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Schrödinger-type equations and unitary highest weight representations of the metaplectic group

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Dedicated to Gestur Olafsson

Abstract. By the work of Kashiwara–Vergne and Enright–Parthasarathy, every unitary highest weight representation of the metaplectic group $\mathrm{Mp}(n,\mathbb{R})$ can be embedded in $\mathcal{L}^2(\mathrm{M}_{n,k})$ for some $k \geq 1$, where $\mathrm{M}_{n,k}$ denotes the space of real $n \times k$ matrices. Furthermore, every unitary highest weight representation can be embedded in a space of sections of a holomorphic vector bundle on the Siegel upper half-space or, via boundary values, in a degenerate principal series representation. In this paper, we give a new realization of unitary highest weight representations in the kernel of a system of Schrödinger-type equations on the space $\mathrm{M}_{n,k} \times \mathrm{Sym}_n$, where Sym_n denotes the space of symmetric real $n \times n$ matrices. Our realization has simple intertwining maps to the previously known realizations mentioned above.

1. Introduction

1.1. The metaplectic group, $G := \mathrm{Mp}(n,\mathbb{R})$, plays an important role in many areas of representation theory, physics, and number theory. In 1978, Kashiwara and Vergne [**8**] constructed a large family of irreducible, unitary hightest weight representations for G inside the k-fold tensor product of the oscillator representation. They conjectured that their construction, in fact, gave all irreducible, unitary hightest weight representations for G. This conjecture was proved in 1981 by Enright and Parthasarathy [**3**].

Kashiwara and Vergne work with two explicit realizations of their representations and, briefly, mention a third. This paper will construct a fourth realization based on generalized Schrödinger operators that has simple intertwining maps to the aforementioned pictures and that ties everything together in a single commutative diagram. In addition to the importance of the Schrödinger equation, the motivation for these results comes out of invariant theory (see §4.2) and provides a framework for naturally generalizing all of these results to the tube-type case. This paper is a natural extension of the cluster of ideas found in [**5–7**, **9**, **10**]. More generally, our line of investigation sits within the extensive body of literature devoted to providing explicit realization for unitary representations. There is also some overlap with the work by R. Berndt and R. Schmidt [**1**, **2**].

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1.2. In order to summarize our result in more detail, we first outline the three constructions of Kashiwara and Vergne. For their first construction, they show there is a linear action of G given by (see §5.5 for the precise lift of this from $\mathrm{Sp}(n,\mathbb{R})$ to G) on $\mathcal{L}^2(\mathrm{M}_{n,k})$

$$(\begin{pmatrix} A & 0 \\ 0 & A^{-1,T} \end{pmatrix} \cdot f)(\xi) = (\det A)^{\frac{k}{2}} f(A^T \xi)$$

$$(\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \cdot f)(\xi) = e^{-\frac{i}{2}\operatorname{tr}(\xi^T B \xi)} f(\xi)$$

$$(\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot f)(\xi) = \left(\frac{i}{2\pi}\right)^{\frac{nk}{2}} \int_{\mathrm{M}_{n,k}} f(x) e^{i\operatorname{tr}(\xi^T x)}\, dx.$$

There is also a linear $\mathrm{O}(k)$-action on $\mathcal{L}^2(\mathrm{M}_{n,k})$ given by $(c \cdot f)(\xi) = f(\xi c)$ which gives rise to the decomposition

$$\mathcal{L}^2(\mathrm{M}_{n,k}) = \bigoplus_{\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)} \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \otimes V_\lambda'$$

as a $G \times \mathrm{O}(k)$-module (see §5.4 for the definition of Σ). Here V_λ' is the dual of V_λ and $\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ is the space of square-integrable V_λ-valued functions $f : \mathrm{M}_{n,k} \to V_\lambda$ satisfying $f(xc) = \lambda(c)^{-1} f(x)$ for all $c \in \mathrm{O}(k)$. The space $\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ is shown to be an irreducible, unitary highest weight G-module.

1.3. For their second construction, they consider the differential operators

$$\Delta_{ij} := \sum_{\nu=1}^{k} \partial_{x_{i\nu}} \partial_{x_{j\nu}}$$

and write $\Delta := (\Delta_{ij})_{1 \le i \le j \le n}$ for the resulting family of differential operators. For $\lambda \in \Sigma$, let $\tau = \tau(\lambda)$ be the corresponding representation of $\mathrm{GL}(n,\mathbb{C})$ given by Kashiwara–Vergne (see §5.4 and §6.1 for a precise definition) that is realized on

$$W_\tau := \mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda),$$

the space of all V_λ-valued polynomial functions $f : \mathrm{M}_{n,k} \to V_\lambda$ satisfying

$$\Delta f = 0$$

and $f(xh) = \lambda^{-1}(h) f(x)$ for all $x \in \mathrm{M}_{n,k}$, $h \in \mathrm{O}(k)$, which is shown to be a finite dimensional, irreducible $\mathrm{GL}(n,\mathbb{C})$-module. Write Ω_n for the Siegel upper half-space (all $Z = X + iY$ with $X, Y \in \mathrm{Sym}_n$ and $Y > 0$).

Kashiwara and Vergne show that there is an action of G on

$$\mathcal{O}(\Omega_n, W_\tau),$$

the space of holomorphic W_τ-valued functions on Ω_n, with G-action given by

$$(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot f)(z) = \det(A - ZC)^{-\frac{k}{2}}\, \tau(A - ZC) \cdot f((A - ZC)^{-1}(ZD - B)).$$

They also construct a G-intertwining injection

$$\mathrm{KV}_\lambda : \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \hookrightarrow \mathcal{O}(\Omega_n, W_\tau)$$

given by

$$(\mathrm{KV}_\lambda\, f)(Z) := \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(x^T Z x)} I_\lambda^*(x) f(x)\, dx$$

for a certain $\mathrm{Hom}(V_\lambda, W_\tau)$-valued polynomial in $x \in \mathrm{M}_{n,k}$, $I^*_\lambda(x)$.

1.4. For their third realization (only mentioned in passing in [**8**] on page 3), let $MA\overline{N}$ be the maximal parabolic subgroup of G covering $\{\left(\begin{smallmatrix} A & 0 \\ C & A^{-1,T}\end{smallmatrix}\right)\} \subseteq \mathrm{Sp}(n,\mathbb{R})$ and let $\det^{-1/2} : MA\overline{N} \to \mathbb{C}$ the character whose square is the lift of the inverse of the determinant function on $A \in \mathrm{GL}(n,\mathbb{R})$. Consider the induced representation $\mathrm{Ind}^G_{MA\overline{N}}(W_\tau \otimes \det^{-k/2})$. Using the (lift of the) embedding $\iota : \mathrm{Sym}_n \to G$ where $\iota(t) := \left(\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}\right)$, the noncompact picture for $\mathrm{Ind}^G_{MA\overline{N}}(W_\tau \otimes \det^{-k/2})$ is given by

$$\begin{aligned}\mathcal{E}(\mathrm{Sym}_n,&W_\tau \otimes \det{}^{-k/2}) \\ &:= \{\psi \in \mathcal{C}^\infty(\mathrm{Sym}_n, W_\tau) \mid \exists \phi \in \mathrm{Ind}^G_{MA\overline{N}}(W_\tau \otimes \det{}^{-k/2}) \text{ with } \psi = \iota^*\phi\}.\end{aligned}$$

The action of G on $\psi \in \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2})$ is given by

$$(g \cdot \psi)(t) = \det{}^{-k/2}(A - tC)\, \tau(a - tc) \cdot \psi((A - tC)^{-1}(tD - B))$$

when $\det(A - tC) \neq 0$.

Let

$$\mathrm{BV} : \mathcal{O}(\Omega_n, W_\tau) \longrightarrow \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det{}^{-k/2})$$

by $(\mathrm{BV}\, f)(t) = \lim_{s\to 0^+} f(t + is)$, $(t, s \in \mathrm{Sym}_n,\ s > 0)$. They show that BV is intertwining and injective on the image of KV_λ. Putting all three realizations together, there is a diagram of G-maps:

$$\begin{array}{ccc} & & \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det{}^{-k/2}) \\ & & \uparrow{\scriptstyle \mathrm{BV}} \\ \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) & \xhookrightarrow[\mathrm{KV}_\lambda]{} & \mathcal{O}(\Omega_n, W_\tau) \end{array}$$

1.5. Our new realization will fit in the upper left corner of the diagram above. A key ingredient is to look at induction outside of the semisimple category.

