Inose's construction and elliptic $K3$ surfaces with Mordell-Weil rank 15 revisited

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ABSTRACT. We describe two constructions of elliptic $K3$ surfaces starting from the Kummer surface of the Jacobian of a genus 2 curve. These parallel the base-change constructions of Kuwata for the Kummer surface of a product of two elliptic curves. One of these also involves the analogue of an Inose fibration. We use these methods to provide explicit examples of elliptic $K3$ surfaces over the rationals of geometric Mordell-Weil rank 15.

1. Introduction

Elliptic $K3$ surfaces have been the focus of much work in algebraic geometry and number theory over the last few decades. For such a surface over a field of characteristic 0 with at least one singular fiber, the rank of the Mordell-Weil group can be as large as 18. Cox [**C**] proved in 1989, using the surjectivity of the period map for $K3$ surfaces, that any integer between 0 and 18 actually occurs as the rank of the Mordell-Weil group of an elliptic $K3$ surface over $\mathbb{C}$. However, the proof uses transcendental methods, and does not lead to an algebraic construction of examples of elliptic $K3$ surfaces of a given rank.

In 2001 Kuwata constructed explicit examples of elliptic $K3$ surfaces over $\mathbb{Q}$ whose Mordell-Weil group over $\overline{\mathbb{Q}}$ has rank r, where $0 \leq r \leq 18$ and $r \neq 15$ ([**Kw**]). The proof uses Inose's theorem (see Proposition 2.2), which relies on a transcendental argument. For this reason, it was not clear how to find explicit generators of the Mordell-Weil group of such elliptic surfaces until our recent work [**KK**], where we found a systematic method to describe a set of generators (see also [**Kl2**] for earlier progress on this question). Meanwhile, there have been constructions of elliptic $K3$ surfaces with Mordell-Weil rank 15 ([**Kl1**][**TdZ**]) in order to fill the gap of [**Kw**]. However, these constructions are based on somewhat different ideas from that of [**Kw**], to which the method of [**KK**] does not seem to apply to find generators of the Mordell-Weil group.

The goal of this article is to describe some new constructions of elliptic $K3$ surfaces with high Mordell-Weil rank using ideas closely related to the idea of [**Kw**]. The construction of [**Kw**] is based on the Kummer surfaces $Km(E_1 \times E_2)$ associated with the product of two elliptic curves. For such a Kummer surface Shioda and Inose [**SI**] constructed a double cover with certain properties (now called a Shioda-Inose

structure). We say that a $K3$ surface S has a Shioda-Inose structure if S admits an involution fixing the global 2-form on S, such that the quotient is a Kummer surface $Km(A)$, and provided that the rational quotient map $S \dashrightarrow Km(A)$ induces a Hodge isometry $T_S(2) \simeq T_{Km(A)}$ of (scaled) transcendental lattices. When $A = E_1 \times E_2$ is the product of two elliptic curves, S was constructed in [**SI**] as an elliptic surface having two II* fibers. Shioda and Inose used this construction to prove a beautiful theorem establishing a one-to-one correspondence between singular $K3$ surfaces and even positive definite binary quadratic forms. Later, Inose [**I1**] constructed S explicitly as a quartic surface in $\mathbb{P}^3$ and as the quotient of a Kummer surface by an involution. Thus, we have a chain of rational maps $S \dashrightarrow Km(A) \dashrightarrow S$, which we now call a "Kummer sandwich" ([**Shi4**]). The construction of $Km(E_1 \times E_2) \dashrightarrow S$ in [**I1**] can be viewed as a base change of elliptic surfaces, and it is this fact that [**Kw**] generalized to construct elliptic $K3$ surfaces of high rank. Elliptic $K3$ surfaces of various Mordell-Weil ranks are constructed as a base change from S, which we call the Inose surface.

