CONTEMPORARY MATHEMATICS

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Topology and Quantum Theory in Interaction

NSF-CBMS Regional Conference
in the Mathematical Sciences
Topological and Geometric Methods in QFT
July 31–August 4, 2017
Montana State University, Bozeman, Montana

David Ayala
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Preface

In early August 2017, two of us (Ayala and Grady) organized a CBMS conference on *Topological and Geometric Methods in Quantum Field Theory* at Montana State University (MSU) in Bozeman. The third editor of this volume (Freed) gave a series of ten lectures. Supplementary lectures by several other mathematicians and physicists were also given. Many students and postdocs attended, as did several more senior mathematicians. The beautiful natural setting and relaxed atmosphere were perfect catalysts for interactions among the participants.

This volume brings together texts based on some of the supplementary lectures and related contributed manuscripts. They give a window into some aspects of the vigorous interaction between topology and physics.

We thank Beth Burroughs, Jane Crawford, and Katie Sutich in the Department of Mathematical Sciences at MSU for massive assistance during the planning and execution of the CBMS conference. We also thank the College of Letters and Science and the VP of Research at Montana State University for support. Finally, we are grateful to Christine M. Thivierge, Sergei Gelfand, and the AMS Editorial Committee for encouraging us to publish this volume.

David Ayala
Dan Freed
Ryan Grady

May 2018

Introduction

Quantum theory was developed at the beginning of the 20th century to explain microscopic phenomena in physics, which defied explanation using classical theories. The quantum mechanics of particles inspired the invention of new mathematics, in particular, the rapid development of operator theory by von Neumann in the late 1920s. This provided a solid foundation not only for quantum mechanics but also for many topics in mathematics unrelated to physics. At about that time physicists also initiated the quantum mechanics of fields, in particular, the electromagnetic field, whose classical behavior is governed by Maxwell's equations. From the beginning this subject was plagued by foundational problems. Computational progress came a few decades later in the hands of Feynman, Schwinger, Tomonaga, and others: infinities were tamed and physical predictions—some to spectacular accuracy—were theoretically derived and later experimentally verified. In the 1960s quantum field theory was extended from the electromagnetic field to fields that carry the weak and strong forces. The resulting Standard Model of Glashow, Salam, and Weinberg is the gold standard for fundamental theories of nature with experimental confirmation. Despite these spectacular successes in physics, and despite substantial mathematical work, there is no solid mathematical foundation for quantum field theory analogous to von Neumann's operator theory for quantum mechanics. Over the past several decades mathematical engagement with quantum field theory has turned to the geometric side, which has led to a fruitful interaction among several branches of mathematics and theoretical physics. This volume is part of that enterprise.

In mechanics there is a duality between *states* and *operators*: they pair to give a probability measure on the real line. Approaches to quantum theory can be crudely divided into *Schrödinger* and *Heisenberg* types accordingly as to whether they emphasize states or operators. Thus, in the Schrödinger picture of quantum mechanics states evolve in time, whereas in the Heisenberg picture the operators evolve. There are two mathematical axiom systems for quantum field theory in flat Minkowski spacetime, both developed in the 1960s. We might categorize the Wightman axiom system as Schrödinger type and the Haag–Kastler axiom system as Heisenberg type. Both emphasize analytic aspects of quantum fields and observables, and they were the basis for many further developments and constructions. While this mathematics was advancing, in the 1970s and 1980s physicists pushed in new directions, exploring more geometric theories (σ-models and Yang–Mills theories, among others). Then topological features with important physical ramifications were identified and studied. This work attracted the attention of a few geometers, in particular, Isadore Singer, who foresaw the broad impact that quantum field theory would have on mathematics. In the 1980s string theory, and the closely related two-dimensional conformal field theories, came to the fore. Mathematicians from other fields, such as representation theory, began to engage with this physics as well. In that context Graeme Segal proposed a Schrödinger-type axiom system for quantum field theory, which emphasizes its geometric aspects. Many variations and extensions have been explored by mathematicians since, especially for topological field theories. Michael Atiyah pointed out the structural connection in the topological case with classical bordism theory, an observation with profound consequences. More recently, a mathematical Heisenberg-type axiom system has been developed by Kevin Costello and Owen Gwilliam. It too originates in two-dimensional conformal field theory—the chiral algebras of Beilinson–Drinfeld—and the whole concept is a geometric version of the Haag–Kastler approach to quantum field theory. Examples of factorization algebras are based on Costello's approach to perturbative quantum field theory.

These modern geometric formulations of basic structures in quantum field theory have impacted several parts of mathematics: algebraic and symplectic geometry, low-dimensional topology, geometric representation theory, category theory, etc. They are not definitive definitions; there are constant revisions, variations, refinements, etc. Much remains to be done, and especially the dynamical aspects of quantum field theory remain out of reach. Still, these numerous mathematical ramifications make clear that quantum field theory, once properly formalized, will be an important mathematical structure.