Write the matrices $\mathrm{M}_{2n,k}$ in the form $\binom{v}{w}$, where $v, w \in \mathrm{M}_{n,k}$. Let

$$H := \mathrm{M}_{2n,k} \oplus \mathbb{R}$$

be the Heisenberg group with $\mathrm{Sp}(n,\mathbb{R}) \times \mathrm{O}(k)$-action $(g, c) \cdot (\binom{v}{w}, s) = (g\binom{v}{w}c^{-1}, s)$. Define

$$W := \{(\tbinom{0}{w}, s) \mid w \in \mathrm{M}_{n,k}, s \in \mathbb{R}\} \subseteq H.$$

The stabilizer of W in G is the maximal parabolic subgroup with Langlands decomposition $MA\overline{N}$.

We define

$$\overline{P} := MA\overline{N} \ltimes W$$

with character $\chi : \overline{P} \to \mathbb{C}$ by

$$\chi(ma\overline{n}, (\tbinom{0}{w}, s)) := \det(ma)^{-\frac{k}{2}}\, e^{\frac{is}{2}}.$$

Via the embedding $\iota : \mathrm{M}_{n,k} \times \mathrm{Sym}_n \to (G \ltimes H)$ by (the lift of)

$$\iota(x, t) := (\left(\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}\right), (\tbinom{x}{0}, 0)),$$

we work in the "noncompact" picture of

$$\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n) := \{\psi \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, \mathbb{C}) \mid \exists \phi \in \mathrm{Ind}^{G\ltimes H}_{\overline{P}}\, \chi \text{ so } \psi = \iota^*\phi\}$$

which carries a $(G \times \mathrm{O}(k)) \ltimes H$-action. It can be shown that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ acts as

$$(g \cdot \psi)(x,t) = \det{}^{-k/2}(A - tC)e^{\frac{i}{2}\operatorname{tr}(x^T C(A-tC)^{-1}x)} \times \psi((A-tC)^{-1}x,\, (A-tC)^{-1}(tD - B))$$

when $\det(A - tC) \neq 0$ (see Theorem 3.1).

1.6. Now let $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ be the space of all solutions to the Schrödinger equation[1]

$$i\partial_t \psi(x,t) = \Delta\psi(x,t)$$

with $\psi \in \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ such that the initial condition $\psi|_{t=0} \in \mathscr{S}(\mathrm{M}_{n,k})$. It is a lemma (Lemma 4.2) that $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ is invariant under the action of $(G \times \mathrm{O}(k)) \ltimes H$.

The $\mathrm{O}(k)$ action gives rise to the decomposition

$$\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}} = \bigoplus_{\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)} \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \otimes V_\lambda'$$

as a $G \times \mathrm{O}(k)$-module. Here $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ is the space of smooth functions $\psi : \mathrm{M}_{n,k} \times \mathrm{Sym}_n \to V_\lambda$ satisfying $\psi(xc,t) = \lambda(c)^{-1}\psi(x,t)$ for all $c \in \mathrm{O}(k)$ and $\langle \psi, f \rangle \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ for all $f \in V_\lambda'$.

Define $\mathrm{ev}_{t=0} : \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \to \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ by $(\mathrm{ev}_{t=0}\,\psi)(x) = \psi(x,0)$ and $\mathscr{F} : \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \to \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ be given by $(\mathscr{F} f)(\xi) = \widehat{f}(\xi)$. In Theorem 5.2 we show that the map

$$\begin{array}{c} \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \\ {\scriptstyle \mathscr{F} \circ \mathrm{ev}_{t=0}} \Big\downarrow \\ \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \end{array}$$

is nonzero, injective, G-intertwining, and an isomorphism on K-finite vectors. It follows that $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ completes to an irreducible unitary highest weight representation of G.

1.7. Now recall the $\mathrm{Hom}(V_\lambda, W_\tau)$-valued polynomial in $x \in \mathrm{M}_{n,k}$, $I_\lambda^*(x)$ (used in the map $\mathrm{KV}_\lambda : \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \to \mathcal{O}(\Omega_n, W_\tau))$ and define

$$\nabla_\lambda : \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \to \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, W_\tau)$$

by

$$\nabla_\lambda := I_\lambda^*(-i\partial_x).$$

In Theorem 6.1, we show that the map

$$\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \xhookrightarrow{\nabla_\lambda|_{x=0}} \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det{}^{-k/2})$$

is injective and G-intertwining. For $\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)$, we can combine everything into

[1] The equation here is really a system of partial differential equations. Furthermore, the sign in front of Δ is chosen so that we obtain highest weight representations. The choice

$$i\partial_t \psi(x,t) = -\Delta\psi(x,t)$$

would lead to lowest weight representations. See Section 7 for more details.

one commutative diagram of G-maps:

$$\begin{array}{ccc}
\mathcal{H}(\mathrm{M}_{n,k}\times\mathrm{Sym}_n,V_\lambda)_{\mathrm{pre}} & \xhookrightarrow{\nabla_\lambda|_{x=0}} & \mathcal{E}(\mathrm{Sym}_n, W_\tau\otimes\det^{-k/2}) \\
{\scriptstyle \mathscr{F}\circ \mathrm{ev}_{t=0}}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathrm{BV}} \\
\mathcal{L}^2(\mathrm{M}_{n,k},V_\lambda) & \xhookrightarrow[\mathrm{KV}_\lambda]{} & \mathcal{O}(\Omega_n, W_\tau)
\end{array}$$

1.8. This work generalizes our papers [**5**, **6**] ($k = 1$) and [**9**] ($n = 1$). We recently also obtained analogous results for $\mathrm{U}(n, n)$. We expect that there is a uniform generalization to all groups associated to Hermitian symmetric spaces of tube type via Jordan algebras. Finally, we anticipate that our work will eventually tie in with Enright–Wallach [**4**].

2. The Group

2.1. In the following, we write the matrices $\mathrm{M}_{2n,k} = \mathrm{M}_{2n,k}(\mathbb{R})$ in the form $\binom{v}{w}$, where $v, w \in \mathrm{M}_{n,k}$. We then define the Heisenberg group $H := \mathrm{M}_{2n,k} \oplus \mathbb{R}$ with multiplication given by

$$(\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, s_1)(\begin{pmatrix} v_2 \\ w_2 \end{pmatrix}, s_2) = (\begin{pmatrix} v_1 + v_2 \\ w_1 + w_2 \end{pmatrix}, s_1 + s_2 + \mathrm{Tr}(w_1^T v_2 - v_1^T w_2)).$$

The symplectic group $\mathrm{Sp}(n, \mathbb{R}) \subseteq \mathrm{GL}(2n, \mathbb{R})$ acts on H by $g \cdot (\binom{v}{w}, s) = (g\binom{v}{w}, s)$. Here $\mathrm{Sp}(n, \mathbb{R})$ is realized with respect to the standard symplectic form $\left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. The orthogonal group $\mathrm{O}(k)$ acts on H by $h \cdot (\binom{v}{w}, s) = (\binom{v}{w}h^{-1}, s)$.

2.2. Let $G := \mathrm{Mp}(n, \mathbb{R})$ be the double cover of $\mathrm{Sp}(n, \mathbb{R})$ defined as in [**5**]. We recall that G consists of the set of pairs (g, ε) with $g \in \mathrm{Sp}(n, \mathbb{R})$ and smooth $\varepsilon : \Omega_n \to \mathbb{C}$ satisfying $\varepsilon(Z)^2 = \det(CZ + D)$ where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and Ω_n is the Siegel upper half-space (all $Z = X + iY$ with $X, Y \in \mathrm{Sym}_n := \mathrm{Sym}_n(\mathbb{R})$ and $Y > 0$). We extend ε almost everywhere to the boundary of Ω_n, Sym_n, by $\varepsilon(x) := \lim_{Y\to 0^+} \varepsilon(X + iY)$ when $\det(CX + D) \neq 0$.

The Siegel upper half-space carries a transitive action by $\mathrm{Sp}(n, \mathbb{R})$ by linear fractional transformations,

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

This also extends to an almost everywhere defined action on $X \in \mathrm{Sym}_n$ when $\det(CX + D) \neq 0$.

We form the semidirect product

$$G \ltimes H$$

by having G act on H via its projection to $\mathrm{Sp}(n, \mathbb{R})$. Finally, we will also consider the group $(G \times \mathrm{O}(k)) \ltimes H$.