Our main idea in this article is to replace $A = E_1 \times E_2$ by the Jacobian $J(C)$ of a curve C of genus 2. We propose two different generalizations of Inose's fibration on $Km(E_1 \times E_2)$. A detailed study of Shioda-Inose structures on $K3$ surfaces was carried out by Morrison [**Mo**]. For Kummer surfaces of principally polarized abelian surfaces, double covers with Shioda-Inose structure were studied by Naruki [**N**] and Dolgachev [**GLD**]. In [**Km1**], Kumar explicitly described these surfaces as a family of elliptic $K3$ surfaces with II* and III* fibers, whose Weierstrass coefficients are related to invariants of the genus 2 curve C. In this construction, the rational map $S \dashrightarrow Km(A)$ in [**Km1**] is a 2-isogeny between elliptic surfaces. The dual isogeny sets up the sandwich $S \dashrightarrow Km(A) \dashrightarrow S$. Our first construction is based on the observation that $Km(A) \dashrightarrow S$ can also be described as an Inose-style base change. More precisely, there is an elliptic fibration on the Kummer surface (fibration 13 in [**Km2**]) which can be obtained from the Shioda-Inose surface in [**Km1**] as a base change by a simple change of variables. In §3 we give the details of this construction of elliptic $K3$ surfaces of various Mordell-Weil ranks. In particular, we give two numerical examples that have Mordell-Weil rank 15.

Note that fibration 13 in [**Km2**] has one IV* fiber and one I$_0^*$ fiber, while Inose's fibration on $Km(E_1 \times E_2)$ has two IV* fibers. There does exist an elliptic fibration on $Km(J(C))$ with two IV* fibers, and it can be constructed from known fibrations in [**Km2**] using the "2-neighbor method" of Elkies. However, this fibration does not generically have a section. Nevertheless, taking its Jacobian fibration produces an elliptic fibration with section and the same fiber type. Taking a base change, we can construct elliptic $K3$ surfaces with Mordell-Weil rank up to 18. In §4 we give details of this construction, and give a numerical example with Mordell-Weil rank 15.

In future work, we hope to find explicit generators for the Mordell-Weil groups of the elliptic $K3$ surfaces constructed here, along the lines of [**KK**].

The computer algebra systems `Magma`, `sage`, `gp/PARI`, `Maxima` and `Maple` were used in the calculations for this paper.

2. Preliminaries

By an elliptic surface we mean a smooth projective surface S together with a relatively minimal elliptic fibration $\pi : S \to C$ over a smooth projective curve C, all defined over a field k, which we assume to be a number field throughout. We are interested in the case where S is a $K3$ surface, C is isomorphic to $\mathbb{P}^1$ over k, and π admits a section $\sigma_0 : \mathbb{P}^1 \to S$ also defined over k. Under these assumptions the generic fiber $\mathcal{E}$ of π is an elliptic curve defined over the rational function field $k(t)$. We also assume there is at least one singular fiber (to ensure that the Mordell-Weil group is finitely generated). Denote by $\mathrm{MW}_{\bar{k}}(\mathcal{E})$ the Mordell-Weil lattice $\mathcal{E}(\bar{k}(t))/\mathcal{E}(\bar{k}(t))_{\mathrm{tors}}$. We refer the reader to [**Shi2**] for the theory of Mordell-Weil lattices and [**SS**] for a broader overview of elliptic surfaces.

A theorem of Shioda and Tate connects the Mordell-Weil $\mathcal{E}(\bar{k}(t))$ group with the Picard group or the Néron-Severi group $\mathrm{NS}_{\bar{k}}(S)$ of $\mathcal{E}$ (note that linear equivalence and algebraic equivalence coincide). In particular, we have the Shioda-Tate formula (cf. Shioda[**Shi1**])

$$(2.1) \qquad \mathrm{rank}\, \mathrm{NS}_{\bar{k}}(S) = 2 + \mathrm{rank}\, \mathrm{MW}_{\bar{k}}(E) + \sum_{P \in C} (m_P - 1),$$

where m_P is the number of (geometrically) irreducible components of the fiber $\pi^{-1}(P)$. We call $\rho(S) = \mathrm{rank}\, \mathrm{NS}_{\bar{k}}(S)$ the (geometric) Picard number.

REMARK 2.1. In the absence of a section we would say that π is a genus one fibration. Sometimes in the literature these are called elliptic fibrations and those with section are labeled Jacobian elliptic fibrations.

The following theorem of Inose is essential to our construction.

