

The Mass Transference Principle: Ten years on

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ABSTRACT. In this article we discuss the Mass Transference Principle due to Beresnevich and Velani and survey several generalisations and variants, both deterministic and random. Using a Hausdorff measure analogue of the inhomogeneous Khintchine–Groshev Theorem, proved recently via an extension of the Mass Transference Principle to systems of linear forms, we give an alternative proof of (most cases of) a general inhomogeneous Jarník–Besicovitch Theorem which was originally proved by Levesley. We additionally show that without monotonicity Levesley’s theorem no longer holds in general. Thereafter, we discuss recent advances by Wang, Wu and Xu towards mass transference principles where one transitions from lim sup sets defined by balls to lim sup sets defined by rectangles (rather than from “balls to balls” as is the case in the original Mass Transference Principle). Furthermore, we consider mass transference principles for transitioning from rectangles to rectangles and extend known results using a slicing technique. We end this article with a brief survey of random analogues of the Mass Transference Principle.

1. Introduction

Since its discovery by Beresnevich and Velani in 2006, the Mass Transference Principle has become an important tool in metric number theory. Originally motivated by the desire for a Hausdorff measure version of the Duffin–Schaeffer conjecture, the Mass Transference Principle allows us to transfer a Lebesgue measure statement for a lim sup set defined by a sequence of balls in \mathbb{R}^k to a Hausdorff measure statement for a related lim sup set. Over the past few years a number of generalisations have been proved and more general settings have been considered.¹ In this article we survey several of these recent developments and consider some of their applications, mostly in the field of metric number theory.

1.1. Notation and Basic Definitions. Throughout, by a *dimension function* we mean a continuous, non-decreasing function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f(r) \rightarrow 0$ as $r \rightarrow 0$. Recall that $\mathbb{R}^+ = [0, \infty)$. If there exists a constant $\lambda > 1$ such that for $x > 0$ we have $f(2x) \leq \lambda f(x)$ then we say that f is *doubling*.

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¹Since the initial writing of this paper, further generalisations of the Mass Transference Principle have been established in [1] and [48].

DEFINITION 1.1. Let $F \subseteq \mathbb{R}^k$ and let $\delta > 0$. The δ -Hausdorff pre-measure of F with respect to the dimension function f , denoted $\mathcal{H}_\delta^f(F)$, is given by

$$\mathcal{H}_\delta^f(F) = \inf \left\{ \sum_{i=1}^{\infty} f(\text{diam}(U_i)) : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \text{diam}(U_i) \leq \delta \text{ for all } i \right\},$$

where the infimum is taken over all countable collections $\{U_i\}$ of open sets. The Hausdorff content \mathcal{H}_∞^f with respect to f is

$$\mathcal{H}_\infty^f(F) = \inf_{\delta > 0} \mathcal{H}_\delta^f(F).$$

The Hausdorff measure \mathcal{H}^f with respect to f is defined by

$$\mathcal{H}^f(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^f(F).$$

A simple consequence of the definition of \mathcal{H}^f is the following useful fact (see, for example, [24]).

LEMMA 1.2. If f and g are two dimension functions such that the ratio $f(r)/g(r) \rightarrow 0$ as $r \rightarrow 0$, then $\mathcal{H}^f(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$.

We note that for all dimension functions f , and all bounded subsets $F \subset \mathbb{R}^k$, the Hausdorff content satisfies $\mathcal{H}_\infty^f(F) \leq f(\text{diam}(F))$ and for all $\delta > 0$

$$\mathcal{H}_\infty^f(F) \leq \mathcal{H}_\delta^f(F) < \infty.$$

We also observe that, for a given f , the δ -Hausdorff pre-measure $\mathcal{H}_\delta^f(F)$ is non-decreasing as $\delta \rightarrow 0$. So, using monotone convergence, the limit $\lim_{\delta \rightarrow 0} \mathcal{H}_\delta^f(F)$ exists but may be infinite.

Often we are interested in Hausdorff dimension and the classical Hausdorff s -measure. The Hausdorff s -measure, which we will usually denote by \mathcal{H}^s , can be obtained by letting $f(r) = r^s$. The Hausdorff dimension of a set F is then defined as follows.

DEFINITION 1.3. Let $F \subseteq \mathbb{R}^k$. The Hausdorff dimension of F is

$$\dim_{\text{H}} F = \inf \{s > 0 : \mathcal{H}^s(F) = 0\}.$$

One interesting property of the Hausdorff measure is that for subsets of \mathbb{R}^k , \mathcal{H}^k is comparable to the k -dimensional Lebesgue measure. For a set $X \subset \mathbb{R}^k$ we denote the k -dimensional Lebesgue measure by $|X|$. Lebesgue null sets, *i.e.* sets X with $|X| = 0$, can still have intricate geometric structure and in many cases we are able to appeal to Hausdorff dimension to discriminate between their respective ‘sizes’. For further information regarding Hausdorff measures and dimension we refer the reader to [24, 51, 57]. Finally, we recall the notion of a lim sup set.

DEFINITION 1.4. Let $(A_i)_{i \in \mathbb{N}}$ be a collection of subsets of a set Y . Then

$$\limsup_i A_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$

Equivalently,

$$\limsup_i A_i = \{x \in Y : x \in A_i \text{ for infinitely many } i \in \mathbb{N}\}.$$

2. The Mass Transference Principle

The main object of study in this article is the *Mass Transference Principle* and its generalisations and variants. The original Mass Transference Principle was developed by Beresnevich and Velani in [8] and was motivated by a conjecture of Duffin and Schaeffer.

Given an approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, for $k \in \mathbb{N}$, let

$$\mathcal{A}_k(\psi) := \left\{ \mathbf{x} \in \mathbb{I}^k : \max_{1 \leq i \leq k} \left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N} \right\},$$

where $\mathbf{p} = (p_1, p_2, \dots, p_k)$, be the *simultaneously ψ -approximable points* in the unit cube, $\mathbb{I}^k = [0, 1]^k$, and consider the following classical theorem by Khintchine [46].

THEOREM 2.1 (Khintchine [46]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximating function. Then*

$$|\mathcal{A}_1(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Khintchine also extended this result to the simultaneously ψ -approximable points in higher dimensions.

THEOREM 2.2 (Khintchine [47]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximating function. Then*

$$|\mathcal{A}_k(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q)^k < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q)^k = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

In the one-dimensional case Duffin and Schaeffer [18] constructed a counter-example showing that the full Lebesgue measure statement can fail for non-monotonic ψ . They also posed a conjecture on what should be true when considering general (not necessarily monotonic) approximating functions.

Given an approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ and an integer $k \geq 1$, let us denote by $\mathcal{A}'_k(\psi)$ the set of points $\mathbf{x} \in \mathbb{I}^k$ such that

$$(2.1) \quad \max_{1 \leq i \leq k} \left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q}$$

for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$ with $\gcd(\mathbf{p}, q) := \gcd(p_1, \dots, p_k, q) = 1$.