3. The Induced Representation and the Noncompact Picture

3.1. Define $W := \{(\binom{0}{w}, s) \mid w \in \mathrm{M}_{n,k},\ s \in \mathbb{R}\} \subseteq H$. The stabilizer of W in G is a maximal parabolic subgroup with Langlands decomposition $MA\overline{N}$ as in [**5**]. The Levi factor, MA, is a double cover of $\mathrm{GL}(n, \mathbb{R})$ whose projection to $\mathrm{Sp}(n,\mathbb{R})$ is $\{\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1,T}\end{smallmatrix}\right) \mid A \in \mathrm{GL}(n,\mathbb{R})\}$ while $\overline{N}$ projects isomorphically to $\{\left(\begin{smallmatrix} I & 0 \\ C & I\end{smallmatrix}\right) \mid C \in \mathrm{Sym}_n\}$. We define

$$\overline{P} := MA\overline{N} \ltimes W.$$

In particular, there is a function(s) $\det^{-1/2} : MA \to \mathbb{C}$ whose square is the lift of the inverse of the determinant function on $\mathrm{GL}(n,\mathbb{R})$. Precisely, elements of MA consist of pairs $l_{A,c} := (\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1,T}\end{smallmatrix}\right), Z \mapsto c)$ with $c \in \mathbb{C}$ satisfying $c^2 = \det A^{-1}$. Define the function $\det^{-1/2} : MA \to \mathbb{C}$ by

$$\det{}^{-1/2}(l_{A,c}) := c\,.$$

We extend $\det^{-1/2}$ to $MA\overline{N}$ by having it act trivially on $\overline{N}$. Finally, we define two characters $\chi : \overline{P} \to \mathbb{C}$ by

$$\chi(ma\overline{n}, (\begin{pmatrix} 0 \\ w \end{pmatrix}, s)) := \det{}^{-k/2}(ma)\, e^{\frac{is}{2}}.$$

where, of course, $\det^{-k/2}(ma) := (\det^{-1/2}(ma))^k$. Note that MA is isomorphic to the double cover $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ and that $(MA)_{\mathbb{C}}$ is isomorphic to the double cover $\widetilde{\mathrm{GL}}(n,\mathbb{C})$. Clearly $\det^{-1/2}$ uniquely extends to a holomorphic character of $(MA)_{\mathbb{C}}$. We also remark that, in [**5**] with $k = 1$, the character χ was written as $\chi_{1,-1/2,-i/2}$ and eventually led to the the study of unitary highest weight modules.

3.2. The induced representation

$$\mathrm{Ind}_{\overline{P}}^{G\ltimes H}\chi := \left\{\phi \in \mathcal{C}^\infty(G \ltimes H, \mathbb{C}) \mid \phi(g\overline{p}) = \chi(\overline{p})^{-1}\phi(g)\ \forall g \in G \ltimes H,\, \overline{p} \in \overline{P}\right\}$$

is an analog of a degenerate principal series representation for the group $G \ltimes H$. Our focus will be on the non-compact picture of this representation. Embed $\iota : \mathrm{M}_{n,k} \times \mathrm{Sym}_n \to (G \ltimes H)$ by

$$\iota(x,t) := ((\begin{pmatrix} I_n & t \\ 0 & I_n \end{pmatrix}, Z \mapsto 1), (\begin{pmatrix} x \\ 0 \end{pmatrix}, 0)).$$

The projection of the image of this map to $(G \ltimes H)/\overline{P}$ is open and dense and so the elements of $\mathrm{Ind}_{\overline{P}}^{G\ltimes H}\chi$ are determined by their restriction to the image of ι. Let $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ denote the set of these restrictions,

$$\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n) := \{\psi \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, \mathbb{C}) \mid \exists\phi \in \mathrm{Ind}_{\overline{P}}^{G\ltimes H}\chi \text{ so } \psi = \iota^*\phi\},$$

and view it as a $G \ltimes H$-module for which the map $\phi \mapsto \psi = \iota^*\phi$ is an intertwining isomorphism. Observe that $\mathrm{O}(k)$ also acts on $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ by

$$(h \cdot \psi)(x,t) := \psi(xh, t).$$

This action commutes with the action of G.

3.3. In the following result, we write $g = (\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right), \varepsilon) \in G$. When $\det(A - tC) \neq 0$ for $t \in \mathrm{Sym}_n$, a calculation shows that $g^{-1} \cdot t = (A - tC)^{-1}(tD - B)$ and that $\varepsilon(g^{-1} \cdot t)^2 = \det(A - tC)^{-1}$. In order to make the formulas below more transparent, we will use the imprecise notation (when k is odd) $\det^{-k/2}(A - tC)$ to denote $\varepsilon(g^{-1} \cdot t)^k$. Alternately, one can write g^{-1} as $(\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)^{-1}, \varepsilon_{g^{-1}})$ and show $\varepsilon(g^{-1} \cdot t) = \varepsilon_{g^{-1}}(t)$.

THEOREM 3.1. *For $\psi \in \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$, the action of $(g, \varepsilon) \in G$ on ψ is given by*

$$((g,\varepsilon)\cdot\psi)(x,t) = \det{}^{-k/2}(A - tC)e^{\frac{i}{2}\operatorname{tr}(x^T C(A-tC)^{-1}x)} \cdot \psi((A-tC)^{-1}x,\, (A-tC)^{-1}(tD-B))$$

when $\det(A - tC) \neq 0$.

The action of $(\binom{v}{w}, s) \in H$ *on* ψ *is given by*

$$((\tbinom{v}{w}, s)\cdot\psi)(x,t) = e^{\frac{i}{2}(s+\operatorname{tr}(2x^T w - v^T w + w^T tw))}\psi(x - v + tw, t).$$

The action for $h \in \mathrm{O}(k)$ *is*

$$(h\cdot\psi)(x,t) = \psi(xh,t).$$

PROOF. This result follows by straightforward calculations. For the action of G, we give the details in the most involved case and leave the rest to the reader. Full details may be extracted by the patient reader from Theorem 1 of [**5**].

Suppose $\psi = \iota^*\phi$ with $\phi \in \mathrm{Ind}_{P}^{G\ltimes H}\chi$ and $\overline{n} = ((\begin{smallmatrix} I_n & 0 \\ C & I_n\end{smallmatrix}), \varepsilon(Z)) \in G$. By definition, we calculate

$$\begin{aligned}
&(\overline{n}\cdot\psi)(x,t)\\
&= \phi(\overline{n}^{-1}\iota(x,t))\\
&= \phi(((\begin{smallmatrix} I_n & 0 \\ -C & I_n\end{smallmatrix}), \det(I_n - CZ)^{1/2})((\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{x}{0}, 0))\\
&= \phi(((\begin{smallmatrix} I_n & (I_n-tC)^{-1}t \\ 0 & I_n\end{smallmatrix}), 1)((\begin{smallmatrix} (I_n-tC)^{-1} & 0 \\ -C & (I_n-Ct)\end{smallmatrix}), \det(I_n - C(t+Z))^{1/2})\\
&\quad \times (\tbinom{x}{0}, 0))\\
&= \phi(((\begin{smallmatrix} I_n & (I_n-tC)^{-1}t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{(I_n-tC)^{-1}x}{-CX}, 0)\\
&\quad \times ((\begin{smallmatrix} (I_n-tC)^{-1} & 0 \\ -C & (I_n-Ct)\end{smallmatrix}), \det(I_n - C(t+Z))^{1/2})\\
&= \phi(((\begin{smallmatrix} I_n & (I_n-tC)^{-1}t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{(I_n-tC)^{-1}x}{0}, 0)(\tbinom{0}{-Cx}, -\operatorname{tr}(x^T(I_n - Ct)^{-1}Cx))\\
&\quad \times \det{}^{-k/2}(I_n - tC)\\
&= \det{}^{-k/2}(I_n - tC)e^{\frac{i}{2}\operatorname{tr}(x^T(I_n-Ct)^{-1}Cx)}\psi((I_n - tC)^{-1}x, (I_n - tC)^{-1}t).
\end{aligned}$$

For the action of H, we calculate

$$\begin{aligned}
&((\tbinom{v}{w}, s)\cdot\psi)(x,t)\\
&\quad = \phi((\tbinom{-v}{-w}, -s)((\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{x}{0}, 0))\\
&\quad = \phi(((\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{-v+tw}{-w}, -s)(\tbinom{x}{0}, 0))\\
&\quad = \phi(((\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{x-v+tw}{-w}, -s - \operatorname{tr}(w^T x)))\\
&\quad = \phi(((\begin{smallmatrix} I_n & t \\ 0 & I_n\end{smallmatrix}), 1)(\tbinom{x-v+tw}{0}, 0)(\tbinom{0}{-w}, -s - \operatorname{tr}(2x^T w - v^T w + w^T tw)))\\
&\quad = e^{\frac{i}{2}(s+\operatorname{tr}(2x^T w - v^T w + w^T tw))}\psi(x - v + tw, t).
\end{aligned}$$

□

Exponentiating and differentiating gives the following action of the Lie algebra. Here we write $\partial^x_M := \sum_{ij} M_{ij}\partial_{x_{ij}}$ and $\partial^t_M := \sum_{i\le j} M_{ij}\partial_{(t)_{ij}}$ and identify $\mathrm{Lie}(G)$ with $\mathfrak{sp}(n,\mathbb{R})$.

