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Poncelet's porism and projective fibrations

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Abstract. Poncelet's porism theorem is used to produce a natural compactification of several moduli spaces. The monodromy of the polygons, viewed as torsion points on a fibration by elliptic curves, can be tested computationally for an action of the full symmetric group. Analogous constructions can be implemented for hyperelliptic fibrations corresponding to higher-dimensional versions of Poncelet's porism.

Introduction

The goal of this note is to present certain configurations of projective geometry (over the complex numbers), compute their invariants in the sense of classical invariant theory, and give them an interpretation in terms of elliptic fibrations, with applications to classical mechanics.

The theory of invariants for (sets of) quadric hypersurfaces has a long and ongoing history. So does Poncelet's Porism, a classical result which affords perhaps the first example of a link between projective geometry and abelian functions. In this paper, I offer a few observations on common aspects of these two theories. In Section 1, I set up and classify an interpretation of the Segre symbol of a pencil of space quadrics in relation to a plane Poncelet configuration. My motivation is the construction of natural projective varieties that represent a natural compactification of certain moduli spaces. For example, while there is no natural elliptic fibration that encodes certain data of the Poncelet configurations [BM, Sec. 1], there is a surface in $\mathbb{P}^3$ which contains all the relevant elliptic curves (and their limits): to construct it, I use a linear deformation of the pencil. The relationship between moduli of curves and these projective configurations allows for the computation of interesting numerical invariants. A second reason for investigating limits is a dynamical application: it was proved by G.D. Birkhoff that Poncelet's porism is equivalent to the integrability of a billiard with elliptical boundary (cf. e.g. [KT]), as are its higher-dimensional generalizations (cf. e.g. [Pr] and, particularly for Cayley's closure condition, [DR1, 2]). Using the idea of this note for these generalizations will give hyperelliptic, as opposed to elliptic, fibrations. In Section 2, I pose the Galois problem for the set of conics in a pencil n-circumscribed to a given conic; the answer relies on a computer program: I identify the objects to be calculated, give the first few examples, and explain the strategy for performing the general computation.

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The bibliography on Poncelet's porism is immense, and still growing: I limit myself strictly to the works used in this note. An excellent set of classical references can be found in the entry "Poncelet's Porism", at *MathWorld*–A Wolfram Web Resource, by Eric W. Weisstein, `http://mathworld.wolfram.com`.

1. Limits and Invariants

A pair of (distinct) quadrics in $\mathbb{P}^n$ can be put in normal form under the action of the group of projective transformations, as follows. If we denote by A, B a pair of symmetric $(n+1)\times(n+1)$ matrices that define the quadrics (by abuse of notation we will call the quadrics "A" and "B" also), then:

THEOREM 1.1. [**HP**, XIII.10]
(i) If the matrix B is nonsingular, there exists a linear transformation of $\mathbb{P}^n$, given by a matrix P, such that ${}^tPAP = A_1$ and ${}^tPBP = B_1$ if and only if $\lambda A + \mu B$ and $\lambda A_1 + \mu B_1$ have the same elementary divisors.
(ii) If the matrix B is singular, if the rank of the $(n+1)\times 2(n+1)$ matrix $[AB]$ is $n+1-r_0$, and if the elementary divisors of $\lambda A + \mu B$ are: $(\alpha_i\lambda+\mu)^{e_i}(i = 1,\dots,r); \lambda^{2f_i+1}(i=1,\dots,s); \lambda^{2g_i}(i=1,\dots,t)$, then by a projective transformation A, B can be brought in a unique way into the form:

$$\sum^r \theta_1(\alpha_i, e_i) + \sum^s \theta_2(f_i) + \sum^t \theta_3(g_i) + \sum^k \theta_4(\varrho_i)$$
$$\sum^r \phi_1(\alpha_i, e_i) + \sum^s \phi_2(f_i) + \sum^t \phi_3(g_i) + \sum^k \phi_4(\varrho_i)$$

where:

$$\theta_1(\alpha, e) = \alpha \sum_{i=0}^{e-1} X_i X_{e-1-i} + \sum_{i=0}^{e-2} X_{i+1}X_{e-1-i},$$
$$\phi_1(\alpha, e) = \sum_{i=0}^{e-1} X_i X_{e-1-i}$$
$$\theta_2(e) = 2X_0X_1 + 2X_2X_3 + \ldots + 2X_{2e-2}X_{2e-1} + X_{2e}^2$$
$$\phi_2(e) = 2X_1X_2 + 2X_3X_4 + \ldots + 2X_{2e-1}X_{2e}$$
$$\theta_3(e) = 2X_0X_1 + 2X_2X_3 + \ldots + 2X_{2e-2}X_{2e-1}$$
$$\phi_3(e) = 2X_1X_2 + 2X_3X_4 + \ldots + 2X_{2e-3}X_{2e-2} + X_{2e-1}^2$$
$$\theta_4(e) = 2X_0X_1 + 2X_2X_3 + \ldots + 2X_{2e-2}X_{2e-1}$$
$$\phi_4(e) = 2X_1X_2 + 2X_3X_4 + \ldots + 2X_{2e-1}X_{2e}$$

(distinct pairs of forms appearing in the quadrics involve non-overlapping sets of coordinates so that $n+1 = \sum^r e_i + \sum^s(2f_i+1) + \sum^t 2g_i + \sum^k(2\varrho_i+1) + r_0$).