PROPOSITION 2.2 (Inose [**I1**, Cor. 1.2]). *Let S_1 and S_2 be $K3$ surfaces defined over a number field k and $f : S_1 \to S_2$ a rational map of finite degree. Then S_1 and S_2 have the same Picard number.*

PROOF. Inose proves this lemma for $K3$ surfaces over $\mathbb{C}$ using a transcendental argument. Since we know that $\mathrm{NS}_{\mathbb{C}}(S) = \mathrm{NS}_{\bar{k}}(S)$, the conclusion is still valid for our case. $\square$

Due to the transcendental nature of Inose's proof we do not have a very good understanding of how the Néron-Severi lattices of S_1 and S_2 are related. Thus, knowledge of an explicit set of divisors generating the Picard group of S_1, or of the Mordell-Weil group if S_1 is elliptically fibered, does not easily translate to the same for S_2.

Let S be a Kummer surface associated with an abelian surface A. Then,

$$\mathrm{rank}\, \mathrm{NS}_{\bar{k}}(S) = \mathrm{rank}\, \mathrm{NS}_{\bar{k}}(A) + 16.$$

PROPOSITION 2.3 (cf. Birkenhake-Lang [**BL**, Prop. 5.5.7]). *Let A be a simple abelian surface. Then, the Picard number $\rho(A) = \operatorname{rank} NS_{\bar{k}}(S)$ is given as follows.*

$\operatorname{End}_{\mathbb{Q}}(A)$	$\rho(A)$
$\mathbb{Q}$	1
real quadratic field	2
indefinite quaternion algebra	3
CM field of degree 4	2

PROPOSITION 2.4. *Let A be an abelian surface isogenous to the product of two elliptic curves $E_1 \times E_2$. Then, the Picard number $\rho(A)$ is given as follows.*

$A \sim E_1 \times E_2$	$\rho(A)$
$E_1 \not\sim E_2$	2
$E_1 \sim E_2$, *without CM*	3
$E_1 \sim E_2$, *with CM*	4

3. Inose type surface for the Jacobian of a curve of genus 2

Let C be a curve of genus 2 given by the equation

$$y^2 = f(x) = \sum_{i=0}^{6} f_i x^i,$$

where f is a squarefree polynomial. In particular, we allow f_6 to be 0 (but in that case $f_5 \neq 0$). In [**Km1**] it is shown that a generic elliptic $K3$ surface with II* and III* fibers has a Shioda-Inose structure such that the quotient is the Kummer surface of the Jacobian of a genus 2 curve. In fact, this correspondence sets up a birational map between the moduli space of such elliptic $K3$ surfaces and $\mathcal{M}_2$, the moduli space of genus 2 curves. It can also be viewed as an isomorphism between the moduli space of $K3$ surfaces lattice polarized by $U \oplus E_8(-1) \oplus E_7(-1)$ and $\mathcal{A}_2$, the moduli space of principally polarized abelian surfaces. For a genus 2 curve C, we can describe the unique associated elliptic $K3$ surface with II* and III* fibers as follows.

Let (I_2, I_4, I_6, I_8) be the Igusa-Clebsch invariants (see for example [**Km2**]). Then the corresponding $K3$ surface is given by the Weierstrass equation

$$(3.1) \qquad y^2 = x^3 - t^3\Big(\frac{I_4}{12}t + 1\Big)x + t^5\Big(\frac{I_{10}}{4}t^2 + \frac{I_2 I_4 - 3I_6}{108}t + \frac{I_2}{24}\Big)$$

It has a type II* fiber at $t = \infty$ and a type III* at $t = 0$. We say that C is "general" if the other singular fibers are all I_1.

The surface defined by (3.1) is isomorphic to

$$y^2 = x^3 - \Big(\frac{I_4}{12} + \frac{1}{t}\Big)x + \Big(\frac{I_{10}}{4}t + \frac{I_2 I_4 - 3I_6}{108} + \frac{I_2}{24t}\Big),$$

which we call $G^{(1)}$, and define the base change

$$G^{(n)} : y^2 = x^3 - \Big(\frac{I_4}{12} + \frac{1}{t^n}\Big)x + \Big(\frac{I_{10}}{4}t^n + \frac{I_2 I_4 - 3I_6}{108} + \frac{I_2}{24t^n}\Big).$$

Compare with the definition of $F^{(n)}$ in [**Shi3**] and [**KK**]. Straightforward calculations show that the Kodaira-Néron model of $G^{(n)}$ is a $K3$ surface for $n = 1, \ldots, 4$.