THEOREM 3.2. *For* $X = \left(\begin{smallmatrix} a & b \\ c & -a^T \end{smallmatrix}\right) \in \mathfrak{sp}(n,\mathbb{R})$ *and* $\psi \in \mathcal{E}(\mathrm{M}_{n,k}\times \mathrm{Sym}_n)$,

$$\begin{aligned}(X\cdot\psi)(x,t) &= [\frac{k}{2}\operatorname{tr}(tc-a) + \frac{i}{2}\operatorname{tr}(x^T cx)]\psi(x,t)\\ &\quad + \partial^x_{(tc-a)x}\psi(x,t) + \partial^t_{tct}\psi(x,t) - \partial^t_{(ta^T+at)}\psi(x,t) - \partial^t_b\psi(x,t).\end{aligned}$$

PROOF. In $\mathrm{Sp}(n,\mathbb{R})$, $\exp(s\begin{pmatrix} a & b \\ c & -a^T\end{pmatrix}) = \begin{pmatrix} I_n+sa & sb \\ sc & I_n-sa^T\end{pmatrix} + \cdots$ so that

$$\begin{aligned}&(\begin{pmatrix} a & b \\ c & -a^T\end{pmatrix}\cdot\psi)(x,t)\\ &= \frac{d}{ds}\Big|_{s=0}\{\det{}^{-k/2}(I_n - s(tc-a))e^{\frac{i}{2}s\operatorname{tr}(x^T c(I_n-s(tc-a))^{-1}x)}\\ &\qquad \times\psi((I_n - s(tc-a))^{-1}x,\ (I_n-s(tc-a))^{-1}(t-s(ta^T+b)))\,\}\\ &= [\frac{k}{2}\operatorname{tr}(tc-a) + \frac{i}{2}\operatorname{tr}(x^T cx)]\,\psi(x,t)\\ &\quad + \partial^x_{(tc-a)x}\psi(x,t) + \partial^t_{tct}\psi(x,t) - \partial^t_{(ta^T+at)}\psi(x,t) - \partial^t_b\psi(x,t).\end{aligned}$$

□

4. A System of Schrödinger-Type Operators

4.1. The set $\{E_{ij}+E_{ji} \mid 1\le i\le j\le n\}$ forms an orthogonal basis for Sym_n with respect to the trace form. Let $\{t_{ij} \mid 1\le i\le j\le n\}$ be the dual basis for Sym_n^* which we use to coordinatize Sym_n and to identify it with $\mathbb{R}^{n(n+1)/2}$. Note that if we write $(t)_{ij}$ for the ij-matrix entry on Sym_n, then $(t)_{ij} = t_{ij}$ when $i<j$ while $(t)_{ii} = 2t_{ii}$ (and $2\partial_{(t)_{ii}} = \partial_{t_{ii}}$). We write $\partial_{t_{ij}}$ for the corresponding differential operator on $\mathcal{C}^\infty(\mathrm{Sym}_n,\mathbb{C})$ and $\partial_t = (\partial_{t_{ij}})_{1\le i\le j\le n}$ for the resulting family of differential operators.

4.2. Next define $\pi : \mathrm{M}_{n,k}\to\mathrm{Sym}_n$ by $\pi(x) := xx^T$ and let $q_{ij} := \pi^*((t)_{ij}) \in \mathcal{C}^\infty(\mathrm{M}_{n,k})$. Then $q_{ij}(x) = \sum_{\nu=1}^k x_{i\nu}x_{j\nu}$. Write

$$\Delta_{ij} := q_{ij}(\partial_x) = \sum_{\nu=1}^k \partial_{x_{i\nu}}\partial_{x_{j\nu}}$$

for the corresponding polynomial differential operator with constant coefficients and $\Delta := (\Delta_{ij})_{1\le i\le j\le n}$ for the resulting family of differential operators.

Let

$$\mathcal{H}(\mathrm{M}_{n,k}\times\mathrm{Sym}_n)_{\mathrm{pre}}$$

be the space of all solutions to

$$i\partial_t\psi(x,t) = \Delta\psi(x,t)$$

with $\psi\in\mathcal{E}(\mathrm{M}_{n,k}\times\mathrm{Sym}_n)$ such that the initial condition $\psi|_{t=0}\in\mathscr{S}(\mathrm{M}_{n,k})$, the Schwartz space of rapidly decreasing functions on $\mathrm{M}_{n,k}\cong\mathbb{R}^{nk}$. We will shortly see that $\mathcal{H}(\mathrm{M}_{n,k}\times\mathrm{Sym}_n)_{\mathrm{pre}}$ is a pre-Hilbert space and is invariant under $(G\times\mathrm{O}(k))\ltimes H$.

For $\Psi \in L^1(\mathrm{M}_{n,k})$, we write the Fourier transform as

$$\widehat{\Psi}(\xi) := \int_{\mathrm{M}_{n,k}} \Psi(x) e^{-i\,\mathrm{tr}(x^T\xi)}\, dx.$$

For $\psi \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$, we also write $\widehat{\psi}(\xi, t)$ for the Fourier transform of ψ with respect to x with fixed t.

THEOREM 4.1. *Any $\psi \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ may be written in terms of its initial condition by*

$$\psi(x,t) = \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} \widehat{\psi}(\xi, 0) e^{\frac{i}{2}\,\mathrm{tr}(\xi^T t\xi)} e^{i\,\mathrm{tr}(x^T\xi)}\, d\xi.$$

PROOF. By standard Fourier techniques, it follows that

$$-i\partial_{t_{ij}} \widehat{\psi}(\xi, t) = \sum_{\nu=1}^{k} \xi_{i\nu}\xi_{j\nu} \widehat{\psi}(\xi, t)$$

so that (recall $2t_{ii} = (t)_{ii}$)

$$\widehat{\psi}(\xi, t) = \widehat{\psi}(\xi, 0) e^{i\sum_{i\le j}\sum_{\nu=1}^{k} \xi_{i\nu}\xi_{j\nu}t_{ij}} = \widehat{\psi}(\xi, 0) e^{\frac{i}{2}\,\mathrm{tr}(\xi^T t\xi)}.$$

The result now follows by applying the inverse Fourier transform. □

Note that it is not automatic that the solution to the Cauchy problem for an arbitrary Schwartz initial condition lies in $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$.

The next lemma follows from Theorem 3.1 and straightforward calculations as in Theorems 2 and 4 of [**5**].

LEMMA 4.2. *$\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ is invariant under the action of $(G \times \mathrm{O}(k)) \ltimes H$.*

PROOF. Invariance of the set of solutions to $(-i\partial_t + \Delta)\psi = 0$ follows from Theorem 3.2 and a straightforward calculation showing that

$$\frac{1}{2}[X, (-i\partial_t + \Delta)_{ij}] = ((ct - a^T)(i\partial_t - \Delta))_{ij} + ((ct - a^T)(i\partial_t - \Delta))_{ji}$$

for each $X \in \mathfrak{sp}(n, \mathbb{R})$. See Theorem 2 of [**5**] for explicit analogous calculations. Preservation of the Schwartz initial condition follows from Theorems 3.1 and 4.1 and Fourier theory as in Theorem 4 of [**5**]. □

5. $\mathrm{O}(k)$-decomposition of $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$

5.1. For an irreducible representation (λ, V_λ) of $\mathrm{O}(k)$, write the dual space as $V_\lambda' = \mathrm{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C})$ (in this case known to be isomorphic to V_λ). For $f \in V_\lambda'$ and $\psi \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)$, define $\langle \psi, f\rangle \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ by $\langle \psi, f\rangle (x,t) := f(\psi(x,t))$. Now define

$$\begin{aligned}\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) := \{\psi \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \mid \\ \langle \psi, f\rangle \in \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)\ \forall f \in V_\lambda' \text{ and} \\ \psi(xh, t) = \lambda(h)^{-1}\psi(x,t)\ \forall h \in \mathrm{O}(k)\}.\end{aligned}$$

It is easy to see that $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \cong \mathrm{Hom}_{\mathrm{O}(k)}(V'_\lambda, \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n))$. Therefore the canonical decomposition of the $\mathrm{O}(k)$-finite vectors of $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ can be written as

$$\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{O}(k)} \cong \bigoplus_{\lambda \in \widehat{\mathrm{O}}(k)} V'_\lambda \otimes \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda).$$

Write $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\lambda'}$ for the V'_λ-isotypic $\mathrm{O}(k)$-component of $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$. For each λ, the isomorphism is implemented by the map

$$\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \otimes V'_\lambda \xrightarrow{\cong} \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\lambda'}$$

induced by $\psi \otimes f \mapsto \langle \psi, f \rangle$. As G commutes with $\mathrm{O}(k)$, $\mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)$ inherits the structure of a G-module. Specifically, by Theorem 3.1 and with the same conventions, the action of $(g, \varepsilon) \in G$ on ψ is given by

$$\begin{aligned}((g,\varepsilon)\cdot\psi)(x,t) = \det(A-tC)^{-\frac{k}{2}} e^{\frac{i}{2}\operatorname{tr}(x^T C(A-tC)^{-1}x)} \\ \cdot\, \psi((A-tC)^{-1}x,\ (A-tC)^{-1}(tD-B))\end{aligned} \tag{5.1.1}$$

when $\det(A - tC) \neq 0$.