However, we will be interested in pencils of quadrics, so we are free to choose the generators A, B; moreover, for reasons explained below, we will want the pencil to contain at least one (hence, all with a finite number of exceptions) smooth quadric. This implies (but is not equivalent to) $r_0 = 0$; notice also that the case $r_0 > 0$ can be analyzed by projecting from the common vertex of the quadrics to a pencil in $\mathbb{P}^{n-r_0}$, for which the corresponding r_0 is zero.

Segre symbol. It can be shown [HP, XIII.11] that a basis of the pencil can be put into the normal form of Theorem 1.2 not involving the (θ_2, ϕ_2) or (θ_3, ϕ_3) pairs (basically by a change of coordinates in $\mathbb{P}^1$ that moves the point $[\lambda, \mu] = [0, 1]$) and in this case, the Segre symbol of the pencil is defined by writing down the numbers $e_1, \ldots, e_r$ in any order, except that the values of e_i corresponding to elementary divisors with the same α_i are grouped and enclosed in round brackets; after a semicolon, the numbers $\varrho_1, \ldots, \varrho_k$ are listed in any order.

Since we will only be interested in pencils for which there are no θ_4, ϕ_4 parts to the normal form, again for reasons to be explained below, we recall for completeness the invariant theory of pencils of that form.

THEOREM 1.2. [**AM**, 2.2] *Two pencils of quadrics* $\{\lambda A_i + \mu B_i, [\lambda, \mu] \in \mathbb{P}^1\}, i = 1, 2,$ *such that* $\det(\lambda A_i + \mu B_i)$ *is not identically zero, with singular elements corresponding to* $[\lambda_{i,j}, \mu_{i,j}]$ *are projectively equivalent if and only if they have the same Segre symbol and there is an automorphism of* $\mathbb{P}^1$ *taking* $[\lambda_{1,j}, \mu_{1,j}]$ *to* $[\lambda_{2,j}, \mu_{2,j}]$.

THEOREM 1.3. [**AG**][**AM**, 2.2] *The ring of invariants of pairs of* $(n+1)\times(n+1)$ *symmetric matrices under the action of* $SL(n+1, \mathbb{C})$ *is generated by the coefficients of the polynomial* $\det(\lambda A + \mu B)$.

If we let $SL(2, \mathbb{C})$ act on $[\lambda, \mu]$ we can use known facts on binary forms to conclude:

COROLLARY 1.4. [**AM**, 3.4] *The stable (semi-stable) pencils of quadrics in* $\mathbb{P}^n$ *are those for which* $\det(\lambda A + \mu B)$ *is not identically zero and has no root of multiplicity* $\geq \frac{n+1}{2} (> \frac{n+1}{2})$.

A pair of conics in $\mathbb{P}^2$ gives rise to several deep geometric constructions; we briefly recall the relevant ones. The next statement is one version of "Poncelet's Porism".

PONCELET'S CLOSURE THEOREM 1.5. [**BKOR**, 7.11] Given two smooth conics $C, D \subset \mathbb{P}^2$, if for some integer $N \geq 3$ there exists a nontrivial interscribed N-gon between C and D i.e. there exists a $(P_0, L_0) \in C \times D^*, P_0 \in L_0$, such that by taking L_1 to be the second tangent to D from P_0, P_1 the second point on $L_1 \cap C$ and iterating, we obtain $P_N = P_0$, then for any $(P_0', L_0') \in C \times D^*, P_0' \in L_0'$, the construction also closes ater N steps.

For completeness, we recall that the authors of [BKOR] define the trivial case to be that in which $P_{N-j} = P_j$ for all $0 \leq j \leq N$, and it occurs when $N = 2k + 1$ is odd and $L_k \in C^* \cap D^*$, or $N = 2k$ is even and $P_k \in C \cap D$.

ELLIPTIC CURVE ASSOCIATED TO THE CLOSURE THEOREM 1.6. [**BKOR**, 7.12-7.17] The curve $I = \{(P, L) | P \in L\} \subset C \times D^*$ has arithmetic genus 1 and the choice of a point allows us to give it a group structure. Poncelet's iteration is then translation by a group element and the construction closes after N steps if and only if that element has N torsion. Five groups can occur, according to the intersection multiplicities of C and D; a local calculation for the equation of I in two parameters, s for a point P of C, say, and t for a point on D where L is tangent, yields the following table:

Intersection Type	Group	I
1+1+1+1	$\mathbb{C}/\Lambda$	smooth
2+1+1	$\mathbb{C}^*$	node
2+2	$\mathbb{C}^* \times (\mathbb{Z}/2)$	2 components, 2 nodes
3+1	$\mathbb{C}_a$	cusp
4	$\mathbb{C}_a \times (\mathbb{Z}/2)$	tacnode

We can ask whether the different intersection patterns, which correspond to certain multiplicities for the points that give the base locus of the pencil spanned by C and D, can be detected in terms of Segre symbols. We note first that, in order for a Poncelet situation to make sense, the conics of the pencil cannot all be singular, and by inspection [HP, XIII.11] we find that the only such case with $r_0 = 0$ is that of form $\theta_4 = 2X_0X_1, \phi_4 = 2X_1X_2$. Having excluded that case, we can match the curves with the symbols as follows (notice that the curve I may be stable in the sense of moduli, even when the pencil is not).

