog of the Siegel-Klingen theorem. Surprsingly, this was first proved as late as 1973 by Coates and Lichtenbaum [**10**]. A readable exposition of this theorem can be found in [**16**]. But here is a quick summary.

Their result can be stated in the following form. Given a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{C})$, the Artin L-function $L(s, \rho)$ has a functional equation in which the gamma factors appearing in the functional equation are of the form

$$\Gamma(s/2)^a \Gamma((s+1)/2)^b$$

and we say ρ is totally real if $b = 0$ and totally complex if $a = 0$. (One could refer to (a, b) as the "Hodge type" of ρ.) In any case, one can show that $L(2k, \rho)$ is an algebraic multiple of $\pi^{2k\chi(1)}$ if ρ is totally real. Similarly, if ρ is totally complex, then $L(2k+1, \rho)$ is an algebraic multiple of $\pi^{(2k+1)\chi(1)}$. The essential idea for the proofs of these theorems is to use Brauer's induction formula and reduce it to the case of evaluation of Hecke's L-series and then to use a form of the Siegel-Klingen theorem as described in the previous section. I believe that there is no gentle exposition of these facts using classical analysis. The work that comes closest to such a goal is Shintani's paper [**48**]. It would be a good program of research to simplify considerably these proofs into a readable exposition.

12. Schanuel's conjecture and special values at $s = 1$.

The situation with respect to special values of Artin L-series at $s = 1$ has been studied extensively by Stark in a series of papers (see for example, [**52**]). Essentially, the conjecture predicts that the value is (up to an algebraic factor) a power of π and a determinant of logarithms of algebraic numbers (more precisely units).

A special case of the Schanuel conjecture is the following. Let $\alpha_1, ..., \alpha_n$ be non-zero algebraic numbers such that $\log \alpha_1, ..., \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Then these numbers are algebraical-ly independent. Following [**21**], we call this the weak Schanuel conjecture. Baker's theorem asserts that these numbers are linearly independent over $\overline{\mathbb{Q}}$.

Assuming the weak form of Schanuel's conjecture, the authors in [**21**] show that these values are all transcendental numbers. Stark's conjectures should be viewed as generalizations of Dirichlet's class number formula. In some cases, Stark's conjecture can be proved without Schanuel's conjecture and we refer the reader to [**21**] for more details.

13. The Chowla and Erdös conjectures

As mentioned in section 3, Chowla [**12**] asked in 1970 the following question. Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Q}$ be a function defined on the residue classes (mod q), not identically zero. Under what conditions is it true that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0? \tag{13.1}$$

Chowla himself conjectured that this is the case if q is a prime number. This conjecture was then settled by Baker, Birch and Wirsing [**3**] and presumably by Chowla (since Chowla never published his proof and there is a comment by the authors in [**3**] that he had done so along the lines of their paper). The author [**32**] has written an exposition on how Chowla may have proved his theorem with the resources available to him, since the methods of [**3**] involve the theory of linear forms of logarithms and this may not have been the method adopted by Chowla.

Chowla's question is undoubtedly inspired by Dirichlet's theorem regarding the non-vanishing of $L(1,\chi)$. The general programs of special values of L-series have focused on those which admit Euler products and multiplicative structure. It may be fruitful to consider the slightly general framework suggested by Chowla. In this connection, Chatterjee, Murty and Pathak [**9**] have characterized all functions f for which $L(1,f) = 0$ in Chowla's problem. Related to this is a question (conjecture) of Erdös, that (13.1) does not vanish whenever $f(n) = \pm 1$ and $f(q) = 0$. This conjecture is non-trivial only in the case q is odd. Using some algebraic number theory, Murty and Saradha [**38**] settled Erdös's conjecture if $q \equiv 3 \pmod 4$. If $q \equiv 1 \pmod 4$, the conjecture is still open, though it was shown by Chatterjee and Murty [**11**] that the conjecture is true for at least 82 percent of q with $q \equiv 1 \pmod 4$. Most likely, one needs to understand the arithmetic significance of the non-vanishing to settle the conjecture completely.

That this approach has value can be seen in a set of analogous results obtained in the case of an imaginary quadratic field. For instance, if k is an imaginary quadratic field, and f is a function defined on the ideal class group of the ring of integers of k, we may consider

$$L(s,f) = \sum_{0 \neq \mathfrak{a} \in \mathcal{O}_k} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s}$$

and ask under what conditions this is non-zero at $s = 1$. This has been answered by using the Kronecker limit formula, by the author and V. Kumar Murty in [**34**] and further extended to functions on ray class groups of k in [**35**]. A good problem for further research is to study this in general number fields.

14. Concluding remarks

The appearance of the polylogarithm functions is expected in the evaluation of special values of Artin L-series. Zagier formulated a general conjecture for Dedekind zeta functions and this has been extended for Artin L-series by Zagier and Gangl [**56**]. The polylogarithm function is

$$L_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

and one expects that the value of $L(s,\rho)$ at s equal to an integer m is a power of π times a determinant of a matrix whose entries are values of the polylogarithm function evaluated at certain algebraic numbers. It would be interesting to make this more precise. This has been done in some cases but there is definitely a need to specify which power of π we expect, what the size of the determinant should be, and what are the algebraic numbers that appear as arguments of the polylogarithm function. A part of this conjecture already emerges in the work of [**34**] where the authors evaluated the special values of Hecke L-series attached to an imaginary quadratic field.

The polylogarithm function $L_k(z)$ is of course a generalization of the classical logarithm function since for $k = 1$, $L_1(z) = -\log(1-z)$. Much of the success of the work so far on the Chowla problem is due to Baker's theorem about special values of $L_1(z)$ at algebraic arguments. This led Gun, Murty and Rath to make the following polylog conjecture. Suppose that $\alpha_1, ..., \alpha_n$ are algebraic numbers satisfying $|\alpha_i| \leq 1$ such that $L_k(\alpha_1), ..., L_k(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then they are linearly independent over $\overline{\mathbb{Q}}$. They show [**21**] that if the polylog conjecture is true, then the Chowla-Milnor conjecture is true for all $q > 1$ and $k > 1$. Thus, the program to extend Baker's theory of linear forms in logarithms to linear forms in polylogarithms will have tremendous applications in solving many open problems.

In Zagier's formulation of special values, we see the appearance of the polylogarithm function. By contrast, in [**19**], the authors relate special values of zeta and L-functions to the multiple gamma functions. They also study instances of special values of derivatives of the Riemann zeta function and Dirichlet L-functions. In this connection, there is some similarity with other conjectures such as the Birch and Swinnerton-Dyer conjecture. One could consider more generally, L-series of automorphic L-functions or even linear combinations of these. This short survey cannot exhaust the topics or the possibilities of these lines of investigation. But we hope the reader is inspired to explore further this galaxy of special values and behold its stellar beauty.

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