In [**Km1**] the map $G^{(1)} \to Km(J(C))$ is given as a 2-isogeny between two elliptic surfaces. All Jacobian elliptic fibrations (elliptic fibrations with section) on $Km(J(C))$ are classified in [**Km2**]. Fibration 13 in [**Km2**] is

$$y^2 = x^3 - 108t^4\big(48t^2 + I_4\big)x + 108t^4\big(72I_2t^4 + (4I_4I_2 - 12I_6)t^2 + 27I_{10}\big),$$

which is isomorphic to $G^{(2)}$ by $(x, y, t) \mapsto \big(t^2x/9, t^3y/27, 1/(2t)\big)$. Thus, $G^{(2)}$ is isomorphic to $Km(J(C))$, and we have a Kummer sandwich diagram: $G^{(1)} \dashrightarrow G^{(2)} \simeq Km(J(C)) \dashrightarrow G^{(1)}$.

REMARK 3.1. The surface $G^{(1)}$ can be realized as a quartic surface just as Inose's suface in [**I2**]:

$$y^2zw - x^3z + \Big(\frac{I_4}{12}w + z\Big)xzw - \Big(\frac{I_{10}}{4}w^2 + \frac{I_2}{24}z^2\Big)w^2 - \Big(\frac{I_2I_4 - 3I_6}{108}\Big)zw^3 = 0.$$

Therefore we can regard $G^{(1)}$ as a generalization of Inose's surface to the case of Jacobian of the genus 2 curve C. This point of view was also studied in [**CD**].

THEOREM 3.2. *Suppose C is general. Then the rank of the Mordell-Weil group $G^{(n)}(\overline{\mathbb{Q}}(t))$ is given by the table below. Here, $\rho = \rho(J(C))$ is the Picard number of $J(C)$.*

	Singular fibers	Rank
$G^{(1)}$	$II^*, III^*, 5I_1$	$\rho - 1$
$G^{(2)}$	$IV^*, I_0^*, 10I_1$	$4 + \rho$
$G^{(3)}$	$I_0^*, III, 15I_1$	$9 + \rho$
$G^{(4)}$	$IV, 20I_1$	$12 + \rho$

PROOF. Determination of the types of singular fibers is a straightforward application of Tate's algorithm. By Proposition 2.2, the Picard number of each of the $K3$ surfaces $G^{(1)}, \ldots, G^{(4)}$ is equal to that of $Km(J(C))$, namely, $16 + \rho$. Now, calculation of the rank is a straightforward application of the Shioda-Tate formula (2.1). $\qquad\square$

In order to obtain elliptic $K3$ surfaces whose Mordell-Weil rank 15, it now suffices to find curves C such that the Picard number of its Jacobian $J(C)$ is 3. We give two different examples below.

REMARK 3.3. As shown in [**Km2**], a set of five sections forms a basis of $G^{(2)}(\overline{\mathbb{Q}}(t))$ if $\rho(J(C)) = 1$. These sections are defined over the field $k(J(C)[2])$.

EXAMPLE 3.4 (QM case). In the case where the Jacobian $J(C)$ is simple, $\rho(J(C)) = 3$ if and only if it has quaternionic multiplication. Writing down an explicit equation of such a curve C is a delicate number theoretical problem. Here, we use an example of [**DR**] which depends on a result of Hashimoto and Tsunogai [**HT**] (see also [**HM**]). Take the curve in [**HT**, Lemma 4.5] and let $\sigma = \tau = \sqrt{-3/2}$. Then we obtain

$$C : y^2 = \Big(x^2 + \frac{7}{2}\Big)\Big(\frac{83}{30}x^4 + 14x^3 - \frac{1519}{30}x^2 + 49x - \frac{1813}{120}\Big).$$

It has quaternionic multiplication by a maximal order in the quaternion algebra of discriminant 6, which is the smallest possible discriminant for a non-split quaternion

algebra. Calculating Igusa-Clebsh invariants, and simplifying the coefficients by a change of variables, we obtain the elliptic surface

$$G^{(4)} : Y^2 = X^3 + 529200\Big(6 - \frac{5}{t^4}\Big)X - 9261000\Big(4t^4 + 20 - \frac{3431}{t^4}\Big).$$

Simple calculations show that the singular fibers are irreducible except at $t = \infty$ where it has a fiber of type IV. This shows that the rank of $G^{(4)}(\overline{\mathbb{Q}}(t))$ equals 15.