Segre symbols 1.7. Five cases remain and by inspection we find the following intersection patterns [HP,XII.11]

Intersection Type	Segre Symbol
1+1+1+1	[1,1,1]
2+1+1	[2,1]
2+2	[(1,1),1]
3+1	[3]
4	[(2,1)]

A pencil of quadrics in $\mathbb{P}^{2g+1}$ with Segre symbol $[1,\dots,1]$ determines a hyperelliptic curve of genus g:

Theorem 1.8. [**N**, Sec. 1] *If a pencil of quadrics in $\mathbb{P}^{2g+1}$ has generators in normal form: $\sum_0^{2g+1} X_i^2, \sum_0^{2g+1} e_iX_i^2$, then the 2:1 covering of $\mathbb{P}^1 = \{[\lambda,\mu]\}$ that parametrizes the pencil given by the two rulings on the quadric $Q_{[\lambda,\mu]}$ is a hyperellyptic curve X of genus g, which has an affine model $y^2 = \prod_0^{2g+1}(x - e_i)$.*

However, when $g = 1$ the intersection of two generators of the pencil is also a curve of genus 1 in $\mathbb{P}^3$. It is isomorphic to the curve given in 1.8, and rather than check this directly we quote another theorem that implies it, because we will need this result also. The original proof was given by several people independently; references can be found in $[K]$ whose further dynamical applications we use below. With the same notation as in Theorem 1.8,

Theorem 1.9. *(cf. [K]) The variety of $\mathbb{P}^{g-1}s$ contained in the intersection of the quadrics*
$\sum_0^{2g+1} x_i^2, \sum_0^{2g+1} e_ix_i^2$ is isomorphic to $JacX$.

Corollary 1.10. *When $g = 1$, the genus-1 curve given by equation $y^2 = \prod_0^{2g+1}(x-e_i)$ (Th. 1.8) is isomorphic to the intersection of the two quadric surfaces $\sum_0^{2g+1} X_i^2, \sum_0^{2g+1} e_iX_i^2$.*

But we may consider all other possible Segre symbols, and the base loci of these pencils, again in the $g = 1$ case. We will relate them to Poncelet's curves. Indeed, it was observed (several times, independently) that a pencil of quadrics in $\mathbb{P}^3$ gives rise to an equivalent version of Poncelet's theorem by projection. Again, we only cite a recent account, best suited to our purposes, when older references may be found. Another recent reference where such an example is worked out is [CCS].

THEOREM 1.11. [**BB**] *Let $Q_1, Q_2 \subset \mathbb{P}^3$ be quadrics of rank ≥ 3 such that their intersection curve is either a smooth elliptic curve or the union of two conics meeting in two distinct points. Fix rulings $\mathcal{R}_i$ on $Q_i (i = 1, 2)$. Suppose that there exists a sequence of distinct lines $L_1, \ldots, L_{2n+1} = L_1$ such that the line L_j belongs to $\mathcal{R}_1$, resp. $\mathcal{R}_2$ if j is odd, resp. even, and such that consecutive lines L_j, L_{j+1} intersect each other. Then there exist such sequences of length $2n$ through any point on $Q_1 \cap Q_2$.*

A proof of this theorem is based again on the group law for the curve $Q_1 \cap Q_2$. If the situation obtains, the quadrics Q_1, Q_2 or the rulings $\mathcal{R}_1, \mathcal{R}_2$ are said to be in Poncelet n-position. Notice that the theorem still holds if both rulings $\mathcal{R}_1, \mathcal{R}_2$ are on one of the quadrics, if it is smooth. But if we now assume the pencil to be of Segre type $[1, 1, 1, 1]$, take Q_2 to be smooth and Q_1 to be a cone, with vertex v, then the generic projection from v gives rise to a pair of conics, C and D, where C is the image of Q_1 and D is the ramification locus of the projection of Q_2, for which Poncelet's theorem applies.

THEOREM 1.12. [**BB**, 1.3] *The quadrics Q_1 and Q_2 are in Poncelet n-position if and only if the conic D is n-inscribed in C.*

A suitable projection in $\mathbb{P}^{2g+1}$, from a $\mathbb{P}^{g-1}$ spanned by vertices of cones in a pencil of type $[1, \ldots, 1]$, to $\mathbb{P}^{g+1}$, had also been considered in [K], in order to translate the addition law on Jac$X$ into a dynamical flow of lines in $\mathbb{P}^{g+1}$ whereby Jacobi described the completely integrable system of geodesic motion on the g-dimensional ellipsoid (at least, Jacobi treated the $g = 2$ case). Again, a (generalized) Poncelet theorem for g confocal quadrics in $\mathbb{P}^{g+1}$ to be inscribed in another confocal quadric can therefore be implemented by using a point of finite order in Jac$X$ [Pr]. When applied to the higher-genus Poncelet porism, the constructions of this note will provide a link with numerical questions of algebraic geometry [TTZ], which generalize those treated for elliptic fibrations [B].