CONJECTURE 2.3 (Duffin–Schaeffer Conjecture [18]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be any approximating function and denote by $\phi(q)$ the Euler function. If*

$$\sum_{q=1}^{\infty} \phi(q) \frac{\psi(q)}{q} = \infty \quad \text{then} \quad |\mathcal{A}'_1(\psi)| = 1.$$

For $k \geq 2$ the analogous conjecture was formulated by Sprindžuk [61, Chapter 1, Section 8]. The conjecture depends again on slightly different coprimality conditions. Therefore, for any approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, let us denote by $\mathcal{A}''_k(\psi)$ the set of points $\mathbf{x} \in \mathbb{I}^k$ for which the inequality (2.1) is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$ which also have $\gcd(p_i, q) = 1$ for all $1 \leq i \leq k$.

CONJECTURE 2.4 (Higher-Dimensional Duffin–Schaeffer Conjecture [61]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be any approximating function and denote by $\phi(q)$ the Euler function. If*

$$\sum_{q=1}^{\infty} \phi(q)^k \frac{\psi(q)^k}{q^k} = \infty \quad \text{then} \quad |\mathcal{A}_k''(\psi)| = 1.$$

For $k > 1$ Sprindžuk’s conjecture (Conjecture 2.4) was proved in the affirmative by Pollington and Vaughan [56].

Finding a general Hausdorff measure analogue of the Duffin–Schaeffer conjecture inspired the Mass Transference Principle that we will now state. Let f be a dimension function and let $\mathcal{H}^f(\cdot)$ denote Hausdorff f -measure. Given a ball $B = B(x, r)$ in \mathbb{R}^k of radius r centred at x let $B^f = B(x, f(r)^{\frac{1}{k}})$. We write B^s instead of B^f if $f(x) = x^s$ for some $s > 0$. In particular, we have $B^k = B$.

THEOREM 2.5 (**Mass Transference Principle**, Beresnevich – Velani [8]). *Let $\{B_i = B(x_i, r_i)\}_{i \in \mathbb{N}}$ be a sequence of balls in \mathbb{R}^k with $r_i \rightarrow 0$ as $i \rightarrow \infty$. Let f be a dimension function such that $x^{-k}f(x)$ is monotonic and let Ω be a ball in \mathbb{R}^k . Suppose that, for any ball B in Ω ,*

$$\mathcal{H}^k(B \cap \limsup_{i \rightarrow \infty} B_i^f) = \mathcal{H}^k(B).$$

Then, for any ball B in Ω ,

$$\mathcal{H}^f(B \cap \limsup_{i \rightarrow \infty} B_i^k) = \mathcal{H}^f(B).$$

REMARK. *Strictly speaking, the statement of the Mass Transference Principle given initially by Beresnevich and Velani, [8, Theorem 2], corresponds to the case where Ω is taken to be \mathbb{R}^k in Theorem 2.5. The statement we have opted to give above is a consequence of [8, Theorem 2].*

The Mass Transference Principle allows us therefore to transfer a Lebesgue measure statement for a lim sup set of balls to a Hausdorff measure statement for a lim sup set of balls which are obtained by “shrinking” the original balls in a certain manner according to f . This is a remarkable result given that Lebesgue measure can be considered to be much ‘coarser’ than Hausdorff measure.

Before we continue our discussion of the Mass Transference Principle, we take this opportunity to note that some related results, which are more specialised but still similar in spirit to the Mass Transference Principle, were proved earlier by Jaffard in [36, 37]. Jaffard’s results allow for the transference of full Lebesgue measure statements for a certain type of lim sup set in $[0, 1]^k$ to Hausdorff f -measure statements for related lim sup sets where the dimension function f takes a specific form. A one-dimensional statement is given in [36] and is extended to higher dimensions in [37]. More precisely, suppose we are given a sequence of points $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in $[0, 1]^k$ and a sequence of positive real numbers $(r_n)_{n \in \mathbb{N}}$. Given $a > 0$, define

$$E_a := \limsup_{n \rightarrow \infty} B(\mathbf{x}_n, r_n^a).$$

THEOREM 2.6 (Jaffard, [37]). *For $c > 0$, let $f_c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined by $f_c(r) = (\log r)^2 r^c$. If $|E_a| = 1$, then for every $b \in \mathbb{R}$ such that $b > a$ we have*

$$\mathcal{H}^{f_{ka/b}}(E_b) > 0.$$

In particular, for $b > a$, we have $\dim_{\text{H}}(E_b) \geq \frac{ka}{b}$.

Returning to our earlier discussion, the Mass Transference Principle (Theorem 2.5) was used to show that the Duffin–Schaeffer conjecture for Lebesgue measure gives rise to an analogous statement for Hausdorff measures.

CONJECTURE 2.7 (Hausdorff Measure Duffin–Schaeffer Conjecture [8]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be any approximating function and let f be a dimension function such that $r^{-k}f(r)$ is monotonic. If*

$$\sum_{q=1}^{\infty} \phi(q)^k f\left(\frac{\psi(q)}{q}\right) = \infty \quad \text{then} \quad \mathcal{H}^f(\mathcal{A}_k''(\psi)) = \mathcal{H}^f(\mathbb{I}^k).$$

Setting $f(r) = r^k$ in the above we see that we immediately recover Conjecture 2.4. What is much more surprising is that, using the Mass Transference Principle (Theorem 2.5), Beresnevich and Velani proved that Conjecture 2.4 implies Conjecture 2.7 and hence that they are equivalent. In particular, Conjecture 2.7 holds for $k > 1$ for general approximating functions ψ and for $k = 1$ if ψ is furthermore monotonic, see [8].

Two further easy yet surprising consequences of the Mass Transference Principle, which are also mentioned in [8], are that Khintchine’s Theorem implies Jarník’s Theorem and also that Dirichlet’s Theorem implies the Jarník–Besicovitch Theorem. We shall elaborate briefly on these examples here, however for further details and proofs we refer the reader to [7, 8].

Let us consider what Khintchine’s Theorem (Theorem 2.1) can tell us when our approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ is given by $\psi(q) = q^{-\tau}$ for some $\tau > 1$. In this case we will write $\mathcal{A}(\tau)$ in place of $\mathcal{A}_1(\psi)$ and we will refer to the points in $\mathcal{A}(\tau)$ as τ -approximable points. For any $\tau > 1$ it can be seen that the sum of interest in Khintchine’s Theorem converges and so all we can infer is that $|\mathcal{A}(\tau)| = 0$ for all values of $\tau > 1$. However, in this case, we are still able to distinguish the “sizes” of these sets thanks to the Jarník–Besicovitch Theorem. Jarník and Besicovitch both independently proved the following result regarding the Hausdorff dimension of the τ -approximable points.

THEOREM 2.8 (Jarník [38], Besicovitch [12]). *Let $\tau > 1$. Then*

$$\dim_{\text{H}}(\mathcal{A}(\tau)) = \frac{2}{\tau + 1}.$$

In fact it turns out that, using the Mass Transference Principle, the Jarník–Besicovitch Theorem can be extracted from Dirichlet’s theorem. Jarník later proved a much stronger statement, regarding the Hausdorff measures of more general sets of ψ -approximable points, which can be viewed as the Hausdorff measure analogue of Khintchine’s Theorem (Theorem 2.2). We state below a modern version of Jarník’s Theorem, see [5, Theorem 11] for a greater discussion of the derivation of this statement.