5.2. Now let

$$\begin{aligned}\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} := \{\psi \in \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \mid \\ \langle \psi, f\rangle \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}\ \forall f \in V'_\lambda\}.\end{aligned}$$

The above discussion and restriction results in a mapping

$$\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \otimes V'_\lambda \to \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$$

induced by $\psi \otimes f \to \langle \psi, f \rangle$ that is an isomorphism onto the V'_λ-isotypic $\mathrm{O}(k)$-component of $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$. Note that these spaces may be trivial for certain λ. As before, $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ inherits the structure of a G-module with action given by Equation 5.1.1.

5.3. Extend the Fourier Transform in Equation 4.2 from scalar valued integration to V_λ-vector valued integration on $\mathrm{M}_{n,k}$. By a simple modification of Equation 4.1, any $\psi \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ can be written in the form

$$\psi(x,t) = \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} \widehat{\psi}(\xi, 0) e^{\frac{i}{2}\operatorname{tr}(\xi^T t\xi)} e^{i \operatorname{tr}(x^T \xi)}\, d\xi.$$

5.4. It still remains to show that $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \neq 0$ for certain λ. The difficulty is that any particular solution to the Cauchy problem may not originate from a function coming from the induced representation, $\mathrm{Ind}_P^{G \ltimes H} \chi$.

To this end, let $r_s := \exp_G\left(\begin{smallmatrix} 0 & -sI_n \\ sI_n & 0 \end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} I_n \cos s & -I_n \sin s \\ I_n \sin s & I_n \cos s \end{smallmatrix}\right), \varepsilon_s\right)$ where $\varepsilon_s : \Omega_n \to \mathbb{C}$ is determined by the conditions $\varepsilon_s(Z)^2 = \det(I_n \cos s + Z \sin s)$ and $\varepsilon_s(iI_n) = e^{\frac{isn}{2}}$. In particular, $r_{\pi/2} = \left(\left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right), \varepsilon_{\pi/2}\right)$ where $\varepsilon_{\pi/2}$ is determined by the the conditions $\varepsilon_{\pi/2}(Z)^2 = \det Z$ and $\varepsilon_{\pi/2}(iI_n) = e^{\frac{i\pi n}{4}}$. It is easy to see that them map $s \mapsto r_s$ is a 1-parameter subgroup of G.

The *Cartan involution* $\theta : G \to G$ is the anti-involution $\theta(g,\varepsilon) := r_{\pi/2}(g,\varepsilon)^{-1} r_{\pi/2}^{-1}$. We also define $\varepsilon^T : \Omega_n \to \mathbb{C}$ so that

$$\theta(g, \varepsilon) = (g^T, \varepsilon^T).$$

By definition, $\varepsilon^T(Z)^2 = \det(B^T Z + D^T)$. Pinning down the exact square root is a bit messy, but can be done with a straightforward calculation found in [**5**].

By definition, $(\left(\begin{smallmatrix} I_n & t \\ 0 & I_n \end{smallmatrix}\right), 1)^T = (\left(\begin{smallmatrix} I_n & 0 \\ t & I_n \end{smallmatrix}\right), 1^T)$. Here, $1^T(Z)^2 = \det(tZ + I_n)$. Since $1^T(Z)^2 \to 1$ as $t \to 0$, it follows that the function $1^T(Z)$ is the analytic continuation of the function $Z \to \sqrt{\det(tZ + I_n)}$ that is initially defined only for sufficiently small Z. In the following, we will abuse notation and write $\det(I_n + tZ)^{1/2}$ for the function $1^T(Z)$. We will also write $\det(I_n + t\overline{Z})^{1/2}$ for the function $\overline{\det(I_n + tZ)}^{1/2}$ (especially for the case of $Z = iI_n$).

5.5. Let $\mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda)$ be the space of all V_λ-valued polynomial functions $f : \mathrm{M}_{n,k} \to V_\lambda$ satisfying $\Delta f = 0$ and $f(xh) = \lambda^{-1}(h)f(x)$ for all $x \in \mathrm{M}_{n,k}$, $h \in \mathrm{O}(k)$. The λ's for which $\mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda)$ are nonzero are explicitly calculated in [**8**]. We write

$$\Sigma := \{\lambda \in \widehat{\mathrm{O}}(k) \mid \mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda) \neq 0\}.$$

For such λ, by [**8**], $\mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda)$ is known to be a finite dimensional, irreducible representation of $\mathrm{GL}(n, \mathbb{C})$. Note that here we extend the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda)$ holomorphically to $\mathrm{GL}(n, \mathbb{C})$ by identifying complex valued polynomial functions on $\mathrm{M}_{n,k}$ with complex valued polynomial functions on $\mathrm{M}_{n,k}(\mathbb{C})$.

Lemma 5.1. *Let $\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)$ and $f \in \mathfrak{H}(\mathrm{M}_{n,k}, V_\lambda)$. Then the function $\Psi_f \in \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)$ given by*

$$\Psi_f(x,t) = \det(I_n - it)^{-k/2} e^{-\frac{1}{2}\operatorname{tr}(x^T(I_n - it)^{-1}x)}\, f((I_n - it)^{-1}x).$$

lies in $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$.

Proof. We begin by calculating $\varepsilon_{\overline{p}}^T$ explicitly for

$$\overline{p} = (\left(\begin{smallmatrix} A & 0 \\ C & A^{-1,T} \end{smallmatrix}\right), \varepsilon_{\overline{p}}) \in MA\overline{N}.$$

We know that $\varepsilon_{\overline{p}}^T(Z)^2 = \det(A^{-1}) = (\det^{-1/2}\overline{p})^2$. To determine the precise value, first look at the special case of $C = 0$, $A = \lambda I_n$ with $\lambda > 0$, and $\varepsilon_{\overline{p}} = \det_1^{-1/2}\overline{p} = \epsilon_0 \lambda^{-n/2}$ for a fixed $\epsilon_0 \in \{\pm 1\}$. By continuity, there is a choice of $\epsilon \in \{\pm 1\}$ so that $\varepsilon_{\overline{p}}^T = \epsilon\epsilon_0\lambda^{-n/2}$ for all λ. But as $\lambda \to 1$, $r_{\pi/2}\overline{p}^{-1}r_{\pi/2}^{-1} \to (\mathrm{I}_{2n}, \epsilon_0)$. Thus $\epsilon = 1$ and $\varepsilon_{\overline{p}}^T = \det_1^{-1/2}\overline{p} = \overline{\det^{-1/2}}\overline{p}$ for this special case. Next, look at the special case of $C = 0$, $A \in \mathrm{O}(n)$, and $\varepsilon_{\overline{p}} = \det_1^{-1/2}\overline{p} \in \{\pm 1, \pm i\}$. In this case, $r_{\pi/2}\overline{p}^{-1}r_{\pi/2}^{-1} = \overline{p}^{-1}$ so that $\varepsilon_{\overline{p}}^T = \varepsilon_{\overline{p}}^{-1} = \overline{\det^{-1/2}}\overline{p}$ here. Since it is easy to show $\varepsilon_{\overline{p}}^T = 1$ when $\overline{p} \in \overline{N}$, it follows that $\varepsilon_{\overline{p}}^T = \overline{\det^{-1/2}}\overline{p}$ for all $\overline{p} \in MA\overline{N}$. In particular, $((g,\varepsilon)\overline{p})^T = (\left(\begin{smallmatrix} A^T & C \\ 0 & A^{-1} \end{smallmatrix}\right) g^T, \overline{\det^{-1/2}}\overline{p}\,\varepsilon^T)$ for any $(g,\varepsilon) \in G$.