Our goal is now to extend the analysis to pencils of quadrics in $\mathbb{P}^3$ with other Segre symbols, motivated by two issues. The first is invariant-theoretic: what happens in moduli (the relevant moduli, for instance those of the elliptic curve or the Jacobian, the pencil etc.) in the limit? In particular, what happens to the (possible) 13 Segre symbols in $\mathbb{P}^3$ when translated into the 5 Segre symbols we encountered in $\mathbb{P}^2$? In $\mathbb{P}^3$, a natural compactification of a generic configuration will give a projective variety that will exhibit interesting singularities. Again the concept applies to curves, Jacobians, pencils, or other members of a suitable moduli space. One specific construction is that of an elliptic fibration, which we now describe. First we refine the Poncelet construction, in a way that will be used in Section 2, and that was introduced in [BM].

By fixing four generic points in $\mathbb{P}^2$, which we may normalize to be: $P_0 = [1, 1, 1], P_1 = [-1, 1, 1], P_2 = [1, -1, 1], P_3 = [1, 1, -1]$, we determine a pencil of conics of Segre type $[1, 1, 1]$. As we saw, two smooth conics C, D in the pencil and the choice of an origin determine an elliptic curve $I \subset C \times D^*$. However, a different point of view is to associate to any smooth conic in the pencil, by fixing the order $(P_0, \ldots, P_3)$ an elliptic curve with a level-2 structure, namely the double cover of C branched over P_0, P_1, P_2, P_3, with P_0 representing the origin. In fact:

THEOREM 1.13. [**BM**, (1.1)] *Given an elliptic curve E with a level-2 structure there is a unique (smooth) conic C in the pencil so that the corresponding elliptic curve is isomorphic to E compatibly with the level-2 structure. This gives an identification of the parameter space $\mathbb{P}^1$ of the pencil, minus 3 points, with the modular curve X_2 that parametizes elliptic curves with level-2 structure.*

There does not exist a universal elliptic curve with level-2 structure, but [BM] constructs a rational surface S, which is not "natural" because it depends on the choice of a smooth conic C in the pencil, and is an elliptic fibration whose general fibre is the elliptic curve associated to any smooth conic in the pencil other than C. First, the authors of [BM] blow up the four base points of the pencil in $\mathbb{P}^2$; they pull back to this blow up the incidence correspondence $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ (which is the double cover of $\mathbb{P}^2$ with branch locus C), obtaining a surface T. The rational elliptic fibration S is then the minimal desingularization of T, and is fibered by the elliptic curves that correspond to the given pencil, plus singular fibres of Kodaira type I_0^* and I_2. The sections of S corresponding to points of finite order in the elliptic curves are used in [BM] to count the conics in Poncelet n-position as we'll recall in Section 2. However, S is constructed out of a pencil of Segre type $[1, 1, 1]$. The previous point of view allows us to deform other symbols; in [BKOR], the normalization is given by fixing the conic D: $Y = X^2$, and taking C to be of the form: $\alpha Y = X^2 + \beta XY + \gamma Y^2, \alpha \neq 0$. These conics have an intersection of multiplicity at least 2 at the origin, and for suitable values of α, β, γ give pencils of the 4 remaining Segre types. A linear deformation of C then, such as $\lambda(Y - X^2) + \mu[s(\alpha Y - X^2 - \beta XY - \gamma Y^2) + tX]$, will provide a 'pencil of pencils', a line in $Gr(2, 6)$, such that generically the corresponding curve $I_{[s,t]} \subset C_{[s,t]} \times D^*_{[s,t]}$ is smooth (where C denotes as before the conic corresponding to $[\lambda, \mu] = [0, 1]$ and D the one corresponding to [1,0]) while the $t = 0$ curve is singular. While this family does not 'fit' into an elliptic fibration in any natural way that I can see, the corresponding curves in $\mathbb{P}^3$ that will be considered below can be compared with those that fiber the surface S.

With this motivation, in this paper we just compute the Segre symbols which correspond to the projections that could be considered.

CONSTRUCTION 1.14. We consider a projection of this particular type: we choose two generators of a pencil of quadrics in $\mathbb{P}^3$, a cone Q_0 and a smooth quadric Q_1, and we project from a vertex v of Q_0 that does not belong to Q_1. Then: (i) The projection of Q_0 is a conic C, which we can take to be the intersection of Q_0 with a generic plane in $\mathbb{P}^3$, and the ramification locus of the projection of Q_1 is another conic D; (ii) the conic C and all the conics D obtained in this way by varying Q_1 in the pencil (including the singular members) form a pencil of conics. The reason for this particular construction is that we want for a Poncelet polygon in the plane to make sense. As in 1.11, points of C will correspond to lines on Q_0 through the vertex and tangent lines to D will correspond to a plane tangent to Q_1. In particular, this is reason enough to exclude certain pencils in $\mathbb{P}^3$, as explained in the following Remark (1).