THEOREM 2.9 (Jarník [39]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximating function and let f be a dimension function such that $r^{-k}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(\mathcal{A}_k(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^k f\left(\frac{\psi(q)}{q}\right) < \infty, \\ \mathcal{H}^f(\mathbb{I}^k) & \text{if } \sum_{q=1}^{\infty} q^k f\left(\frac{\psi(q)}{q}\right) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Setting $\psi(q) = q^{-\tau}$ and $f(r) = r^s$ in Jarník's Theorem we recover the Jarník–Besicovitch Theorem and additionally gain knowledge of the Hausdorff measure at the critical value $s_0 = 2/(\tau + 1)$, *i.e.* $\mathcal{H}^{s_0}(\mathcal{A}(\tau)) = \infty$. Although it may at first be surprising, (the original statement of) Jarník's Theorem follows directly from (the original statement of) Khintchine's Theorem using the Mass Transference Principle. For a proof see, for example, [7, 8]. We remark here that in the original versions of Theorems 2.1, 2.2 and 2.9 various stronger monotonicity conditions were required and note that this is of some relevance when using the Mass Transference Principle to deduce Jarník's Theorem from Khintchine's Theorem. It is possible to deduce Theorem 2.9 from Theorem 2.2 via the Mass Transference Principle but an additional constraint is required on the monotonicity of ψ in this case.

Apart from these important applications in number theory, the Mass Transference Principle can be used to determine Hausdorff dimension and Hausdorff measure statements for many other constructions.

We end this section by stating the most general variant of the Mass Transference Principle in the original article of Beresnevich and Velani [8] and mentioning one of its applications. Let (X, d) be a locally compact metric space. Let g be a doubling dimension function and suppose that X is g -Ahlfors regular, *i.e.* there exist constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1 g(r) \leq \mathcal{H}^g(B(x, r)) \leq c_2 g(r)$$

for any ball $B = B(x, r)$ with centre $x \in X$ and radius $r \leq r_0$. In this case, given a ball $B = B(x, r)$ and any dimension function f we define $B^{f \cdot g} = B(x, g^{-1}f(r))$. Note that $B^{g \cdot g} = B$.

THEOREM 2.10 (Beresnevich – Velani [8]). *Let (X, d) be a locally compact metric space and let g be a doubling dimension function as above. Let $\{B_i = B(x_i, r_i)\}_{i \in \mathbb{N}}$ be a sequence of balls in X with $r_i \rightarrow 0$ as $i \rightarrow \infty$ and let f be a dimension function such that $f(x)/g(x)$ is monotonic. Suppose that, for any ball B in X ,*

$$\mathcal{H}^g(B \cap \limsup_{i \rightarrow \infty} B_i^{f \cdot g}) = \mathcal{H}^g(B).$$

Then, for any ball B in X , we have

$$\mathcal{H}^f(B \cap \limsup_{i \rightarrow \infty} B_i^{g \cdot g}) = \mathcal{H}^f(B).$$

As an example, Theorem 2.10 is applicable when X is, say, the standard middle-third Cantor set which we denote by K (*i.e.* K is the set of $x \in [0, 1]$ which have a ternary expansion containing only 0s and 2s). In fact, in this case, Levesley, Salp, and Velani [50] have used Theorem 2.10 as a tool for proving an assertion of Mahler on the existence of very well approximable numbers in the middle-third Cantor set. It is well known that

$$|K| = 0 \quad \text{and} \quad \dim_{\text{H}} K = \frac{\log 2}{\log 3}.$$

As a result of Dirichlet's Theorem, we know that for any $x \in \mathbb{R}$ there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ for which

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

If the exponent in the denominator of the right-hand side of the above can be improved (*i.e.* increased) for some $x \in \mathbb{R}$ then x is said to be *very well approximable*; that is, a real number x is said to be very well approximable if there exists some $\varepsilon > 0$ such that

$$(2.2) \quad \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}$$

for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We will denote the set of very well approximable numbers by \mathcal{W} . If, further, (2.2) is satisfied for every $\varepsilon > 0$ for some $x \in \mathbb{R}$ then x is called a *Liouville number*. We will denote by \mathcal{L} the set of all Liouville numbers.

It is known that

$$\begin{aligned} |\mathcal{W}| &= 0, \quad \dim_{\mathbb{H}}(\mathcal{W}) = 1, \\ |\mathcal{L}| &= 0, \quad \text{and } \dim_{\mathbb{H}}(\mathcal{L}) = 0. \end{aligned}$$

Regarding the intersection of \mathcal{W} with the middle-third Cantor set, Mahler is attributed with having made the following claim.

MAHLER'S ASSERTION. *There exist very well approximable numbers, other than Liouville numbers, in the middle-third Cantor set; i.e.*

$$(\mathcal{W} \setminus \mathcal{L}) \cap K \neq \emptyset.$$

REMARK. *We refer the reader to [50] for discussion of the precise origin of this claim and also for some discussion regarding why it is natural/necessary to exclude Liouville numbers from Mahler's assertion.*

Now, let $\mathcal{B} = \{3^n : n = 0, 1, 2, \dots\}$ and, given an approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, consider the set

$$\mathcal{A}_{\mathcal{B}}(\psi) := \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathcal{B} \right\}.$$

Levesley, Salp and Velani have used the general Mass Transference Principle (Theorem 2.10) as a tool for establishing the following statement regarding Hausdorff measures of the set $\mathcal{A}_{\mathcal{B}}(\psi) \cap K$ in [50].

THEOREM 2.11 (Levesley – Salp – Velani [50]). *Let f be a dimension function such that $r^{-\frac{\log 2}{\log 3}} f(r)$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}_{\mathcal{B}}(\psi) \cap K) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\frac{\log 2}{\log 3}} < \infty, \\ \mathcal{H}^f(K) & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\frac{\log 2}{\log 3}} = \infty. \end{cases}$$

As a consequence of Theorem 2.11 the following corollary may be deduced, for details of how we refer the reader to [50].

COROLLARY 2.12 (Levesley – Salp – Velani [50]). *We have*

$$\dim_{\mathbb{H}}((\mathcal{W} \setminus \mathcal{L}) \cap K) \geq \frac{\log 2}{2 \log 3}.$$

The truth of Mahler's assertion follows immediately from this corollary.

We conclude this section by noting that both the original Mass Transference Principle (Theorem 2.5) and its generalisation given by Theorem 2.10 concern

lim sup sets arising from sequences of balls. In subsequent sections we will explore what happens when this condition is relaxed. More precisely, we will consider linear forms (Section 3) and rectangles (Section 4) in the deterministic setting and arbitrary Lebesgue measurable sets in the random setting (Section 5).

Inevitably, there are various aspects of the Mass Transference Principle that are not covered in this survey. For example, we have not touched upon the fundamental connections between the Mass Transference Principle set up and the ubiquitous systems framework as developed in [6] — in short, the so-called KGB-Lemma [8, Lemma 5] is very much at the heart of both. Although a ubiquitous framework was developed in [6], we remark that the idea of a ubiquitous system was introduced earlier in [17] and was further developed in [11]. For an overview of ubiquity and some of its applications also see [6, 20] and references within. Another omission from this survey is any mention of mass transference principles in the multifractal setting — see, for example, [4, 28]. In the interest of brevity we have ultimately opted against the inclusion of such topics and chosen here to only focus on the aspects mentioned above.