Identifying f with a complex valued polynomial on $\mathrm{M}_{n,k}(\mathbb{C})$ when necessary, consider the function $\Phi \in \mathcal{C}^\infty(G \ltimes H, V_\lambda)$ given by

$$\Phi((g,\varepsilon)\,(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), s)) := \overline{\varepsilon^T}(iI_n)^{-k}\, e^{-\frac{i}{2}(s + \operatorname{tr}(y^T x + x^T(g^T \cdot (iI_n))x))} \times f((D_g + iB_g)^{-1,T}x)$$

where $g = \left(\begin{smallmatrix} A_g & B_g \\ C_g & D_g \end{smallmatrix}\right)$. Calculate

$$\begin{aligned}
&\Phi((g,\varepsilon)\,(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), s)\,\overline{p}\,(\left(\begin{smallmatrix} 0 \\ y_0 \end{smallmatrix}\right), s_0)) \\
&= \Phi((g,\varepsilon)\,\overline{p}\,(\left(\begin{smallmatrix} A^{-1}x \\ A^T y - Cx + y_0 \end{smallmatrix}\right), s + s_0 - \operatorname{tr}(y_0^T A^{-1}x))\,) \\
&= [(\det{}^{-k/2}\,\overline{p})e^{\frac{is_0}{2}}]^{-1}\overline{\varepsilon^T}(iI_n)^{-k}e^{-\frac{i}{2}(s+\operatorname{tr}(y^T x + x^T(g^T\cdot(i\sigma I_n))x))} \\
&\quad\times f((-iB_g + D_g)^{-1,T}x) \\
&= \chi(\overline{p}\,(\left(\begin{smallmatrix} 0 \\ y_0 \end{smallmatrix}\right), s_0))^{-1}\Phi((g,\varepsilon)\,(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), s)).
\end{aligned}$$

From this it easily follows that $\Psi := \iota^*\Phi \in \mathcal{E}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)$.

A quick examination shows that

$$\Psi(x,t) = \det(I_n - it)^{-k/2}e^{-\frac{1}{2}\operatorname{tr}(x^T(I_n - it)^{-1}x)}\, f((I_n - it)^{-1}x).$$

It remains to see that $\Psi \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$. Using only the fact that $\Delta f = 0$, it is a straightforward exercise with the chain rule to verify that Ψ satisfies the Schrödinger Equation 1.6. We omit the calculation (see Theorem 5 of [**5**] for a similar calculation in the scalar-valued case). □

5.6. Let $\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ be the space of square-integrable V_λ-valued functions $f : \mathrm{M}_{n,k} \to V_\lambda$ satisfying $f(xh) = \lambda(h)^{-1}f(x)$ for all $h \in \mathrm{O}(k)$ equipped with the G-action generated by

$$\begin{aligned}
(l_{A,c}\cdot f)(\xi) &= \det{}^{k/2}(A)f(A^T\xi) \\
(\iota(0,B)\cdot f)(\xi) &= e^{-\frac{i}{2}\operatorname{tr}(\xi^T B\xi)}f(\xi) \\
(r_{\pi/2}\cdot f)(\xi) &= (\frac{i}{2\pi})^{\frac{nk}{2}}\int_{\mathrm{M}_{n,k}} f(x)e^{i\operatorname{tr}(\xi^T x)}\,dx
\end{aligned}$$

where $(i)^{nk/2}$ is shorthand for $e^{i\pi nk/4}$. In [**8**], Kashiwara and Vergne showed that $\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ is an irreducible unitary highest weight representation of G.

Let $\mathrm{ev}_{t=0} : \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \to \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ be given by $(\mathrm{ev}_{t=0}\,\psi)(x) = \psi(x,0)$ and $\mathscr{F} : \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \to \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ be given by $(\mathscr{F}f)(\xi) = (2\pi)^{-nk}\,\widehat{f}(\xi)$.

THEOREM 5.2. *For $\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)$, the map*

$$\mathscr{F}\circ\mathrm{ev}_{t=0} \qquad \begin{array}{c} \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \\ \downarrow \\ \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \end{array}$$

is nonzero, injective, and G-intertwining.

PROOF. This result follows by standard Fourier analysis calculations. For example, by Theorem 3.1,

$$((\mathscr{F}\circ\mathrm{ev}_{t=0})(l_{A,c}\cdot\psi))(\xi) = \frac{\det^{-k/2}A}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(A^{-1}x,0)e^{-i\operatorname{tr}(x^T\xi)}\,dx$$

while

$$\begin{aligned}
(l_{A,c}\cdot((\mathscr{F}\circ\mathrm{ev}_{t=0})(\psi)))(\xi) &= \frac{\det^{k/2}A}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(x,0)e^{-i\operatorname{tr}(x^T A^T\xi)}\,dx \\
&= \frac{\det^{k/2}A\ |\det A|^{-k}}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(A^{-1}x,0)e^{-i\operatorname{tr}(x^T\xi)}\,dx.
\end{aligned}$$

As it is straightforward to show that $\det^{-k/2} A = \det^{k/2} A\ |\det A|^{-k}$ (see Theorem 6 of [**5**] for an explicit calculation), it follows that the action of MA commutes with $\mathscr{F} \circ \mathrm{ev}_{t=0}$.

For N, observe that Equation 5.3 may be rewritten as

$$\psi(x,t) = \mathscr{F}^{-1}[e^{\frac{i}{2}\operatorname{tr}(\cdot^T t\cdot)}\,(\mathscr{F} \circ \mathrm{ev}_{t=0})(\psi)](x)$$

and calculate that

$$\begin{aligned}
((\mathscr{F} \circ \mathrm{ev}_{t=0})(\iota(0,B)\cdot\psi))(\xi) &= \frac{1}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(x,-B)e^{-i\operatorname{tr}(x^T\xi)}\,dx \\
&= \mathscr{F}(\mathscr{F}^{-1}[e^{-\frac{i}{2}\operatorname{tr}(\cdot^T B\cdot)}\,(\mathscr{F} \circ \mathrm{ev}_{t=0})(\psi)])(\xi) \\
&= \frac{e^{-\frac{i}{2}\operatorname{tr}(\xi^T B\xi)}}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(x,0)e^{-i\operatorname{tr}(x^T\xi)}\,dx \\
&= (\iota(0,B)\cdot((\mathscr{F} \circ \mathrm{ev}_{t=0})(\psi)))(\xi).
\end{aligned}$$

Finally turn to $r_{\pi/2}$. The relation $\psi(x,t) = \mathscr{F}^{-1}[e^{\frac{i}{2}\operatorname{tr}(\cdot^T t\cdot)}\,(\mathscr{F} \circ \mathrm{ev}_{t=0})(\psi)](x)$ can be written as (for $\lambda \neq 0$)

$$\begin{aligned}
&\psi(x,-\lambda^{-1}I_n) \\
&= \frac{1}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \widehat{\psi}(\xi,0)e^{-\frac{i}{2\lambda}\operatorname{tr}(\xi^T\xi)}e^{i\operatorname{tr}(x^T\xi)}\,d\xi \\
&= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \widehat{\psi}(\xi,0)e^{(-\epsilon-\frac{i}{2\lambda})\operatorname{tr}(\xi^T\xi)}e^{i\operatorname{tr}(x^T\xi)}\,d\xi \\
&= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}}\int_{\mathrm{M}_{n,k}} \psi(w,0)e^{-i\operatorname{tr}(\xi^T w)}e^{(-\epsilon-\frac{i}{2\lambda})\operatorname{tr}(\xi^T\xi)}e^{i\operatorname{tr}(x^T\xi)}\,dw\,d\xi \\
&= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^{nk}}\int_{\mathrm{M}_{n,k}} \psi(w,0)\int_{\mathrm{M}_{n,k}} e^{(-\epsilon-\frac{i}{2\lambda})\operatorname{tr}(\xi^T\xi)}e^{i\operatorname{tr}((x-w)^T\xi)}\,d\xi dw \\
&= \lim_{\epsilon\to 0^+}\frac{\pi^{\frac{nk}{2}}}{(2\pi)^{nk}(\epsilon+\frac{i}{2\lambda})^{\frac{nk}{2}}}\int_{\mathrm{M}_{n,k}} \psi(w,0)e^{\frac{1}{4(-\epsilon-\frac{i}{2\lambda})}\operatorname{tr}((x-w)^T(x-w))}\,dw \\
&= (-\frac{i\lambda}{2\pi})^{\frac{nk}{2}}\int_{\mathrm{M}_{n,k}} \psi(w,0)e^{\frac{i\lambda}{2}\operatorname{tr}((x-w)^T(x-w))}\,dw.
\end{aligned}$$