REMARK 1.15. (1) The only pencils in $\mathbb{P}^3$ in which all the members are singular are either the one where they have a common vertex, which by intersection reduces to a plane pencil, or the one corresponding to the normal form $\theta_1(a,1)+\theta_4(1), \phi_1(a,1)+\phi_4(1)$ with generators $2X_1X_2+aX_3^2, 2X_0X_1+X_3^2$ having the line $X_1 = X_3 = 0$ in common, tangent to the plane $X_1 = 0$ at every point on that line, and with vertex $[1,0,0,0], [0,0,1,0]$, resp. This is not a situation considered in Section 1, because whatever vertex we choose to project from, it will belong to the quadric Q_1.

(2) A different kind of projection is considered in [G] in order to exhibit a geometric model of the elliptic curve. First, the curve $I \subset C \times D^*$ is embedded in $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, then from a point of Q not on I the curve is projected to a smooth cubic, whereas from a point on I the curve is projected to a quartic with two singular points, for which a Poncelet theorem is proved analytically using abelian integrals. We do not consider this type of projection since we are interested in comparing Segre symbols of pencils of quadrics.

(3) The type of the plane pencil will depend on the vertex we choose, even if belonging to the same singular quadric. We give one example, to serve as an illustration of the calculations which we otherwise do not reproduce here.

EXAMPLE 1.16. The pencil [(11)11] has normal-form generators:

$$A = \begin{bmatrix} a & & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix} \qquad B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad a, b, c \text{ distinct}$$

The curve $A \cap B$ is the union of two conics meeting transversely in two points. The singular quadrics of the pencil are two cones with a 1-point vertex such as

$$\begin{bmatrix} a-c & & & \\ & a-c & & \\ & & b-c & \\ & & & 0 \end{bmatrix} \text{ and a cone with a line vertex, } \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & b-a & \\ & & & c-a \end{bmatrix}.$$

The corresponding pencil in $\mathbb{P}^2$, if we choose a vertex $[0,0,0,1]$, is indeed of type [(11)1], because the cone meets a generic $\mathbb{P}^2$, which can be taken to be $X_3 = 0$, in $(a-c)X_0^2 + (a-c)X_1^2 + (b-c)X_2^2 = 0$, and the branch locus of the projection of $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ from $[0,0,0,1]$, which consists of the points with $X_3^2 = 0$, is $X_0^2 + X_1^2 + X_2^2$. So, the elliptic curve is the same. However, if we choose a vertex $v = [1,0,0,0]$, and consider the intersection of $\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & b-a & \\ & & & c-a \end{bmatrix}$ with $X_0 = 0$ and the branch locus of the projection of B, we obtain the pencil $\langle \begin{bmatrix} 0 & & \\ & b-a & \\ & & c-a \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rangle$, with symbol [111]. Thus, unless the pencil in $\mathbb{P}^3$ is of type [1111], the curve X of Theorem 1.8 may not be the same as a corresponding plane Poncelet curve. Notice that the 3-dimensional Poncelet construction of

Theorem 1.11 does not quite make sense when $A = (b-a)X_2^2 + (c-a)X_3^2$ is the union of two planes, for a point in $A \cap B$ will not belong to a unique line on A; however, for the plane conics $A' = (b-a)X_2^2 + (c-a)X_3^2$ and $B' = X_1^2 + X_2^2 + X_3^2$, say, a Poncelet configuration will make sense: from a point on A', which is the union of two lines, we can send a tangent to B' and iterate; the polygon will never close or one line would be tangent to B', and it can be lifted to a polygon of lines in $\mathbb{P}^3$ through v and tangent to B, by taking the inverse image of the vertices under projection.

As we tabulate the type of elliptic curve to which the 13 possible Segre symbols in $\mathbb{P}^3$ give rise, together with the plane model, which in some cases will depend on the choice of vertex, we record the Kodaira-type notation, (cf. e.g. [BPvdV, V.7]) in view of possible applications to the theory of elliptic fibrations: however, the terminology "Kodaira-type" is only used as an analogy, as this projective surface is not a minimal elliptic fibration, as demonstrated by the table (cf. the comment that follows it).