3. Extension to systems of linear forms

In this section we will consider an extension of the Mass Transference Principle to systems of linear forms and mention some of the associated consequences in the theory of Diophantine approximation.

3.1. A mass transference principle for systems of linear forms. Let $k, m \geq 1$ and $l \geq 0$ be integers such that $k = m + l$. Let $\mathcal{R} = (R_n)_{n \in \mathbb{N}}$ be a family of planes in \mathbb{R}^k of common dimension l . For every $n \in \mathbb{N}$ and $\delta \geq 0$, define

$$\Delta(R_n, \delta) := \{\mathbf{x} \in \mathbb{R}^k : \text{dist}(\mathbf{x}, R_n) < \delta\},$$

where $\text{dist}(\mathbf{x}, R_n) = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in R_n\}$ and $\|\cdot\|$ is a norm on \mathbb{R}^k .

Let $\Upsilon : \mathbb{N} \rightarrow \mathbb{R}$ be a non-negative real-valued function $n \mapsto \Upsilon_n$ on \mathbb{N} such that $\Upsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Consider the lim sup set

$$\Lambda(\Upsilon) := \{\mathbf{x} \in \mathbb{R}^k : \mathbf{x} \in \Delta(R_n, \Upsilon_n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

In 2006, Beresnevich and Velani also established the following extension of the Mass Transference Principle to systems of linear forms [9].

THEOREM 3.1 (Beresnevich – Velani [9]). *Let \mathcal{R} and Υ be as given above. Let V be a linear subspace of \mathbb{R}^k such that $\dim V = m = \text{codim } \mathcal{R}$,*

- (i) $V \cap R_n \neq \emptyset$ for all $n \in \mathbb{N}$, and
- (ii) $\sup_{n \in \mathbb{N}} \text{diam}(V \cap \Delta(R_n, 1)) < \infty$.

Let f and $g : r \mapsto g(r) := r^{-l}f(r)$ be dimension functions such that $r^{-k}f(r)$ is monotonic and let Ω be a ball in \mathbb{R}^k . Suppose for any ball B in Ω that

$$\mathcal{H}^k \left(B \cap \Lambda \left(g(\Upsilon) \frac{1}{m} \right) \right) = \mathcal{H}^k(B).$$

Then, for any ball B in Ω ,

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B).$$

REMARK. *Note that when $l = 0$ in Theorem 3.1 we recover the Mass Transference Principle (Theorem 2.5).*

Conditions (i) and (ii) essentially say that V should intersect every plane and that the angle of intersection between V and each plane should be bounded away from 0. In other words, every plane R_n ought not to be parallel to V and should intersect V in precisely one place. These conditions are technical and come about as a consequence of the “slicing” technique used by Beresnevich and Velani to prove Theorem 3.1 in [9] (for a simple demonstration of the idea of “slicing” see the proofs of Propositions 4.7 and 4.8 in Section 4). It was conjectured by Beresnevich et al. [5, Conjecture E] that Theorem 3.1 should also be true without conditions (i) and (ii). Recently, this conjecture has been settled by Allen and Beresnevich in [2] by using a proof closer in strategy to that used by Beresnevich and Velani to prove the original Mass Transference Principle in [8], rather than “slicing”.

THEOREM 3.2 (Allen – Beresnevich [2]). *Let \mathcal{R} and Υ be as given above. Let f and $g : r \mapsto g(r) := r^{-l}f(r)$ be dimension functions such that $r^{-k}f(r)$ is monotonic and let Ω be a ball in \mathbb{R}^k . Suppose for any ball B in Ω that*

$$\mathcal{H}^k \left(B \cap \Lambda \left(g(\Upsilon)^{\frac{1}{m}} \right) \right) = \mathcal{H}^k(B).$$

Then, for any ball B in Ω ,

$$\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B).$$

Although Theorem 3.1 itself has some interesting consequences, see [5, 9], it seems likely that there will be a number of applications of Theorem 3.2 which are unachievable using Theorem 3.1. In particular, in Section 3.4 we record some very general statements obtained in [2] which essentially rephrase Theorem 3.2 as statements for transferring Lebesgue measure statements to Hausdorff measure statements for sets of ψ -approximable (and Ψ -approximable) points.

Before that we state some more concrete applications of Theorems 3.1 and 3.2; namely, we mention Hausdorff measure analogues of the homogeneous and inhomogeneous Khintchine–Groshev Theorems obtained in [2, 5]. We also use the Hausdorff measure analogue of the inhomogeneous Khintchine–Groshev Theorem to make some remarks on a theorem of Levesley [49].

3.2. Hausdorff measure Khintchine–Groshev statements. Throughout this section, let $n \geq 1$ and $m \geq 1$ be integers and denote by \mathbb{I}^{nm} the unit cube $[0, 1]^{nm} \subset \mathbb{R}^{nm}$.

Given a function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, let $\mathcal{A}_{n,m}(\psi)$ denote the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}| < \psi(|\mathbf{q}|)$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

Here, $|\cdot|$ denotes the supremum norm and we think of $\mathbf{x} = (x_{ij})$ as an $n \times m$ matrix and of \mathbf{p} and \mathbf{q} as row vectors. So $\mathbf{q}\mathbf{x}$ represents a point in \mathbb{R}^m given by the system

$$q_1x_{1j} + \cdots + q_nx_{nj} \quad (1 \leq j \leq m)$$

of m real linear forms in n variables. We say that the points in $\mathcal{A}_{n,m}(\psi)$ are ψ -approximable. As with many sets of interest in Diophantine approximation, $\mathcal{A}_{n,m}(\psi)$ satisfies an elegant zero-one law with respect to Lebesgue measure.

THEOREM 3.3 (Khintchine–Groshev Theorem [10]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ and let $nm > 1$. Then*

$$|\mathcal{A}_{n,m}(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty. \end{cases}$$

The earliest versions of this theorem are attributed to Khintchine and Groshev [33, 46]. These were subject to extra assumptions including monotonicity of ψ . Due to the famous counter-example of Duffin and Schaeffer [18] we know that if we have $m = n = 1$ then monotonicity cannot be removed. However, when we insist that $nm > 1$ monotonicity of ψ is unnecessary. In the case that $n = 1$ or $n \geq 3$ this follows, respectively, from results due to Gallagher [31] and Schmidt [59]. In the case that $n \geq 3$ this also follows from a result of Sprindžuk [61, Chapter 1, Section 5], stated as Theorem 3.5 below. For further information we refer the reader to the detailed survey [5]. It was conjectured by Beresnevich et al. in [5, Conjecture A] that monotonicity should also be unnecessary when $n = 2$. This conjecture was finally settled by Beresnevich and Velani in [10] leaving the above modern statement of the Khintchine–Groshev Theorem, which is the best possible.

Regarding the Hausdorff measure theory, combining Theorem 3.2 with Theorem 3.3 yields the following.