REMARK 3.5. If we take the quadratic twist of $G^{(4)}$ by $\mathbb{Q}(\sqrt{2 \cdot 3 \cdot 5 \cdot 7})$, the elliptic surface becomes simpler:

$$Y^2 = X^3 + 12\Big(6 - \frac{5}{t^4}\Big)X - \Big(4t^4 + 20 - \frac{3431}{t^4}\Big).$$

However, it is possible that the field of definition of the Mordell-Weil group of the original surface may be easier to describe.

EXAMPLE 3.6 (Split case). As a second example, we consider the case where $J(C)$ is isogenous to the product of two elliptic curves. Since we want $\rho(J(C))$ to be equal to 3, we need to consider the case where $J(C) \sim E \times E$ with E without complex multiplication.

Take two elliptic curves

$$E_1 : y^2 + xy + y = x^3 + 4x - 6 \qquad \text{(Cremona label 14a1)},$$
$$E_2 : y^2 + xy + y = x^3 - 36x - 70 \quad \text{(Cremona label 14a2)},$$

These both have rational 6-torsion points, and are 2-isogenous to each other. Moreover, we have $E_1[3] \simeq E_2[3] \simeq \mathbb{Z}/3\mathbb{Z} \times \mu_3$ as Galois modules. Take the quadratic twist $E_2^{(-3)}$ of E_2 by -3. Then, the Galois isomorphism $E_1[3] \to E_2[3]$ isometric with respect to the Weil pairing induces an anti-isometry $E_1[3] \to E_2^{(-3)}[3]$. Therefore, by Frey-Kani [**FK**], there exists a curve C of genus 2 admitting morphisms $C \to E_1$ and $C \to E_2^{(-3)}$ of degree 3. To find an explicit equation, we use Shaska's result [**Sha**, §3]. Using the fact that $j(E_1) = (215/28)^3$ and $j(E_2) = (1705/98)^3$, we find that the curve C corresponds to the values $\mathfrak{u} = 107553525/12595352$, $\mathfrak{v} = -11619959625/1032401161$ in equation (8) in [**Sha**]. After making some changes of variables, we obtain the following Weierstrass equation for C:

$$C : y^2 = -96393(13x + 12)(7x - 13)(107x^2 - 273x + 252)(56x^2 + 104x + 31).$$

The maps $\phi_1 : C \to E_1$ and $\phi_2 : C \to E_2^{(-3)}$ are given by

$\phi_1 : (x, y) \mapsto (x_1, y_1) =$
$$\Big(-\frac{156260x^3 + 3627x + 86895}{3(7x - 13)(56x^2 + 104x + 31)}, -\frac{4(13x - 6)(214x^2 + 273x + 126)y}{9(7x - 13)^2(56x^2 + 104x + 31)^2} \Big),$$
$\phi_2 : (x, y) \mapsto (x_2, y_2) =$
$$\Big(\frac{47485x^3 + 4173x^2 - 211380}{(13x + 12)(107x^2 - 273x + 252)}, -\frac{(12(7x + 26))(14x^2 - 52x + 31)y}{(13x + 12)^2(107x^2 - 273x + 252)^2} \Big),$$

where we have used the following Weierstrass forms for E_1 and $E_2^{(-3)}$:

$$E_1 \quad : y_1^2 = x_1^3 + 5805x_1 - 285714,$$
$$E_2^{(-3)} : y_2^2 = x_2^3 - 5115x_2 + 115414.$$

The elliptic $K3$ surface $G^{(n)}$ corresponding to this C is given by

$$G^{(n)} : Y^2 = X^3 + 33\Big(2933005 - \frac{1126255812}{t^n}\Big)X$$
$$- 2\Big(28449792t^n - 8690133815 - \frac{274280846290470}{t^n}\Big).$$

(Here, we have scaled X, Y, and t differently from the definition of $G^{(n)}$ to clear denominators.) It is readily verified that the singular fibers other than the IV fiber at $t = \infty$ are all irreducible. Since E_1 and $E_2^{(-3)}$ are isogenous over $\mathbb{Q}(\sqrt{-3})$ and do not have complex multiplication, the rank of $G^{(4)}(\overline{\mathbb{Q}}(t))$ is exactly 15.