Thus

$$\begin{aligned}
&(r_{\pi/2}\cdot\psi)(x,\lambda I_n) \\
&= \det{}^{-k/2}(-\lambda I_n)^{-\frac{k}{2}}e^{-\frac{i}{2\lambda}\operatorname{tr}(x^Tx)}\psi(-\lambda^{-1}x,-\lambda^{-1}I_n) \\
&= (-\frac{i\lambda}{2\pi})^{\frac{nk}{2}}\det{}^{-k/2}(-\lambda I_n) \\
&\qquad\times\int_{\mathrm{M}_{n,k}} \psi(w,0)e^{-\frac{i}{2\lambda}\operatorname{tr}(x^Tx)}e^{\frac{i\lambda}{2}\operatorname{tr}((\lambda^{-1}x+w)^T(\lambda^{-1}x+w))}\,dw \\
&= (\frac{i}{2\pi})^{\frac{nk}{2}}\int_{\mathrm{M}_{n,k}} \psi(w,0)e^{\frac{i}{2}(\lambda\operatorname{tr}(w^Tw)+2\operatorname{tr}(x^Tw))}\,dw
\end{aligned}$$

so

$$(r_{\pi/2}\cdot\psi)(x,0) = (\frac{i}{2\pi})^{\frac{nk}{2}}(\mathscr{F}^{-1}\psi)(x,0)$$

and

$$((\mathscr{F}\circ \mathrm{ev}_{t=0})(r_{\pi/2}\cdot\psi))(\xi) = (\frac{i}{2\pi})^{\frac{nk}{2}}\,\psi(\xi,0).$$

This will finish the proof since

$$\begin{aligned}(r_{\pi/2}\cdot((\mathscr{F}\circ \mathrm{ev}_{t=0})(\psi)))(\xi) &= (\frac{i}{2\pi})^{\frac{nk}{2}}\int_{\mathrm{M}_{n,k}}((\mathscr{F}\circ \mathrm{ev}_{t=0})(\psi))(x)e^{i\,\mathrm{tr}(\xi^T x)}\,dx\\ &= (\frac{i}{2\pi})^{\frac{nk}{2}}\,\psi(\xi,0).\end{aligned}$$

□

For $\lambda\in\Sigma\subseteq\widehat{\mathrm{O}}(k)$, it follows that $\mathcal{H}(\mathrm{M}_{n,k}\times\mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ completes to a irreducible unitary highest weight representation of G.

6. The Commutative Diagram

6.1. Recall from Subsection 5.5 that $\mathfrak{H}(\mathrm{M}_{n,k},V_\lambda)$ is the space of all V_λ-valued polynomial functions $f:\mathrm{M}_{n,k}\to V_\lambda$ satisfying $\Delta f=0$ and $f(xh)=\lambda^{-1}(h)f(x)$ for all $x\in\mathrm{M}_{n,k}$, $h\in\mathrm{O}(k)$. By the structure of the differential operators formed out of rows and since the ring of $\mathrm{O}(k)$-invariant constant coefficient differential operators on $\mathrm{M}_{n,k}$ is generated by the Δ_{ij}'s, $\mathfrak{H}(\mathrm{M}_{n,k},V_\lambda)$ has the structure of a $\mathrm{GL}(n,\mathbb{C})\times\mathrm{O}(k)$ modules under the action $((g,h)\cdot f)(x)=f(g^{-1}xh)$ for $g\in\mathrm{GL}(n,\mathbb{C})$, $h\in\mathrm{O}(k)$. Note that here we extend τ holomorphically to $\mathrm{GL}(n,\mathbb{C})$ by identifying complex valued polynomial functions on $\mathrm{M}_{n,k}$ with complex valued polynomial functions on $\mathrm{M}_{n,k}(\mathbb{C})$.

6.2. By [**8**], $\mathfrak{H}(\mathrm{M}_{n,k},V_\lambda)$, $\lambda\in\Sigma$, is a finite dimensional, irreducible representation of $\mathrm{GL}(n,\mathbb{C})$. Write (τ,W_τ) for this $\mathrm{GL}(n,\mathbb{C})$-representation (suppressing the dependence of τ on λ) and, following [**8**], define a map $I_\lambda:\mathrm{M}_{n,k}\to\mathrm{Hom}(W_\tau,V_\lambda)$ by

$$(I_\lambda(x))f := f(x),$$

where $x\in\mathrm{M}_{n,k}$, $f\in W_\tau=\mathfrak{H}(\mathrm{M}_{n,k},V_\lambda)$.

6.3. Fix an inner product $(\cdot,\cdot)_{W_\tau}$ on W_τ so that the adjoint of the $\mathrm{GL}(n,\mathbb{C})$-action is given by the conjugate transpose and fix an inner product $(\cdot,\cdot)_{V_\lambda}$ on V_λ that is $\mathrm{O}(k)$-invariant. Given $I_\lambda(x)\in\mathrm{Hom}(W_\tau,V_\lambda)$, we define $I_\lambda^*(x)\in\mathrm{Hom}(V_\lambda,W_\tau)$ by the condition

$$(I_\lambda(x)w,v)_{V_\lambda}=(w,I_\lambda^*(x)v)_{W_\tau}$$

for any $w\in W_\tau$, $v\in V_\lambda$. This gives a map

$$I_\lambda^*:\mathrm{M}_{n,k}\to\mathrm{Hom}(V_\lambda,W_\tau)$$

by $x\mapsto I_\lambda^*(x)$.

6.4. Holomorphically extend the character $\det^{-k/2}$ of MA to a character of $(MA)_\mathbb{C}$, isomorphic to the double cover $\widetilde{\mathrm{GL}}(n,\mathbb{C})$. Writing

$$\boldsymbol{c}=\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n\\ iI_n & I_n\end{pmatrix}$$

for the Cayley transform in $\mathrm{Sp}(n,\mathbb{C})$, pull back and restrict the character $\det^{-k/2}$ of $(MA)_\mathbb{C}$ to a character of the usual maximal compact subgroup K of G, isomorphic to a double cover $\widetilde{\mathrm{U}}(n)$, by defining $\det^{-k/2} : K \to \mathbb{C}$ as $\det^{-k/2}(k) := \det^{-k/2}(\boldsymbol{c}^{-1} k \boldsymbol{c})$. Similarly using the Cayley transform, pull back the representation of $\mathrm{GL}(n, \mathbb{C})$ on W_τ to a representation of the maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$, isomorphic to $\mathrm{U}(n)$, which may then be lifted to a nongenuine representation of K via the projection map. In this way, we may view $W_\tau \otimes \det^{-k/2}$ as a representation of K. Write $\mathcal{O}(W_\tau \otimes \det^{-k/2})$ for the space of holomorphic sections of $G \times_K (W_\tau \otimes \det^{-k/2})$ with the usual G-action and write $\mathcal{O}(\Omega_n, W_\tau)$ for the space of holomorphic W_τ-valued functions on Ω_n.

6.5. Kashiwara and Vergne showed that $\mathcal{O}(W_\tau \otimes \det^{-k/2})$ is equivalent to $\mathcal{O}(\Omega_n, W_\tau)$ as a G-module ([**8**, Prop. 3.3]), where $(g, \varepsilon) \in G$ acts on $f \in \mathcal{O}(\Omega_n, W_\tau)$ by

$$(g \cdot f)(z) = \det\nolimits_1^{-k/2}(a - zc)\, \tau(a - zc) \cdot f((a - zc)^{-1}(zd - b))$$

where we view $a - zc \in \mathrm{GL}(n, \mathbb{C})$ (for use in τ) and as shorthand for $l_{a-zc,\varepsilon'(z)}$ (for use in $\det^{-1/2}$) where $(g, \varepsilon)^{-1} = (g^{-1}, \varepsilon')$. They also showed that the map $\mathrm{KV}_\lambda : \mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) \to \mathcal{O}(\Omega_n, W_\tau)$, given by

$$(\mathrm{KV}_\lambda\, f)(z) := \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(x^T z x)} I_\lambda^*(x) f(x)\, dx$$

is G-intertwining and injective.