Symbol	A B	$X = A \cap B$	v	Symbol	$I \subset C \times D^*$
[1,1,1,1]	$\sum_{i=0}^3 a_i X_i^2$ $\sum_{i=0}^3 X_i^2$	I_0	$[1,0,0,0]$	[1,1,1]	I_0
[(1,1),1,1]	$a(X_0^2+X_1^2)+bX_2^2+cX_3^2$ $\sum_{i=0}^3 X_i^2$	I_2	$[0,0,0,1]$	[(1,1),1]	I_2
			[1,0,0,0]	[1,1,1]	I_0
[(1,1,1),1]	$a(X_0^2+X_1^2+X_2^2)+bX_3^2$ $\sum_{i=0}^3 X_i^2$	double conic	$[0,0,0,1]$	[(1,1),1]	I_2
			[1,0,0,0]	[(1,1),1]	I_2
[(1,1),(1,1)]	$a(X_0^2+X_1^2)+b(X_2^2+X_3^2)$ $\sum_{i=0}^3 X_i^2$	I_4	$[1,0,0,0]$	[(1,1),1]	I_2
[2,1,1]	$2aX_0X_1+X_1^2+bX_2^2+cX_3^2$ $2X_0X_1+X_2^2+X_3^2$	I_1	$[0,0,1,0]$	[2,1]	I_1
[(2,1),1]	$a(2X_0X_1+X_2^2)+X_1^2+bX_3^2$ $2X_0X_1+X_2^2+X_3^2$	III	$[0,0,1,0]$	[2,1]	I_1
			[0,0,0,1]	[(2,1)]	III
[2,(1,1)]	$2aX_0X_1+X_1^2+b(X_2^2+X_3^2)$ $2X_0X_1+X_2^2+X_3^2$	I_3	$[0,0,0,1]$	[2,1]	I_1
[(2,1,1)]	$a(2X_0X_1+X_2^2+X_3^2)+X_1^2$ $2X_0X_1+X_2^2+X_3^2$	double singular conic	$[0,0,0,1]$	[(2,1)]	III
[2,2]	$2aX_0X_1+X_1^2+2bX_2X_3+X_3^2$ $2X_0X_1+2X_2X_3$	I_2	no suitable v	—	—
[(2,2)]	$2a(X_0X_1+X_2X_3)+X_1^2+X_3^2$ $2X_0X_1+2X_2X_3$	I_0^*	no suitable v	—	—
[3,1]	$a(2X_0X_2+X_1^2)+2X_1X_2+bX_3^2$ $2X_0X_2+X_1^2+X_3^2$	II	$[0,0,0,1]$	[3]	II
[(3,1)]	$a(2X_0X_2+X_1^2+X_3^2)+2X_1X_2$ $2X_0X_2+X_1^2+X_3^2$	IV	$[0,0,0,1]$	[3]	II
[4]	$2a(X_0X_3+X_1X_2)+2X_1X_3+X_2^2$ $2(X_0X_3+X_1X_2)$	III	no suitable v	—	—

To summarize the main point of this calculation, the advantage of the model in $\mathbb{P}^3$ is that $\mathbb{P}^3$ contains the curves. There is no way to embed $I \subset C \times D^*$ consistently in projective space over a family of pairs (C, D), at least this seems to be the problem confronted in [BM, Section 1] also. In $\mathbb{P}^3$, we can deform a pencil with 'nongeneric' Segre symbol, for example as follows: given two generators A, B of the pencil, it can easily be seen that the pencil $\langle A_t = (1-t)A + tR, B_t = (1-t)B + tT \rangle$ will have symbol $[1,1,1,1]$ for an appropriate choice of R, T. The family of base loci of the pencils $\langle A_t, B_t \rangle$ will be a surface S in $\mathbb{P}^3$. Notice however that it may not be an elliptic fibration, since the curves $X_t = A_t \cap B_t$ may intersect for different t's. The set $\tilde{S} = \{(p,t) | p \in X_t\} \subset \mathbb{P}^3 \times \mathbb{P}^1$, cut out by the same equations, is an elliptic fibration; however, it may fail to be smooth and/or minimal-elliptic, as can be seen from the table above, where for the two symbols [(1,1,1),1] and [(2,1,1)] the curve X does not correspond to a Kodaira type for a smooth minimal-elliptic fibration, in which a multiple fibre cannot be simply connected, e.g. [BPvdV, Lemma III.8.3 and V.7 c)]. It seems that these surfaces might still be of interest, in case they give concrete models of elliptic pencils with few singular fibres [B].

The topic of elliptic fibrations is indeed of relevance to moduli theory (cf. [HL], [Pe], [V]), and there are subtle issues related to compactifications. In [V], an elliptic fibration is defined as a flat family of reduced genus-one curves over a smooth curve, with only irreducible nodal singular fibers, smooth total space, and a choice of section. We will assume the base curve to be $\mathbb{P}^1$. This places them in the open part of the moduli space of rational elliptic surfaces, three compactifications of which are described in [HL]; in turn, [Pe] lists all possible configurations of degenerate fibers on a minimal rational elliptic surface.

More invariants? By analogy with the projection from 3 dimensions to 2, which links two Poncelet theorems, we could project $\mathbb{P}^2$ to $\mathbb{P}^1$ from the vertex of one of the singular conics of a (generic) pencil. We would obtain a set of n points in $\mathbb{P}^1$ corresponding to the vertices of each Poncelet n-gon; it might be interesting to associate the invariants of these n points (or, of the 4 points given by the branch loci of the projection restricted to C and D) to the corresponding pair (C, D), in view of the fact that in the [BM] model a varying $C_{[\lambda,\mu]}$ can be viewed as a point of the modular curve X_2 and the corresponding D's in n-Poncelet position, roughly speaking as n-torsion points of the corresponding elliptic curve.