In a different direction one can ask to test these axiom systems against *physics*. Can they be used to rederive known facts about quantum theories or, better yet, be the framework in which to solve open problems? The main lecture series at the CBMS conference in Bozeman[1] recounts a solution to a classification problem in condensed matter theory, which is ultimately based on Segal's field theory axioms. The first three papers in this volume amplify various aspects of these lectures. Another paper in this volume develops some category theory, which lies behind the cobordism hypothesis, the major structure theorem for topological field theories. The final three papers are related to Costello's approach to perturbative quantum field theory and the Costello–Gwilliam factorization algebras. Two of these papers use this framework to recover fundamental results about some physical theories: two-dimensional σ-models and the bosonic string. Perhaps it is surprising that such sparse axiom systems encode enough structure to prove important results in physics. These successes can be taken as encouragement that the axiom systems are at least on the right track toward articulating what a quantum field theory is.

With the broad context understood we now introduce each of the papers with more specificity. David Morrison starts us off by recounting crucial episodes in the interaction among geometry, topology, and gauge theory. As in his lecture at the CBMS conference, Morrison uses these episodes to frame general observations on the relationship between mathematics and physics over a 65-year period. On the journey from Dirac quantization to Seiberg–Witten equations a geometer meets many old friends: fiber bundles, Chern classes, algebraic bundles, index theory, nonlinear elliptic equations, and Chern–Simons invariants. The applications to physics trigger a backreaction in which physical theories profoundly influence mathematics. The surprises generated by this fertile mix continue unabated.

[1] These lectures will be published in a separate volume in the American Mathematical Society CBMS Regional Conference Series in Mathematics.

The next three papers in the volume are motivated by work of Freed–Hopkins, which we now summarize briefly. Just as in geometry one constructs moduli spaces of geometric objects with fixed discrete parameters, so one can contemplate moduli spaces of quantum systems of a fixed type. After removing a locus of "singular" systems (phase transitions, gapless theories) one defines path components to be *phases*: quantum systems connected by a path should exhibit the same qualitative features at low energies. This is a very general picture applicable to quantum field theories, string theories, as well as to quantum systems defined discretely on a lattice. It is this latter type that has recently been the subject of intensive research in condensed matter physics. Physicists move fluidly between discrete lattice models and continuous field theories, and for the particular case of "invertible" systems the mathematical understanding of the field theory side is effective in producing a complete classification. The invertibility moves the problem to stable homotopy theory, and the final result is expressed as the dual to a Thom bordism group, which depends on the parameters of dimension and symmetry type. This formula tests perfectly against classifications derived using physics arguments, often based on lattice models.

Ingmar Saberi's manuscript, loosely based on a lecture of Max Metlitski at the CBMS conference, leads us through basic insights and examples relevant to the classification of phases. From the beginning he emphasizes that lattice systems and field theories are two approximate models of an underlying physical reality, neither expected to be exact. Saberi covers basics of quantum mechanics and field theory before turning to phases of matter. Throughout he illustrates with detailed examples, such as the Heisenberg spin chain and Kitaev's toric code. He includes a thorough discussion of the central problem—classification of phases—which gives perspective and grounding to the more abstract discussions elsewhere.

Arun Debray and Sam Gunningham develop one example in detail, using Gunningham's lecture at the CBMS conference as a starting point. The spacetime dimension in question is 2 and the Wick-rotated symmetry group is Pin_2^-, which is to say the theories of interest include fermionic states and exhibit time-reversal symmetry. The paper begins with field theory, which as stated above leads to bordism theory, and so there is a review of basics of spectra and the Pontrjagin–Thom link to bordism. The relevant invertible field theories have partition functions, which are bordism invariants; for the generating theory the partition function is the Arf–Brown invariant of a closed Pin^- surface for which the authors recount several constructions. The invertible *Arf–Brown topological field theory* with that partition function is realized with codomain the Picard 2-groupoid of complex central simple superalgebras. The last section of the paper switches gears to the lattice perspective. There is a geometric account of the Majorana chain—a lattice model—which in a sense made precise and proved has a precise relationship to the Arf–Brown theory at low energies.

Computations of Thom bordism groups for general symmetry types use well-developed techniques in stable homotopy theory. This is the basis for computations from field theory, which agree with classification results based on lattice models. Agnès Beaudry and Jonathan Campbell give a thorough introduction to these homotopy theoretic techniques, and they use them to compute pertinent examples. This account, based on Beaudry's lecture at the CBMS conference, will be valuable to any student of the Adams spectral sequence, whether for this application to physics or for other purposes. They too begin with a review of spectra in stable homotopy theory. The mod 2 Steenrod algebra plays a central role since the mod 2 cohomology over a space is an $\mathcal{A}$-module. There is a distinguished subalgebra $\mathcal{A}_1$ that suffices for the computations done here. The $\mathcal{A}_1$-module structure is encoded concisely in pictorial form, as is illustrated in relevant examples. The next step is some homological algebra over $\mathcal{A}_1$, which is used as input into the Adams spectral sequence. The entire process is meticulously explained, illustrated, and applied.