THEOREM 3.4 (Hausdorff measure Khintchine–Groshev Theorem [2, 5]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be any approximating function and let $nm > 1$. Let f and $g : r \mapsto g(r) = r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty, \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

For completeness, we remark here that before the above statement appeared in [5], the Hausdorff measures and dimension of the sets $\mathcal{A}_{n,m}(\psi)$ had already been studied by a number of people. Indeed, earlier similar results, albeit subject to further constraints, had already been established. In particular, the first Hausdorff measure result in this direction was obtained by Dickinson and Velani [15] and, even before that, the first Hausdorff dimension results had already been established by Bovey and Dodson [13].

Returning to Theorem 3.4 we note that the statement in [5] additionally required ψ to be monotonic when $n = 2$. At that time it was still unproven that the Khintchine–Groshev Theorem was true without monotonicity in the case that $n = 2$. However, it was conjectured in [5] that, subject to the validity of the Khintchine–Groshev Theorem without assuming monotonicity when $n = 2$ (i.e. Theorem 3.3), it should be possible to use Theorem 3.1 to remove this final monotonicity condition also from the Hausdorff measure version of the Khintchine–Groshev theorem, giving Theorem 3.4. This conjecture has been verified in [2] where, in fact, two proofs of Theorem 3.4 are given. The first uses a combination of Theorem 3.1 and “slicing”, thus verifying the conjecture, and the second uses Theorem 3.2.

In [2], a Hausdorff measure version of the inhomogeneous analogue of the Khintchine–Groshev Theorem is also established. If we are given an approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ and a fixed $\mathbf{y} \in \mathbb{I}^m$ then we will denote by $\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p} - \mathbf{y}| < \psi(|\mathbf{q}|)$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

Supposing we are given an approximating function $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+$ which can depend on \mathbf{q} rather than just $|\mathbf{q}|$, we will let $\mathcal{A}_{n,m}(\Psi)$ be the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p}| < \Psi(\mathbf{q})$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$, we will call these Ψ -approximable points. If we are also given a fixed $\mathbf{y} \in \mathbb{I}^m$ we will denote by $\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p} - \mathbf{y}| < \Psi(\mathbf{q})$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

For $n \geq 2$ we will represent the set of primitive vectors in \mathbb{Z}^n by $P(\mathbb{Z}^n)$; that is, non-zero integer vectors $(v_1, v_2, \dots, v_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $\gcd(v_1, v_2, \dots, v_n) = 1$. Regarding the Lebesgue measure of sets of inhomogeneously Ψ -approximable points, we have the following result due to Sprindžuk [61, Chapter 1, Section 5].

THEOREM 3.5 (Sprindžuk [61]). *Let $m \geq 1$ and $n \geq 2$ be integers. Let $\Psi : \mathbb{Z}^n \rightarrow \mathbb{R}^+$ be an approximating function such that $\Psi(\mathbf{q}) = 0$ whenever $\mathbf{q} \notin P(\mathbb{Z}^n)$ and let $\mathbf{y} \in \mathbb{I}^m$ be fixed. Then,*

$$|\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)| = \begin{cases} 0 & \text{if } \sum_{\mathbf{q} \in \mathbb{Z}^n} \Psi^m(\mathbf{q}) < \infty, \\ 1 & \text{if } \sum_{\mathbf{q} \in \mathbb{Z}^n} \Psi^m(\mathbf{q}) = \infty. \end{cases}$$

Restricting the approximating function Ψ so that it only depends on $|\mathbf{q}|$ yields the following inhomogeneous analogue of the Khintchine–Groshev Theorem (Theorem 3.3).

THEOREM 3.6 (Inhomogeneous Khintchine–Groshev Theorem). *Let $m, n \geq 1$ be integers and let $\mathbf{y} \in \mathbb{I}^m$. If $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ is an approximating function which is assumed to be monotonic if $n = 1$ or $n = 2$, then*

$$|\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty. \end{cases}$$

When $n \geq 3$ the above theorem is a consequence of Theorem 3.5. In the other cases, where monotonicity of ψ is imposed, the above statement follows from results of Beresnevich, Dickinson and Velani [6, Section 12]. For more detailed discussion we refer the reader to, for example, [5].

By combining Theorem 3.2 with Theorem 3.6 the following Hausdorff measure analogue of Theorem 3.6 may be obtained.

THEOREM 3.7 (Allen–Beresnevich [2]). *Let $m, n \geq 1$ be integers, let $\mathbf{y} \in \mathbb{I}^m$, and let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximating function. Let f and $g : r \mapsto g(r) := r^{-m(n-1)} f(r)$*

be dimension functions such that $r^{-nm}f(r)$ is monotonic. In the case that $n = 1$ or $n = 2$ suppose also that $rg\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}$ is monotonic. Then,

$$\mathcal{H}^f(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) < \infty, \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}g\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

REMARK 3.8. In the case when $n = 2$, it is possible to prove Theorem 3.7 subject to the more aesthetically pleasing monotonicity constraint that ψ is monotonically decreasing. This can be achieved by combining Theorem 3.2 with Theorem 3.5, rather than Theorem 3.6.

REMARK 3.9. We note here that in both Theorems 3.6 and 3.7 the monotonicity condition on ψ when $n = 1$ or $n = 2$ is only required for the divergence cases. For both of these theorems the proofs of the convergence parts follow from standard covering arguments for which no monotonicity conditions need to be imposed.

In the next section we show how we can use Theorem 3.7 to provide an alternative proof of most cases of a general inhomogeneous Jarník–Besicovitch Theorem proved by Levesley [49]. Furthermore, we are able to comment on the necessity of the monotonicity condition imposed in this theorem of Levesley.

3.3. A Theorem of Levesley. The Hausdorff dimension of $\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)$, in the general inhomogeneous setting, was determined by Levesley in [49]. Given a function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, the lower order at infinity of f , usually denoted by λ , is

$$\lambda(f) = \liminf_{q \rightarrow \infty} \frac{\log(f(q))}{\log(q)}.$$

THEOREM 3.10 (Levesley, [49]). Let $m, n \in \mathbb{N}$ and let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a monotonically decreasing function. Let λ be the lower order at infinity of $1/\psi$. Then, for any $\mathbf{y} \in \mathbb{I}^m$,

$$\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \begin{cases} m(n-1) + \frac{m+n}{\lambda+1} & \text{when } \lambda \geq \frac{n}{m}, \\ nm & \text{when } \lambda < \frac{n}{m}. \end{cases}$$

REMARK. In the homogeneous case, when $\mathbf{y} = \mathbf{0}$, this result was previously established by Dodson [16].

Levesley proved the above theorem by considering the cases of $n = 1$ and $n \geq 2$ separately. In both cases his argument uses ideas from ubiquitous systems. These are combined with ideas from uniform distribution in the former case and with a more statistical (“mean-variance”) argument in the latter case.

Using Theorem 3.7, we can give an alternative proof of this theorem in the case that $n \geq 2$. That is, we will prove:

THEOREM 3.11. Let $m \geq 1$ and $n \geq 2$ be integers. Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a monotonically decreasing function and let λ be the lower order at infinity of $1/\psi$. Then, for any $\mathbf{y} \in \mathbb{I}^m$,

$$\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \begin{cases} m(n-1) + \frac{m+n}{\lambda+1} & \text{when } \lambda \geq \frac{n}{m}, \\ nm & \text{when } \lambda < \frac{n}{m}. \end{cases}$$

Recall that in Remark 3.8 we noted that it was sufficient in Theorem 3.7 to assume that ψ is monotonically decreasing in the case that $n = 2$. Throughout this section, we shall assume any mention of Theorem 3.7 refers to a statement including this nicer monotonicity condition for the $n = 2$ case.