A second reason for being interested in 'natural' compactifications of Poncelet configurations is the dynamical interpretation that the theorem was given, continuous (geodesic motion) or discrete, as an integrable billiard. As such, the configuration corresponds to a 1-dimensional family of Liouville tori, the real part of an elliptic curve varying in the family. The real monodromy of the systems is then related to the singular curves in the family. Degenerate billiard motion is analyzed in [KT, IV.1, e.g.]; in this analysis the 'confocal' pencil in $\mathbb{P}^{g+1}$ is fixed, but the pencil in $\mathbb{P}^{2g+1}$ that carries the invariants of the elliptic curve does vary. There are indeed quite different ways to encode the invariants in the configuration of motion, and the link between the plane billiard and the points of the abelian variety where the flows are linear (in $\mathbb{P}^3$ for the $g = 1$ case) is provided in Knörrer's work [K] by projecting and dualizing. Notice that in the billiard interpretation the list of possible degenerations is restricted by the fact that choosing a confocal pencil fixes two (g+1 for the higher-dimensional model) of the divisors of the discriminant curve $\det(\lambda A + \mu B)$.

2. Monodromy of n-gons

In this section the two conics C, D of Poncelet's porism will be assumed to meet transversely. If C is n-circumscribed to D we say that C and D are in Poncelet n-position. In Section 1 we alluded to the fact that a given point of the elliptic curve $I \subset C \times D^*$ has order n. To investigate the arithmetic implications of the theorem, Barth and Michel identify the modular curve X_2 that parametrizes elliptic curves with level-2 structure, with the smooth conics of the pencil spanned by C and D, after fixing an order of the base points of the pencil P_0, P_1, P_2, P_3, as in 1.12 above. Moreover, after fixing a smooth conic in the pencil, D say, they construct an elliptic fibration S whose general fibre corresponds to the elliptic curve as an element of X_2, and a plane curve Π'_n which is a birational image of the modular curve $X_{00}(n, 2) = X_{00}(n) \times_{\mathbb{P}^1} X_2$; its closure Π_n corresponds, roughly speaking, to the curve $T_n \subset S$ that cuts out on each smooth fibre the n^2 points of n torsion and does not contain any fibre, with the zero section (resp. the 4 sections corresponding to the 2-torsion points if n is even) removed. What is of interest here is that they are able to compute interesting numerical invariants by means of the following observation:

PROPOSITION 2.1. [**BM**, (5.1)] *C and D are in Poncelet n position if and only if the point $T_{P_0}D \cap C$ belongs to the curve Π_n,*

Barth and Michel deduce that the number of conics in the pencil that are n-circumscribed about C is $c(n) = t(n)/4$, where $t(n)$ is the number of primitive n-torsion points of an elliptic curve. In addition, they observe that Cayley's theorem provides an equation for Π_n defined over the rational integers.

My question is then: any generic pencil gives rise to an enumerative problem that has $c(n)$ solutions: what is the Galois group of this problem? One way to set it up is to fix P_1, P_2, P_3 on C, and let P_0 vary; the solution we are considering are then the $c(n)$ points $T_{P_0}D \cap \Pi_n$. Other settings could be, the hyperplane in Gr(2,6) given by the pencil of conics that contain C (pencils of conics are lines in $\mathbb{P}^5$); or the set of ordered 4-tuples of points on a conic. But in order to use the normalized equation of Π_n, given in [BM] for small n, we will fix the pencil, and change the line through P_0. The conic D will change, so we are moving the elliptic curve and, roughly speaking, 'solving' for its $n-$torsion points. I first learned about the Galois problem in the context of enumerative geometry from the beautiful paper of Joe Harris [H1], who points out that some such questions were considered classically by C. Jordan, e.g., and who devises a method of solution based on the identification of the Galois group with the monodromy group of a finite covering of varieties. Guided by the remark at the end of [H1], on the geometric significance of the Galois group, "in every case in which current theory had failed to discern any intrinsic structure in the set of solutions – it is proved here – there is none", I hope that the Galois group is the full symmetric group $\mathcal{S}_{c(n)}$. The method, as devised by Harris, which proved itself highly effective [H2], [GH], would consist of showing two facts:

Step I. The monodromy group is 2-transitive. This is typically achieved by checking that the parameter space of solutions (respectively, those fixed by the stabilizer of a point of the fibre) is an irreducible variety.

Step II. The monodromy group contains a single transposition. This can be achieved by exhibiting a point of the parameter space whose fibre contains exactly one double point and $(c(n)-2)$ simple points, provided the solution space is locally irreducible at the double point.

Here I set up the corresponding objects in the case at hand and list the relevant properties; I implement the above steps in the simplest cases ($n = 3, 4, 5$), by brute force. The full answer will be achieved by computation for each n, and I give the technique for doing so, although of course there may be other methods that elude me.