REMARK 3.7. The reader may wonder why we did not use a curve C such that $J(C)$ is $(2,2)$-isogenous to the product $E_1 \times E_2$. The reason is that the resulting elliptic surface $G^{(1)}$ has at least one I_2 fiber other than II* and III* fibers. Thus, the rank of $G^{(4)}(\overline{\mathbb{Q}}(t))$ cannot be 15. Similarly, Example 1 in [**Sha**, §5] yields $G^{(1)}$ with two I_2 fibers, and Example 2 yields one with a I_3 fiber.

4. Fibration with two IV* fibers

A Kummer surface $Km(J(C))$ can be realized as a complete intersection of three quadrics in $\mathbb{P}^5$. It contains two sets of sixteen lines intersecting each other as shown in [**GH**, p.767, Figure 21]. Classically, these sets are labeled *nodes* and *tropes*; in the model of the Kummer surface as a quartic surface in $\mathbb{P}^3$ with sixteen singular points, the (blown-down) nodes are the singular points, while the tropes correspond to the sixteen planes tangent to the Kummer surface along plane conics (which are the transforms of the tropes).

As indicated in Figure 1, we find a configuration of a pair of divisors of type IV*. We thus have an elliptic fibration on $Km(J(C))$ having these two IV* fibers. It turns out, however, that such an elliptic fibration does not have a section. Thus, it is not in the list of elliptic fibrations studied in [**Km2**].

The elliptic divisor corresponding to the fibration is $F = 3T_0 + 2N_3 + 2N_4 + 2N_5 + T_3 + T_{14} + T_{25}$.

To obtain an explicit equation of this fibration, we start with Fibration 1 in [**Km2**], two I_0^* fibers and six I_2 fibers, with Mordell-Weil group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. The zero section in [**Km2**] is taken to be T_2. However, we will take T_3 to be the zero section; this does not affect the Weierstrass equation, however changes the identity components of the reducible fibers, for instance. We can now take a 2-neighbor step to the elliptic fibration corresponding to the fiber $F' = T_3 + N_3 + T_0 + N_4 + T_{14} + N_{23}$ (this fiber class is in the orbit of fibration 10 of [**Km2**]). We omit the details of the neighbor-step, apart from giving the elliptic parameter

$$t_1 = \frac{1}{t-a}\left(\frac{y+y_s}{x-x_s} - 2a(b-1)(c-a)\right)$$

where

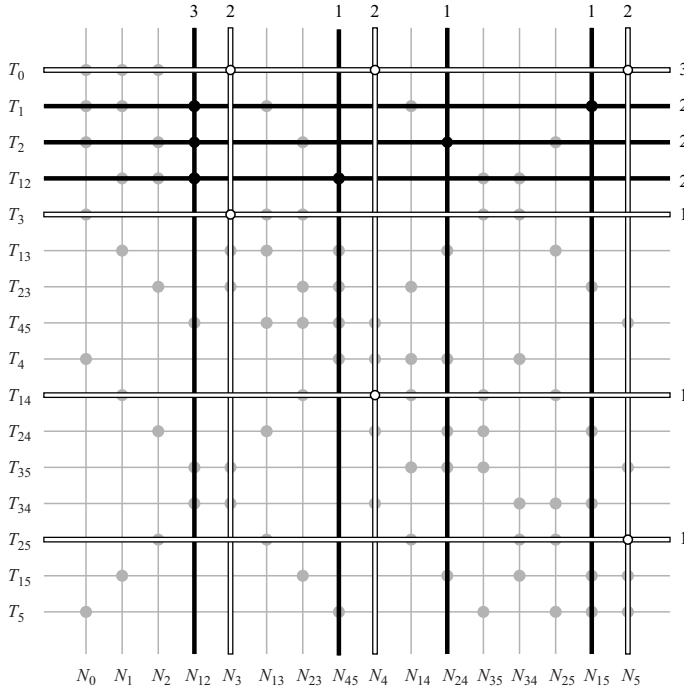$$(x_s, y_s) = \big(-4(a-1)(b-1)t(t-c)(ct-ab),\ -8(a-1)(b-1)(c-a)(c-b)t^2(t-ab)(t-c)\big)$$