To prove Theorem 3.11 using Theorem 3.7 we first establish a useful lemma.

LEMMA 3.12. *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be monotonic and bounded. Then,*

$$\liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} = \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

PROOF. Assume first that ψ is non-increasing. Note that $(2^t)_{t=1}^\infty$ is a subsequence of $(q)_{q=1}^\infty$ and so

$$\liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} \leq \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

It remains to prove the reverse inequality. Suppose for now that $\psi(q) \geq 1$ for all $q \in \mathbb{N}$. In this case, since $\psi(q) \rightarrow c$ for some $c \geq 1$ by monotone convergence,

$$\liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} = 0 = \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

Thus, we may assume that $\psi(q) < 1$ for all sufficiently large q . Given $q \in \mathbb{N}$, set t_q to be the unique integer satisfying $2^{t_q} \leq q < 2^{t_q+1}$. Then $\psi(2^{t_q}) \geq \psi(q)$ and $\log(\psi(2^{t_q})) \geq \log(\psi(q))$. Since further $q < 2^{t_q+1}$ and so $\log q < \log 2^{t_q+1}$, we obtain

$$\begin{aligned} \liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} &\geq \liminf_{q \rightarrow \infty} \frac{-\log(\psi(2^{t_q}))}{\log 2^{t_q+1}} \\ &= \liminf_{q \rightarrow \infty} \frac{-\log(\psi(2^{t_q}))}{\log 2^{t_q} + \log 2} \\ &= \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}, \end{aligned}$$

as required.

For non-decreasing ψ the proof is similar. By the same argument as above, it is again sufficient to show that

$$\liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} \geq \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

We note that if $\psi(q) \geq 1$ for all sufficiently large q then, since ψ is bounded,

$$\liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} = 0 = \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}.$$

Therefore, we may assume that $\psi(q) < 1$ for all $q \in \mathbb{N}$. Now, along the same lines as the argument given above, given $q \in \mathbb{N}$ let t'_q be the unique integer for which $2^{t'_q-1} \leq q < 2^{t'_q}$. Thus, we have

$$\log 2^{t'_q} > \log q \quad \text{and} \quad \log \psi(2^{t'_q}) \geq \log(\psi(q)).$$

Hence, it follows that

$$\begin{aligned} \liminf_{q \rightarrow \infty} \frac{-\log(\psi(q))}{\log q} &\geq \liminf_{q \rightarrow \infty} \frac{-\log(\psi(2^{t'_q}))}{\log 2^{t'_q}} \\ &= \liminf_{t \rightarrow \infty} \frac{-\log(\psi(2^t))}{\log 2^t}, \end{aligned}$$

and the proof is thus complete. \square

PROOF OF THEOREM 3.11 USING THEOREM 3.7. To avoid confusion throughout the proof, for approximating functions $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ we will write λ_ψ to denote the lower order at infinity of $1/\psi$. However, when there is no ambiguity we will just write λ and omit the additional subscript.

We observe that, since ψ is assumed to be monotonically decreasing, we must have $\lambda_\psi \geq 0$. To see this, suppose that $\lambda_\psi < 0$. Then, by the definition of the lower order at infinity, it follows that for any $\varepsilon > 0$ we must have $\psi(q) \geq q^{-(\lambda_\psi + \varepsilon)}$ for infinitely many values of q . In particular, this is true for every $0 < \varepsilon < |\lambda_\psi|$ and so we conclude that ψ cannot be monotonically decreasing if $\lambda_\psi < 0$.

We will now show that if the result stated in Theorem 3.11 is true for approximating functions with $\lambda = \frac{n}{m}$, then this implies the validity of the result for approximating functions with $0 \leq \lambda < \frac{n}{m}$. We will then establish the result for approximating functions with $\lambda \geq \frac{n}{m}$.

For the time being, assume that the conclusion in Theorem 3.11 holds for any monotonically decreasing approximating function with $\lambda = \frac{n}{m}$ and let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be a monotonically decreasing approximating function such that $\lambda_\psi < \frac{n}{m}$. Consider the function $\Psi : \mathbb{N} \rightarrow \mathbb{R}^+$ defined by $\Psi(q) = \min\{\psi(q), q^{-\frac{n}{m}}\}$. Note that Ψ is a monotonically decreasing function (since it is the minimum of two monotonically decreasing functions) and that $\Psi(q) \leq \psi(q)$ for all $q \in \mathbb{N}$. In particular, we have $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)) \leq \dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi))$. Next, note that it follows from the fact that $\Psi(q) \leq q^{-\frac{n}{m}}$ for all $q \in \mathbb{N}$ that $\lambda_\Psi \geq \frac{n}{m}$. On the other hand, since $\lambda_\psi < \frac{n}{m}$ we know that $\psi(q) \geq q^{-\frac{n}{m}}$ for infinitely many values of q . In particular, this implies that we must have $\Psi(q) = q^{-\frac{n}{m}}$ infinitely often and, consequently, that $\lambda_\Psi \leq \frac{n}{m}$. Hence, $\lambda_\Psi = \frac{n}{m}$ and so, by our assumption, we see that

$$\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \geq \dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)) = m(n-1) + \frac{n+m}{\lambda_\Psi + 1} = nm.$$

Combining this with the trivial upper bound we see that $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = nm$, as required.

It remains to be shown that $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = m(n-1) + \frac{n+m}{\lambda+1}$ for monotonically decreasing approximating functions $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ with $\lambda_\psi = \lambda \geq \frac{n}{m}$. To this end, suppose ψ is such an approximating function.

Let $s_0 = m(n-1) + \frac{m+n}{\lambda+1}$ and consider $f_\delta(r) = r^{s_0+\delta}$ where $-\frac{n+m}{\lambda+1} < \delta < \frac{n+m}{\lambda+1}$. We aim to show that

$$\mathcal{H}^{s_0+\delta}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \begin{cases} 0 & \text{if } \delta > 0, \\ \mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) & \text{if } \delta < 0, \end{cases}$$

from which the result would follow.

Note that $f_\delta(r)$ is a dimension function and $r^{-nm} f_\delta(r)$ is monotonic. Let $g_\delta(r) = r^{-m(n-1)} f_\delta(r) = r^{-m(n-1)+s_0+\delta}$. Since $\delta > -\frac{n+m}{\lambda+1}$, and consequently

$-m(n-1) + s_0 + \delta > 0$, the function $g_\delta(r)$ is a dimension function. Thus, f_δ and g_δ satisfy the hypotheses of Theorem 3.7.