The Galois group of our problem is the monodromy group of the finite cover $\pi : Y \to X$, where X is the $\mathbb{P}^1$ that parametrizes the lines through P_0, one of the 4 fixed points P_0, P_1, P_2, P_3, and Y is the variety corresponding to the extension field over $\mathbb{C}(X)$ generated by the solutions of the Poncelet $n-$position problem, namely the points $T_{P_0}D \cap \Pi_n$, as we recalled above. A remark at the end of Section 1 in [H1] allows us to consider the case in which Y is not irreducible, which as we will see may occur: "Note that the group M of a map $\pi : Y \to X$ is well-defined even in case Y is reducible. In this case we simply define the Galois groups of π to be M, so that we can discuss G without first checking irreducibility of Y. G is a subgroup of, but is not always equal to, the product of the Galois group G_α of the irreducible components Y_α of Y dominating X; the action of G on the fibres of Y, however, is the same as that of G_α."

In our case the variety Y is a linear system of dimension 1 on the curve Π_n, namely the supports of the divisors cut out by the lines through P_0. Thus, we consider the map:

$$\Pi_n \to \mathbb{P}^1$$

such that a point P of the curve Π_n is mapped to the parameter $[\lambda, \mu]$ of the conic tangent to the line P_0P. The curve Π_n in general is not irreducible, in particular it contains as a component the curve Π_m if $m|n$, but, as in [BM], we can consider the curve Π'_n which cuts the general fibre of the surface S (cf. 1.12) in its primitive n-torsion points. In this way, since Π'_n is a birational image of the modular curve $X_{00}(n, 2)$ [BM, (4.10)], we can get transitivity as in Step I. To get Step II, we must find a line through P_0 which is simply tangent to Π'_n at only one point.

Cayley gave an algebraic condition for two conics to be in Poncelet n-position, and the equation g_n of the curve Π_n can be derived from it. The first few equations are worked out in [BM] for the given normalization of the coordinates of the points P_i, and I am able to check that the Galois groups is the full symmetric group in the cases of lowest degree. In general, the formula for g_n "seems however too complicated to be evaluated by hand" [BM, Section 4]; in particular, in [BM] g_n is computed for $3 \leq n \leq 12$ except for $n = 11$ which the authors "did not manage to compute". However, a simple computer program gives g_n for any fixed n from Cayley's formula; specifically, $g_{11} = -1024 s_3^6 s_2^4 s_1^4 + (-64 s_3^3 s_2^9 + 1536 s_3^5 s_2^6 + 4096 s_3^7 s_2^3) s_1^3 + (48 s_3^2 s_2^{11} - 384 s_3^4 s_2^8 - 5376 s_3^6 s_2^5 - 6144 s_3^8 s_2^2) s_1^2 + (-12 s_3 s_2^{13} - 32 s_3^3 s_2^{10} + 1152 s_3^5 s_2^7 + 6400 s_3^7 s_2^4 + 4096 s_3^9 s_2) s_1 + (s_2^{15} + 12 s_3^2 s_2^{12} - 16 s_3^4 s_2^9 - 704 s_3^6 s_2^6 - 2560 s_3^8 s_2^3 - 1024 s_3^{10})$, where the $s_i,\ i = 1, ..., 3$ are the symmetric functions in the homogeneous coordinates of $\mathbb{P}^2$.

THEOREM 2.2. *The monodromy group of the n-th Poncelet closure is the full symmetric group for $n = 3, 4, 5$.*

PROOF. Case $n = 3$. The equation of Π_3 is $\frac{1}{2}(X_0X_1 + X_1X_2 + X_0X_2)$; we expect indeed 2 solutions, the homogenous pairs $[s, t]$ for which the line $s[1, 1, 1] + t[X_0, X_1, X_2]$ meets Π_3. The Galois groups is $\mathbb{Z}_2$.

Case $n = 4$. The equation of Π_4 is $X_0X_1X_2$, reducible as was to be expected.

Case $n = 5$. The equation of Π_5 is of degree $6 = c(5)$:

$(X_0 + X_1 + X_2)(X_0X_1 + X_1X_2 + X_0X_2)X_0X_1X_2 - \frac{1}{4}(X_0X_1 + X_1X_2 + X_0X_2)^3 - X_0^2X_1^2X_2^2$. To check that it is irreducible and that from $P_0 = [1, 1, 1]$ there can be drawn lines simply tangent to Π_5 at only one point, we can use the smooth cubic computed in [BM]:

$$X_0^3 + X_1^3 + X_2^3 - (X_0^2X_1 + X_0X_1^2 + X_1X_2^2 + X_2X_1^2 + X_0X_2^2 + X_0^2X_2) - 2X_0X_1X_2,$$

via the Cremona transformation:

$$[X_0, X_1X_2] \mapsto [X_1X_2, X_2X_0, X_0X_1],$$

which more generally improves the singularities of Π_n, which occur only at the coordinate points. Lastly, it can be checked that a transitive subgroups of $\mathcal{S}_6$ that contains a simple transposition, in particular those coming from the tangents from P_0 to the given cubic, must coincide with $\mathcal{S}_6$. □

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