6.6. Notice that if elements of $\mathrm{Hom}(W_\tau, V_\lambda)$ and $\mathrm{Hom}(V_\lambda, W_\tau)$ are identified with matrices by a choice of bases for W_τ and V_λ, then, by construction, both $I_\lambda(x)$ and $I_\lambda^*(x)$ have polynomial entries in $\mathfrak{H}(\mathrm{M}_{n,k})$, the space of $\mathbb{C}$-valued harmonic polynomials on $\mathrm{M}_{n,k}$. Therefore, we may define

$$\nabla_\lambda : \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda) \to \mathcal{C}^\infty(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, W_\tau)$$

by

$$\nabla_\lambda := I_\lambda^*(-i\partial_x).$$

Here we again recall that we identify complex valued polynomial functions on $\mathrm{M}_{n,k}$ with complex valued polynomial functions on $\mathrm{M}_{n,k}(\mathbb{C})$. Thus

$$\nabla_\lambda|_{x=0}\, e^{i\operatorname{tr}(x\xi^T)} \psi(\xi) = I_\lambda^*(\xi)\psi(\xi).$$

6.7. Lift τ by the projection map to a representation of MA by $\tau(l_{A,c}) := \tau(A)$ and then extend the action trivially to $\overline{N}$. We define

$$\mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2}) := \{\psi \in \mathcal{C}^\infty(\mathrm{Sym}_n, W_\tau) \mid \exists \phi \in \mathrm{Ind}_{MA\overline{N}}^G(\tau \otimes \chi) \text{ with } \psi = \iota^*\phi\}$$

where here $\iota(t) := (\left(\begin{smallmatrix} I_n & t \\ 0 & I_n \end{smallmatrix}\right), Z \mapsto 1)$ and we identify $W_\tau \otimes \det^{-k/2}$ with W_τ as vector spaces. It is easy to see from the proof of Theorem 3.1 that the action of $(g, \varepsilon) \in G$ on $\psi \in \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2})$ is given by

$$((g, \varepsilon) \cdot \psi)(t) = \det^{-k/2}(A - tC)\, \tau(a - tc) \cdot \psi((A - tC)^{-1}(tD - B))$$

when $\det(A - tC) \neq 0$.

6.8. We will see below that $\nabla_\lambda|_{x=0} : \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} \hookrightarrow \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2})$. We also define $\mathrm{BV} : \mathcal{O}(\Omega_n, W_\tau) \to \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2})$ by $(\mathrm{BV}\, f)(t) = \lim_{s\to 0^+} f(t+is)$, $(t, s \in \mathrm{Sym}_n,\ s >> 0)$. This will certainly be well-defined on the image of $\mathcal{S}(\mathrm{M}_{n,k}, V_\lambda)$ under KV_λ which will suffice for our purposes.

THEOREM 6.1. *For $\lambda \in \Sigma \subseteq \widehat{\mathrm{O}}(k)$, there is a commutative diagram of G-maps*

$$\begin{array}{ccc}
\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}} & \xhookrightarrow{\nabla_\lambda|_{x=0}} & \mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2}) \\
{\scriptstyle \mathscr{F} \circ \mathrm{ev}_{t=0}} \Big\downarrow & & \Big\uparrow {\scriptstyle \mathrm{BV}} \\
\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda) & \xhookrightarrow[\mathrm{KV}_\lambda]{} & \mathcal{O}(\Omega_n, W_\tau)
\end{array}$$

where $\mathscr{F} \circ \mathrm{ev}_{t=0}$ is an isomorphism on K-finite vectors and KV_λ and $\nabla_\lambda|_{x=0}$ are injective.

PROOF. We have already seen that $\mathscr{F} \circ \mathrm{ev}_{t=0}$ is G-intertwining in Theorem 5.2 and injective by Equation 5.3. It is shown in [**8**] that $\mathrm{KV}_{\lambda,\sigma}$ is injective and G-intertwining and that BV is injective. Since it is obvious that BV is also a G-map, it only remains to show that the diagram is commutative. For this we simply compute. Let $\psi \in \mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$. Then

$$((\mathrm{KV}_\lambda \circ \mathscr{F} \circ \mathrm{ev}_{t=0})\psi)(z) = \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(\xi^T z \xi)} I_\lambda^*(\xi) \widehat{\psi}(\xi, 0)\, d\xi$$

so

$$((\mathrm{BV} \circ \mathrm{KV}_\lambda \circ \mathscr{F} \circ \mathrm{ev}_{t=0})\psi)(t) = \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(\xi^T t \xi)} I_\lambda^*(\xi) \widehat{\psi}(\xi, 0)\, d\xi.$$

On the other hand, by Equation 5.3,

$$\psi(x,t) = \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} \widehat{\psi}(\xi, 0) e^{\frac{i}{2}\operatorname{tr}(\xi^T t \xi)} e^{i \operatorname{tr}(x\xi^T)}\, d\xi$$

so the proof is finished by using Equation 6.6 to calculate

$$\begin{aligned}
(\nabla_\lambda|_{x=0}\psi)(t) &= \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(\xi^T t \xi)} I_\lambda^*(-i\partial_x)|_{x=0} e^{i\operatorname{tr}(x\xi^T)} \widehat{\psi}(\xi, 0)\, d\xi \\
&= \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\operatorname{tr}(\xi^T t \xi)} I_\lambda^*(\xi) \widehat{\psi}(\xi, 0)\, d\xi.
\end{aligned}$$

□

6.9. It is shown in [**8**] that the highest weight vector in $\mathcal{L}^2(\mathrm{M}_{n,k}, V_\lambda)$ is the function

$$e^{-\frac{1}{2}\operatorname{tr}(x^T x)} f_\tau(x)$$

where $f_\tau \in W_\tau$ is the highest weight vector of W_τ. It is also shown in [**8**] that the image of $e^{-\frac{1}{2}\operatorname{tr}(x^T x)} f_\tau(x)$ in $\mathcal{O}(\Omega_n, W_\tau)$ under KV_λ is the function

$$\det(I_n - iz)^{-\frac{k}{2}}\, \tau(I_n - iz) \cdot f_\tau$$

where $I_n - iz \in \mathrm{GL}(n, \mathbb{C})$ acts via τ on $f_\tau \in W_\tau$. It is obvious that the image of this function in $\mathcal{E}(\mathrm{Sym}_n, W_\tau \otimes \det^{-k/2})$ under BV is the function

$$\det(I_n - it)^{-\frac{k}{2}}\, \tau(I_n - it) \cdot f_\tau.$$

6.10. Finally, for $f \in W_\tau$, use [**8**] 4.5 to calculate that

$$\begin{aligned}((\mathscr{F} \circ \mathrm{ev}_{t=0})(\Psi_f))(\xi) &= \frac{1}{(2\pi)^{nk}} \widehat{\Psi}_f(\xi, 0) \\ &= \frac{1}{(2\pi)^{nk}} \int_{\mathrm{M}_{n,k}} f(x) e^{-\frac{1}{2}\operatorname{tr}(x^T x)} e^{-i \operatorname{tr}(x^T \xi)}\, dx \\ &= (2\pi)^{-\frac{nk}{2}} e^{-\frac{1}{2}\operatorname{tr}(\xi^T \xi)} f(-i\xi).\end{aligned}$$

Since elements of W_τ are homogeneous, it follows that, up to a scalar, the function in $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n, V_\lambda)_{\mathrm{pre}}$ corresponding to $e^{-\frac{1}{2}\operatorname{tr}(x^T x)} f_\tau(x)$ is

$$\det(I_n - it)^{-k/2} e^{-\frac{1}{2}\operatorname{tr}(x^T (I_n - it)^{-1} x)}\, f_\tau((I_n - it)^{-1} x).$$

7. Lowest Weights

7.1. We remark that if one instead begins with the character $\overline{\chi} : \overline{P} \to \mathbb{C}$ given by

$$\overline{\chi}(ma\overline{n}, (\left(\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right), s)) := \overline{\det^{-k/2}(ma)}\, e^{-\frac{is}{2}},$$

constructs $\mathrm{Ind}_{\overline{P}}^{G \ltimes H} \overline{\chi}$ and its noncompact picture, $\mathcal{E}_{\overline{\chi}}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$, as usual, and then looks at $\mathcal{H}_{\overline{\chi}}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ defined as the space of all solutions to

$$i\partial_t \psi(x, t) = -\Delta \psi(x, t)$$

with $\psi \in \mathcal{E}_{\overline{\chi}}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)$ such that the initial condition $\psi|_{t=0} \in \mathscr{S}(\mathrm{M}_{n,k})$, it is possible to show $\mathcal{H}_{\overline{\chi}}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$ is a pre-Hilbert space and is invariant under $(G \times \mathrm{O}(k)) \ltimes H$ that is dual to $\mathcal{H}(\mathrm{M}_{n,k} \times \mathrm{Sym}_n)_{\mathrm{pre}}$. By extending to antiholomorphic functions, there is an analogous result to Theorem 6.1 except with lowest weight representations.

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