CONTEMPORARY MATHEMATICS

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Higher Genus Curves in Mathematical Physics and Arithmetic Geometry

AMS Special Session
Higher Genus Curves and Fibrations in Mathematical Physics and Arithmetic Geometry
January 8, 2016
Seattle, Washington

Andreas Malmendier
Tony Shaska
Editors

AMERICAN MATHEMATICAL SOCIETY

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Preface

Algebraic curves and their fibrations have played a major role in both mathematical physics and arithmetic geometry. The goal of this book is to focus on the role of higher genus curves; in particular, hyperelliptic and superelliptic curves in algebraic geometry and mathematical physics.

These proceedings are based on the Special Session entitled Higher Genus Curves and Fibrations in Mathematical Physics and Arithmetic Geometry that was held at the AMS Joint Mathematics Meetings on January 8, 2016, in Seattle, Washington.

The first three papers of this book are focused on automorphism groups of curves. In the first paper, Russell and Wootton study the action of a cyclic group C_p on a compact, oriented surface S of genus $\sigma \geq 2$. Such action is said to be finitely maximal if there is no finite supergroup of homeomorphisms $G > C_p$. The authors prove that for sufficiently large genus σ, the number of topologically distinct finitely maximal C_p-actions on a surface of genus σ is at least linear in σ.

Broughton studies the quasi-platonic action of the group G on the Riemann surface S. This is a conformal action of G on S such that S/G is a sphere and the projection $\pi_G : S \to S/G$ is branched over $\{0, 1, \infty\}$. The projection π_G is a regular Belyi function and induces a regular dessin d'enfant on S, and so S is defined over a number field. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts on regular dessins, hence quasi-platonic actions, by acting on the coefficients of a defining equation of S. The author reconstructs the Galois action from the branch cycle description of the action and the structure of the group G.

Swirnaski presents an algorithm for computing equations of canonically embedded Riemann surfaces with automorphisms. This is used to produce equations of Riemann surfaces with large automorphism groups for genus 7. The main tools are the Eichler trace formula for the character of the action of the automorphism group on holomorphic differentials, algorithms for producing matrix generators of a representation of a finite group with a specified irreducible character, and Gröbner basis techniques for computing flattening stratifications.

A superelliptic curve $\mathcal{X}$ of genus $g \geq 2$ is not necessarily defined over its field of moduli, but it can be defined over a quadratic extension of it. While a lot of work has been done by many authors to determine which hyperelliptic curves are defined over their field of moduli, less is known for superelliptic curves. Hidalgo and Shaska observe that if the reduced group of a genus $g \geq 2$ superelliptic curve $\mathcal{X}$ is different from the trivial or cyclic group, then $\mathcal{X}$ can be defined over its field of moduli; in the cyclic situation we provide a sufficient condition for this to happen. We also determine those families of superelliptic curves of genus at most 10 which might not be definable over their field of moduli.

Beshaj studies the Weierstrass equations for genus 2 curves defined over a ring of integers $\mathcal{O}_{\mathbb{F}}$ which correspond to reduced binary sextics. This is done via reduction theory and Julia quadratic of binary sextics. The author shows that when a binary sextic has extra automorphisms, then it is usually easier to compute its Julia quadratic. Moreover, she shows that when the curve is given in the standard form $y^2z^4 = f(x^2, z^2)$ and defined over $\mathcal{O}_{\mathbb{F}}$, then the binary form f is reduced. Such curves have minimal height among integral models defined by sextics in x^2, z^2, even up to twist.

Continuing the study of genus 2 curves, Beshaj, Hidalgo, Kruk, Malmendier, Quispe, and Shaska describe how to build a database of genus 2 curves defined over $\mathbb{Q}$ which contains all curves with minimal absolute height $\mathfrak{h} \leq 5$, all curves with moduli height $\mathfrak{H} \leq 20$, and all curves with extra automorphisms in standard form $y^2 = f(x^2)$ defined over $\mathbb{Q}$ with height $\mathfrak{h} \leq 101$. Each isomorphism class of genus 2 curves in the database is characterized by its automorphism group and Clebsch and Igusa invariants. Moreover, an equation over its minimal field of definition is provided. The distribution of rational points in the moduli space $\mathcal{M}_2$ for which the field of moduli is a field of definition is discussed and some open problems are presented.

Magyar and Whitcher study strong arithmetic mirror symmetry and toric isogenies. A mirror pair of Calabi-Yau varieties exhibits strong arithmetic mirror symmetry if the number of points on each variety over a finite field is equivalent, modulo the order of that field. The authors search for strong mirror symmetry in pencils of toric hypersurfaces generated using polar dual pairs of reflexive polytopes. They characterize the pencils of elliptic curves where strong arithmetic mirror symmetry arises and provide experimental evidence that the phenomenon generalizes to higher dimensions and that pencils of $K3$ surfaces with the same Picard-Fuchs equation have related point counts.

Kumar and Kuwata describe two constructions of elliptic $K3$ surfaces starting from the Kummer surface of the Jacobian of a genus 2 curve. These parallel the base-change constructions of Kuwata for the Kummer surface of a product of two elliptic curves. One of these also involves the analogue of an Inose fibration. The authors use these methods to provide explicit examples of elliptic $K3$ surfaces over the rationals of geometric Mordell-Weil rank 15.

Shor considers the problem of calculating the higher-order Weierstrass weight of the branch points of a superelliptic curve C. For any $q > 1$, he gives an exact formula for the q-weight of an affine branch point and also finds a formula for the q-weight of a point at infinity in the case where n and d are relatively prime.

Previato studies Poncelet's porism and projective fibrations. Poncelet's porism theorem is used to produce a natural compactification of several moduli spaces. The monodromy of the polygons, viewed as torsion points on a fibration by elliptic curves, can be tested computationally for an action of the full symmetric group. Analogous constructions can be implemented for hyperelliptic fibrations corresponding to higher-dimensional versions of Poncelet's porism.

Levin formulates and proves a general version of Runge's method, suited to combination with other methods for integral points on varieties. He gives some applications of the main theorem, including results for integral points on curves which recover as a special case some known results on rational points on elliptic curves with prime power denominators.

Joyner and Shaska study connections between self-inversive and self-reciprocal polynomials, reduction theory of binary forms, and minimal models of curves. The authors prove that if $\mathcal{X}$ is a superelliptic curve defined over $\mathbb{C}$ and its reduced automorphism group is nontrivial or not isomorphic to a cyclic group, then we can write its equation as $y^n = f(x)$ or $y^n = xf(x)$, where $f(x)$ is a self-inversive or self-reciprocal polynomial.

Deopurkar and Patel describe a sequence of effective divisors on the Hurwitz space $H_{d,g}$ for d dividing $g-1$ and compute their cycle classes on a partial compactification. These divisors arise from vector bundles of syzygies canonically associated to a branched cover. They find that the cycle classes are all proportional to each other. These computations are motivated by the question of determining the effective cone and ultimately the birational type of $H_{d,g}$.

There is considerable activity in the area of algebraic curves of higher genus due to their importance in pure mathematics and applications. We hope that this volume will help further our understanding of algebraic curves and their connections to other areas of mathematics.

Both editors want to thank all authors for their contributions to this volume. We would especially like to thank the referees for their tireless work that they put toward this volume.

Andreas Malmendier

Tony Shasksa

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The articles in this volume investigate the automorphism groups of curves and superelliptic curves and results regarding integral points on curves and their applications in mirror symmetry. Moreover, geometric subjects are addressed, such as elliptic $K3$ surfaces over the rationals, the birational type of Hurwitz spaces, and links between projective geometry and abelian functions.

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