CONTEMPORARY MATHEMATICS

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Vertex Algebras and Geometry

AMS Special Session on
Vertex Algebras and Geometry
October 8–9, 2016
Denver, Colorado

Mini-Conference on
Vertex Algebras
October 10–11, 2016
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Thomas Creutzig
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Contents

Introduction

This book is the proceedings of the AMS Special Session on Vertex Algebras and Geometry, held at the University of Denver (October 8–9, 2016), which was followed by a mini-conference on Vertex Algebras (October 10–11, 2016). The purpose of these two meetings was to bring together experts in the area of vertex algebras in order to discuss recent developments with a focus on geometry and tensor categories.

It is exciting that vertex algebras play a substantial role in the physics of four-dimensional GL-twisted $\mathcal{N} = 4$ supersymmetric gauge theories as well as three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. Moreover, these two connect to interesting symplectic geometry and to the quantum geometric Langlands program on Riemann surfaces. The first highlight is in the physics paper [B-vR], which explains how two-dimensional chiral algebras appear in four-dimensional supersymmetric gauge theories. Chiral algebra is the physicist's vocabulary for vertex algebra. A key conjecture is that the Higgs branch, a symplectic variety, of the gauge theory is the associated variety of the chiral algebra. Tomoyuki Arakawa's contribution reviews this conjecture and how it relates to quasi-lisse vertex algebras. These are vertex algebras whose associated variety has only finitely many symplectic leaves. This condition seems to be needed in the physics setting but is also rather interesting from the vertex algebra point of view as it implies modularity of characters of ordinary modules [AK].

In 2007, Kapustin and Witten suggested how electric-magnetic duality relates to the geometric Langlands program on Riemann surfaces. Electric-magnetic duality (or S-duality) originally referred to the belief that strongly coupled $U(1)$-gauge theory is dual to the weakly coupled one, but the roles of electrically charged objects and magnetic ones were interchanged. This is especially the case in the strongly coupled regime, where point-like magnetically charged objects are supposed to exist. The general setup is then a gauge theory with gauge group and some Lie group G, and the duality interchanges weight and coweight lattices, so the dual theory is a gauge theory with Lie group the Langlands dual ${}^LG$ of G. Now, one is interested in a gauge theory to quantum geometric Langlands correspondence; see, e.g., [AFO]. On the geometric side, one is concerned with equivalences of twisted D-modules over punctured Riemann surfaces. These are sheaves of sections of spaces of conformal blocks of some affine vertex algebra or W-algebra. The twist refers to the level shifted by the dual Coxeter number. Another relation to vertex algebras has recently been described in [CG]. One associates three-dimensional topological boundary conditions to the gauge theory. Line defects ending on these boundary conditions are interpreted as categories of vertex algebra modules and topological boundary conditions intersecting in two-dimensional vertex algebras.

Standard vertex algebra constructions such as the quantum Hamiltonian reduction then relate the vertex algebras and tensor categories associated to different boundary conditions and their intersections. Physics, in particular, advocates a picture that predicts isomorphisms of vertex algebras and equivalences of vertex tensor categories.

This brings us to the second focus of the conference: Using tensor categories of vertex algebra modules. It is usually rather difficult to prove general theorems for vertex algebras, and now we have another tool to do so, namely, braided tensor categories. The reason is that vertex algebra extensions are in one-to-one correspondence to commutative, associative algebras with an injective unit in the vertex tensor category. Moreover, the category of local algebra modules is braided equivalent to the category of modules of the extended vertex algebra [HKL, CKM]. It is now possible to study coset and orbifold theory of vertex algebras using tensor categories. Our contribution, for example, uses simple current extensions to prove that the Heisenberg coset of the simple subregular W-algebra of $\mathfrak{sl}_4$ at certain levels is a regular and rational W-algebra of type A. Two more works in this volume are concerned with vertex tensor categories: Jean Auger and Matt Rupert explain how the categorical framework works for simple current extensions of infinite order, and Jinwei Yang finds conditions for the applicability of the theory of vertex tensor categories.

Let us summarize the contributions in this volume.

(1) Feodor Malikov introduces and classifies strongly homotopy chiral algebroids. These are generalizations of algebras of chiral differential operators and are defined over a general commutative $\mathbb{C}$-algebra, possibly singular. The language of Beilinson-Drinfeld pseudo-tensor categories is essential in this paper.
(2) Tomoyuki Arakawa reviews the relation between the Higgs branch of supersymmetric gauge theories and the associated variety of vertex algebras. An important new concept is quasi-lisse vertex algebras and modularity of characters of ordinary modules.
(3) Geoffrey Mason develops the axiomatic theory vertex rings, which are vertex algebras defined over a general commutative ring, not necessarily a field. There are many subtleties when the ring is not a field; for example, the translation operator does not work in general and must be replaced with a Hasse-Schmidt derivation.
(4) Thomas Creutzig and Andrew R. Linshaw illustrate in the example of the subregular W-algebra of $\mathfrak{sl}_4$ how one can study deformable families of coset vertex algebras and show a strategy to prove vertex algebra isomorphisms.
(5) Jinwei Yang gives a sufficient condition for the convergence and extension of correlation functions of strongly graded vertex operator algebras. This is the crucial condition to establish in order to prove the existence of a vertex tensor category structure on a given category of vertex algebra modules.
(6) Jean Auger and Matt Rupert explain simple current extensions of infinite order. Since an infinite direct sum of objects is not an object of the category anymore but in a completion, one has to be careful in studying infinite order simple current extensions. Here, subtleties are clarified.

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Thomas Creutzig and Andrew R. Linshaw

CONM 711

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This book contains the proceedings of the AMS Special Session on Vertex Algebras and Geometry, held from October 8–9, 2016, and the mini-conference on Vertex Algebras, held from October 10–11, 2016, in Denver, Colorado.

The papers cover vertex algebras in connection with geometry and tensor categories, with topics in vertex rings, chiral algebroids, the Higgs branch conjecture, and applicability and use of vertex tensor categories.

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