The next paper in the volume, by David Ayala and John Francis, is a contribution to higher category, part of their ambitious program centered around topological field theories and the cobordism hypothesis. This technical article is a first step in an anticipated fantastic duality between certain topological quantum field theories: between certain σ-models (with algebraic target) and state-sum theories (factorization homology). A perturbative instance of this was established in the authors' earlier work, which intertwined Koszul duality for $\mathcal{E}_n$-algebras and Poincaré duality for n-manifolds. The anticipated nonperturbative situation then concerns Koszul duality for higher categories. The input for this deformation theory of higher categories, instantiated as Koszul duality, is a *flagged n-category*, which is introduced and characterized in this work. A notable example of such is the bordism n-category.

The remaining three papers pivot from Schrödinger to Heisenberg, that is, from Segal's axioms based on bordisms to the Costello–Gwilliam axioms for factorization algebras and, more prominently, the Costello approach to perturbation theory. The first, by Owen Gwilliam and Theo Johnson-Freyd, is a novel take on the origin of Feynman diagrams, which are a fundamental tool in traditional perturbative quantum field theory to compute asymptotic expansions of oscillatory integrals. In Costello's approach it is rather Batalin–Vilkovisky (BV) algebras that are the more fundamental starting point. Gwilliam and Johnson-Freyd bridge the gap by first defining a particular example of a BV algebra in the form of an explicit chain complex. They demonstrate that the computation of its lowest degree homology can be done using Feynman diagrams.

The final two papers resume the theme of applying mathematical axiom systems for quantum theory back to physics. The first, by Ryan Grady and Brian Williams, proves the formula for the β-function of the two-dimensional σ-model, first derived in Friedan's thesis. Up to a constant it is the Ricci curvature, which means that the renormalization group flow is closely related to Ricci flow. (Indeed, that link is one inspiration for Perelman's work.) To begin, Grady and Williams formulate an effective BV theory that describes the low-energy regime of the σ-model. Via an explicit cohomological computation, they show that the obstruction to quantization vanishes at one-loop, i.e., a solution to the quantum master equation is constructed modulo $\hbar^2$. The authors then briefly recall the essentials of the β-function in Costello's formalism. Several illustrative computations in scalar field theory are given before the article finishes with the identification of the β-function and the Ricci tensor in the two-dimensional σ-model.

The last paper in this volume, by Owen Gwilliam and Brian Williams, takes up the bosonic string, the simplest string theory. A distinguishing—and disqualifying—feature is that it is only consistent in spacetime dimension 26. This is one of the results they recover in this paper using Costello's perturbation theory. Their first task is to formulate the bosonic string in the BV formalism. The natural home for the construction is derived algebraic geometry, as is explained. The authors also check that it is consistent with the Polyakov action in the traditional approach. After investigating deformations, they turn to quantization on a disk. The quantum master equation leads to the dimension restriction: a BV quantization only exists in that dimension. With local quantization in hand, there are local algebras of operators in the form of factorization algebras. The cosheaf of algebras of observables is proved to be locally constant, so essentially topological. This recovers results of Getzler, Lian–Zuckerman, and others on Gerstenhaber algebras in bosonic string theory. The paper concludes with global aspects. In particular, the usual anomaly of the bosonic string arises in the Costello–Gwilliam framework.

In aggregate these papers provide an entrée to modern geometric formulations of quantum field theory and their applications. As befits a mathematical theory in its early stages, there are many heuristic explanations and even more examples treated in detail. We invite the active reader to use them as a springboard to further explorations.

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In recent decades, there has been a movement to axiomatize quantum field theory into a mathematical structure. In a different direction, one can ask to test these axiom systems against physics. Can they be used to rederive known facts about quantum theories or, better yet, be the framework in which to solve open problems? Recently, Freed and Hopkins have provided a solution to a classification problem in condensed matter theory, which is ultimately based on the field theory axioms of Graeme Segal.

Papers contained in this volume amplify various aspects of the Freed–Hopkins program, develop some category theory, which lies behind the cobordism hypothesis, the major structure theorem for topological field theories, and relate to Costello's approach to perturbative quantum field theory. Two papers on the latter use this framework to recover fundamental results about some physical theories: two-dimensional sigma-models and the bosonic string. Perhaps it is surprising that such sparse axiom systems encode enough structure to prove important results in physics. These successes can be taken as encouragement that the axiom systems are at least on the right track toward articulating what a quantum field theory is.

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