It follows from the definition of the lower order at infinity that, for any $\varepsilon > 0$,

$$\psi(q) \leq q^{-(\lambda-\varepsilon)} \text{ for all large enough } q \text{ and}$$

$$(3.1) \quad \psi(q) \geq q^{-(\lambda+\varepsilon)} \text{ for infinitely many } q \in \mathbb{N}.$$

Combining this with Lemma 3.12, we have

$$\psi(2^t) \leq 2^{-t(\lambda-\varepsilon)} \text{ for large enough } t, \text{ and}$$

$$(3.2) \quad \psi(2^t) \geq 2^{-t(\lambda+\varepsilon)} \text{ for infinitely many } t.$$

By Theorem 3.7 it follows that to determine $\mathcal{H}^{f_\delta}(\mathcal{A}_{n,m}^y(\psi))$ we are interested in the behaviour of the sum

$$(3.3) \quad \sum_{q=1}^{\infty} q^{n+m-1} g_\delta \left(\frac{\psi(q)}{q} \right) = \sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta}.$$

Observe that, by the conditions imposed on δ , $-m(n-1) + s_0 + \delta > 0$ and also that, by (3.1), we have $\psi(q) \leq q^{-(\lambda-\varepsilon)}$ for sufficiently large q . Thus, (3.3) will converge if

$$(3.4) \quad \sum_{q=1}^{\infty} q^{n+m-1} (q^{-(\lambda-\varepsilon)-1})^{-m(n-1)+s_0+\delta} = \sum_{q=1}^{\infty} q^{n+m-1+(\lambda+1-\varepsilon)(m(n-1)-s_0-\delta)} < \infty.$$

This will be the case if

$$n + m - 1 + (\lambda + 1 - \varepsilon)(m(n - 1) - s_0 - \delta) < -1,$$

which is true if and only if

$$\frac{n + m}{\lambda + 1 - \varepsilon} + m(n - 1) < s_0 + \delta.$$

If $\delta > 0$ we can force the above to be true by taking ε to be sufficiently small. Thus we conclude that, for $\delta > 0$, (3.3) converges and consequently $\mathcal{H}^{s_0+\delta}(\mathcal{A}_{n,m}^y(\psi)) = 0$.

Next we establish that (3.3) diverges when $-\frac{n+m}{\lambda+1} < \delta < 0$. First we note, since ψ is monotonically decreasing, that

$$(3.5) \quad \begin{aligned} \sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta} &= \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq q < 2^t} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta} \\ &\geq \sum_{t=1}^{\infty} \sum_{2^{t-1} \leq q < 2^t} (2^{t-1})^{n+m-1} \left(\frac{\psi(2^t)}{2^t} \right)^{-m(n-1)+s_0+\delta} \\ &= \sum_{t=1}^{\infty} 2^{t-1} (2^{t-1})^{n+m-1} \left(\frac{\psi(2^t)}{2^t} \right)^{-m(n-1)+s_0+\delta} \\ &= \frac{1}{2^{m+n}} \sum_{t=1}^{\infty} 2^{t(n+m)} \left(\frac{\psi(2^t)}{2^t} \right)^{-m(n-1)+s_0+\delta}. \end{aligned}$$

We proceed by showing that, when $\delta < 0$, we have for infinitely many t that

$$(3.6) \quad 2^{t(m+n)} \left(\frac{\psi(2^t)}{2^t} \right)^{-m(n-1)+s_0+\delta} \geq 1.$$

For any $\delta < 0$ we can choose $\varepsilon > 0$ small enough such that

$$\frac{m+n}{\lambda+1+\varepsilon} + m(n-1) \geq s_0 + \delta.$$

Note that such an ε exists since we are assuming that δ is negative. Rearranging,

$$m+n - (\lambda + \varepsilon + 1)(-m(n-1) + s_0 + \delta) \geq 0$$

and then, exponentiating,

$$2^{t(m+n)} \left(\frac{2^{-t(\lambda+\varepsilon)}}{2^t} \right)^{-m(n-1)+s_0+\delta} \geq 1.$$

Now, by (3.2) we have $\psi(2^t) \geq 2^{-t(\lambda+\varepsilon)}$ infinitely often and so (3.6) holds, thus proving the divergence of (3.5) and hence also the divergence of (3.3).

Hence, we have shown that

$$\mathcal{H}^{s_0+\delta}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \begin{cases} 0 & \text{if } \delta > 0, \\ \mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) & \text{if } \delta < 0. \end{cases}$$

If $s_0 \leq nm$ then $\mathcal{H}^{s_0+\delta}(\mathbb{I}^{nm}) = \infty$ whenever $\delta < 0$. From this it would follow that $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = s_0$. The proof is completed upon noting that $s_0 \leq nm$ is equivalent to $\lambda \geq \frac{n}{m}$. \square

In Theorems 3.10 and 3.11 the approximating function ψ is assumed to be monotonic. However, the main tool in our proof of Theorem 3.11 is Theorem 3.7 which requires no monotonicity assumptions on ψ for $n \geq 3$. This leads immediately to the natural question of whether this monotonicity assumption is indeed necessary in Levesley's Theorem (Theorem 3.10).

In an attempt to address this question, let us consider general (not necessarily monotonic) approximating functions $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ with λ , the lower order at infinity of $1/\psi$, satisfying $\lambda > n/m$. Assuming no monotonicity conditions on ψ , and applying similar arguments to those which we have employed here to prove Theorem 3.11, we obtain the following bounds on the Hausdorff dimension of $\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)$. Although, in the interest of brevity, we omit proof.

PROPOSITION 3.13. *Let $m \geq 1$ and $n \geq 3$ be integers. If $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ is any function and λ is the lower order at infinity of $1/\psi$ then, for any $\mathbf{y} \in \mathbb{I}^m$, if $\lambda > n/m$ we have*

$$m(n-1) + \frac{m+n-1}{\lambda+1} \leq \dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \leq m(n-1) + \frac{m+n}{\lambda+1}.$$

We see that the upper and lower bounds for $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi))$ in Proposition 3.13 do not coincide. Interestingly, it turns out that these bounds are the best possible if one does not assume monotonicity of ψ — as we will now show. To the best of our knowledge the following result has not been considered before, even in the homogeneous setting.

THEOREM 3.14. *Let $m, n \geq 1$ be integers. Let $\alpha > n/m$ be arbitrary and let s_0 be such that*

$$m(n-1) + \frac{m+n-1}{\alpha+1} < s_0 < m(n-1) + \frac{m+n}{\alpha+1}.$$

There exists an approximating function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ such that for every $\mathbf{y} \in \mathbb{I}^m$ we have $\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = s_0$ and $\lambda_{\psi} = \alpha$ (where λ_{ψ} is the lower order at infinity of $1/\psi$).

PROOF. Fix s_0 satisfying the inequality in the statement of the theorem and let $\mathbf{y} \in \mathbb{I}^m$ be arbitrary. Then, let $J := \{a_k : k \in \mathbb{N}\}$, where $a_k = \lceil k^{-\gamma} \rceil$,

$$\gamma := \frac{2}{n+m-1 - (\alpha+1)\left(\frac{n+m}{\beta+1}\right)} \quad \text{and} \quad \beta := \frac{n+m}{s_0 - m(n-1)} - 1.$$

Note that $\gamma < 0$. Define $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ by

$$\psi(q) = \begin{cases} q^{-\alpha} & \text{if } q \in J, \\ q^{-\beta} & \text{if } q \notin J. \end{cases}$$

We show that ψ is an approximating function which satisfies the desired properties of the theorem. First, note that

$$m(n-1) + \frac{n+m}{\alpha+1} > s_0,$$

which implies that

$$\frac{n+m}{s_0 - m(n-1)} - 1 > \alpha.$$

In turn, this implies that $\beta > \alpha$ and so $\liminf_{q \rightarrow \infty} -\log(\psi(q))/\log(q) = \alpha$, giving $\lambda_{\psi} = \alpha$, as required.