FIGURE 1. Two IV* fibers in the 16–6 configuration of NS($Km(J(C))$)

is the equation of the section T_{14}. The new elliptic fibration has Weierstrass equation.

$$
\begin{aligned}
y_1^2 = x_1 \big(x_1^2 &+ x_1(t_1^4/16 + t_1^3(b-1)(c-a)/2 \\
&+ t_1^2(3b^2c^2 - abc^2 - 4bc^2 + 2ac^2 - 4ab^2c - b^2c + 2a^2bc \\
&+ 3abc + 2bc - 4a^2c + 2ac + 2ab^2 + 2a^2b - 4ab)/2 \\
&+ 2t_1(b-1)b(b-a)(c-1)c(c-a) + b^2(b-a)^2c^2(c-1)^2) \\
&- (a-1)a(b-1)(c-a)(c-b)t_1^3(t_1 + 2bc - 2b)(t_1 + 2bc - 2ac)/2 \big)
\end{aligned}
$$

The fiber F' is at ∞, while there is another fiber of type I_6 at $t_1 = 0$, given by $T_1 + N_{12} + T_{12} + N_{45} + T_5 + N_{15}$. There are also I_2 fibers at $t_1 = -2(b-a)c$ and $t_1 = -2b(c-1)$. The first is $T_{25} + N'_{35}$ (in the notation of Fibration 1 of [**Km2**]) and the second has N_{24} as a component. We can now move by another 2-neighbor

step to the elliptic fibration with F as a fiber. The elliptic parameter is

$$t_2 = \frac{x_1}{t_1 + 2(b-a)c};$$

we omit the remaining details. The Jacobian of this new fibration is given by

$$Y^2 = X^3 + AX^2 + \Big(B_1 T + B_2 + \frac{B_3}{T}\Big)X + \Big(C_1 T + \frac{C_2}{T}\Big)^2,$$

where $T = t_2$ and

$$
\begin{aligned}
A &= 4\big(abc^2 + bc^2 - 2ac^2 + ab^2 c - 2b^2 c - 2a^2 bc + bc + 4a^2 c - 2ac + ab^2 - 2a^2 b \\
&\quad +ab\big), \\
B_1 &= (a-1)c - a(b-1), \\
B_2 &= -16\big(ab^2 c^4 - a^2 bc^4 - abc^4 + a^2 c^4 + a^2 b^3 c^3 - ab^3 c^3 - b^3 c^3 - a^3 b^2 c^3 \\
&\quad + 2a^2 b^2 c^3 - 2ab^2 c^3 + b^2 c^3 + a^2 bc^3 + 2abc^3 - 2a^2 c^3 - ab^4 c^2 + b^4 c^2 \\
&\quad - a^3 b^3 c^2 - 2a^2 b^3 c^2 + 6ab^3 c^2 - b^3 c^2 + a^4 b^2 c^2 + a^3 b^2 c^2 - 3a^2 b^2 c^2 \\
&\quad - 2ab^2 c^2 + a^2 bc^2 - abc^2 + a^2 c^2 + a^2 b^4 - ab^4 c + 2a^3 b^3 c - 2a^2 b^3 c - ab^3 c \\
&\quad - 2a^4 b^2 c + a^3 b^2 c + 2a^2 b^2 c + ab^2 c - a^2 bc - a^3 b^3 + a^2 b^3 + a^4 b^2 - a^3 b^2\big), \\
B_3 &= -ab^2 c(a-1)(b-1)(c-1)^2(a-b)(b-c)(c-a)\big((a-1)c - a(b-1)\big), \\
C_1 &= 1, \\
C_2 &= ab^2 c(a-1)(b-1)(c-1)^2(a-b)(b-c)(c-a).
\end{aligned}
$$

It has IV^* fibers at $T = 0$ and ∞.

Define

$$H^{(n)} : Y^2 = X^3 + AX^2 + \Big(B_1 t^n + B_2 + \frac{B_3}{t^n}\Big)X + \Big(C_1 t^n + \frac{C_2}{t^n}\Big)^2.$$

The Kodaira-Néron model of $H^{(n)}$ is a $K3$ surface for $n = 1, 2, 3$.

THEOREM 4.1. *Suppose C is general. Then the rank of the Mordell-Weil group $H^{(n)}(\overline{\mathbb{Q}}(t))$ is given by the table below. Here, $\rho = \rho(J(C))$ is the Picard number of $J(C)$.*

	Singular fibers	Rank
$H^{(1)}$	$2\text{IV}^*, 8\text{I}_1$	$2 + \rho$
$H^{(2)}$	$2\text{IV}, 12\text{I}_1$	$10 + \rho$
$H^{(3)}$	24I_1	$14 + \rho$

PROOF. Since $H^{(1)}$ is the Jacobian of an elliptic fibration on $Km(J(C))$, there is a rational map $H^{(1)} \dashrightarrow Km(J(C))$ of finite degree. Thus, by Proposition 2.2, the Picard number of $H^{(1)}$ is equal to $16 + \rho$. The rest of the proof is similar to that of Theorem 3.2. □

Thus, $H^{(3)}$ for any simple abelian surface without extra endomorphisms gives an elliptic $K3$ surface of rank 15. However, although a very general abelian surface (in the moduli space of principally polarized abelian surfaces) has this property, and we expect a "random" numerical example to have $\rho = 1$, we must verify it for any putative example explicitly. To do so, we first use Leprévost's criterion [**L**, Lemme 3.1.2]: a genus 2 curve C over $\mathbb{Q}$ has simple Jacobian if there is a prime ℓ of good reduction such that the Galois group of the characteristic polynomial of Frobenius of the reduced curve is the 8-element dihedral group D_4. We can compute the characteristic polynomial of Frobenius in `magma` or `sage`; the latter has an

optimized implementation of the p-adic Harvey-Kedlaya algorithm for computing the matrix of Frobenius for a curve over a finite field.

EXAMPLE 4.2. Let $a = -1$, $b = 1/7$, and $c = -6/7$. Then, we can verify that the Jacobian $J(C)$ of the curve

$$C : y^2 = x(x-1)(x+1)\Big(x - \frac{1}{7}\Big)\Big(x + \frac{6}{7}\Big)$$

is simple and $\rho(J(C)) = 1$ by the lemma below. After a certain change of variables, we have

$$H^{(n)} : y^2 = x^3 - \frac{1354}{7}x^2 + \frac{936}{7}\Big(t^n + \frac{42989}{819} + \frac{4}{t^n}\Big)x + \frac{6084}{49}\Big(t^n - \frac{4}{t^n}\Big)^2.$$

The Mordell-Weil rank of $H^{(3)}$ is 15.

LEMMA 4.3. *Let C be the curve of Example 4.2. Then $\rho(J(C)) = 1$.*

PROOF. The characteristic polynomial of Frob_{37} and Frob_{41} on $A = J(C)$ are given by

$$p_{37} = x^4 - 4x^3 + 46x^2 - 148x + 1369$$
$$p_{41} = x^4 + 4x^3 + 6x^2 + 164x + 1681.$$

We can directly verify (for instance in `gp/PARI` or `Magma`) that p_{37} has Galois group D_4. Therefore, the Jacobian is simple. Next, since p_{37} and p_{41} are irreducible, it follows by [**Ta**, Theorem 2] that the endomorphism rings of the reductions $A \otimes \mathbb{F}_{37}$ and $A \otimes \mathbb{F}_{41}$ are orders in quartic fields defined by p_{37} and p_{41}. These fields are easily checked to be linearly disjoint. Since $\mathrm{End}(A) \otimes \mathbb{Q}$ is a subfield of each, it must be $\mathbb{Q}$, and $\mathrm{End}(A)$ must be $\mathbb{Z}$.   □

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