Recall that if $\lambda_{\psi} = \alpha$ then for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $\psi(q) \leq q^{-(\alpha-\varepsilon)}$ for all $q \geq N$, and $\psi(q) \geq q^{-(\alpha+\varepsilon)}$ for infinitely many $q \in \mathbb{N}$.

To establish that the Hausdorff dimension is s_0 we note that $\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \geq \dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(q \mapsto q^{-\beta}))$ since $\psi(q) \geq q^{-\beta}$ for all q . Furthermore, since $q \mapsto q^{-\beta}$ is a monotonic function with $\lambda_{(q \mapsto q^{-\beta})} = \beta$, by Theorem 3.10 we have

$$\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(q \mapsto q^{-\beta})) = m(n-1) + \frac{m+n}{\beta+1} = s_0.$$

Therefore, $\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \geq s_0$ and it remains to show that $\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \leq s_0$.

As a consequence of Theorem 3.7 (and Remark 3.9), we only need to verify that for all $\delta > 0$ we have

$$\sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta} < \infty$$

since this would imply that $\mathcal{H}^{s_0+\delta}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = 0$ and $\dim_{\text{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \leq s_0 + \delta$.

We note that

$$\begin{aligned}
 & \sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta} \\
 &= \sum_{q \in J} q^{n+m-1} (q^{-\alpha-1})^{-m(n-1)+s_0+\delta} + \sum_{q \notin J} q^{n+m-1} (q^{-\beta-1})^{-m(n-1)+s_0+\delta} \\
 (3.7) \quad &= \sum_{q \in J} q^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} + \sum_{q \notin J} q^{n+m-1-(\beta+1)(s_0+\delta-m(n-1))}.
 \end{aligned}$$

We consider each of the terms on the right-hand side of (3.7) separately and show that each of them converges. We first consider the second sum on the right-hand side of (3.7). Since $\delta > 0$ we have $s_0 - m(n-1) < s_0 + \delta - m(n-1)$ and hence

$$n + m < \left(\frac{n + m}{s_0 - m(n-1)} \right) (s_0 + \delta - m(n-1)).$$

Recalling that

$$\beta = \frac{n + m}{s_0 - m(n-1)} - 1$$

it follows that

$$n + m - 1 - (\beta + 1)(s_0 + \delta - m(n-1)) < -1,$$

which is sufficient for the second sum on the right-hand side of (3.7) to converge.

For the first sum on the right-hand side of (3.7) we make the following observations. First of all, notice that

$$n + m - 1 - (\alpha + 1) \left(\frac{n + m}{\beta + 1} \right) = n + m - 1 - (\alpha + 1)(s_0 - m(n-1)).$$

Also note that

$$\frac{n + m - 1}{\alpha + 1} + m(n-1) < s_0 \quad \text{gives} \quad n + m - 1 - (\alpha + 1)(s_0 - m(n-1)) < 0.$$

Thus, provided that δ is sufficiently small,

$$\begin{aligned}
 \sum_{q \in J} q^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} &= \sum_{k=1}^{\infty} a_k^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} \\
 &= \sum_{k=1}^{\infty} \lceil k^{-\gamma} \rceil^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} \\
 (3.8) \quad &\leq \sum_{k=1}^{\infty} (k^{-\gamma})^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))}
 \end{aligned}$$

as $n + m - 1 - (\alpha + 1)(s_0 + \delta - m(n-1)) < 0$ and $\gamma < 0$.

Now, for $\delta > 0$,

$$\begin{aligned}
 \frac{2}{\gamma} &= n + m - 1 - (\alpha + 1)(s_0 - m(n-1)) \\
 &> n + m - 1 - (\alpha + 1)(s_0 + \delta - m(n-1)).
 \end{aligned}$$

Hence,

$$2 < \gamma(n + m - 1 - (\alpha + 1)(s_0 + \delta - m(n-1)))$$

since $\gamma < 0$. Therefore $(k^{-\gamma})^{n+m-1-(\alpha+1)(s_0+\delta-m(n-1))} < k^{-2}$ and so (3.8) converges. Consequently, since both the component sums converge, it follows that (3.7) converges, *i.e.*

$$\sum_{q=1}^{\infty} q^{n+m-1} \left(\frac{\psi(q)}{q} \right)^{-m(n-1)+s_0+\delta} < \infty,$$

and we conclude that $\dim_{\mathbb{H}}(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) \leq s_0 + \delta$. The desired result follows upon noticing that $\delta > 0$ can be taken to be arbitrarily small. \square

3.4. General statements. We conclude this section on the extension of the Mass Transference Principle to systems of linear forms by recording a couple of very general statements established in [2]. Recall that given an approximating function $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+$, which can depend on \mathbf{q} rather than just $|\mathbf{q}|$, and a fixed inhomogeneous parameter $\mathbf{y} \in \mathbb{I}^m$, we define $\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)$ to be the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$|\mathbf{q}\mathbf{x} + \mathbf{p} - \mathbf{y}| < \Psi(\mathbf{q})$$

for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

Considering the Ψ -approximable points we have the following statement.

THEOREM 3.15 (Allen – Beresnevich [2]). *Let $\Psi : \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+$ be an approximating function and let $\mathbf{y} \in \mathbb{I}^m$ be fixed. Let f and $g : r \mapsto g(r) = r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Let*

$$\Theta : \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+ \quad \text{be defined by} \quad \Theta(\mathbf{q}) := |\mathbf{q}|g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right)^{\frac{1}{m}}.$$

Then

$$|\mathcal{A}_{n,m}^{\mathbf{y}}(\Theta)| = 1 \quad \text{implies} \quad \mathcal{H}^f(\mathcal{A}_{n,m}^{\mathbf{y}}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

Supposing we are interested in the case where we have approximating functions $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ which depend only on $|\mathbf{q}|$ (*i.e.* $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$) we can extract the following statement as a corollary to Theorem 3.15.

THEOREM 3.16 (Allen – Beresnevich [2]). *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximating function, let $\mathbf{y} \in \mathbb{I}^m$ be fixed, and let f and $g : r \mapsto g(r) = r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Let*

$$\theta : \mathbb{N} \rightarrow \mathbb{R}^+ \quad \text{be defined by} \quad \theta(r) := r g\left(\frac{\psi(r)}{r}\right)^{\frac{1}{m}}.$$

Then

$$|\mathcal{A}_{n,m}^{\mathbf{y}}(\theta)| = 1 \quad \text{implies} \quad \mathcal{H}^f(\mathcal{A}_{n,m}^{\mathbf{y}}(\psi)) = \mathcal{H}^f(\mathbb{I}^{nm}).$$

It is observed in [2] that Theorems 3.4 and 3.7 follow as corollaries from Theorem 3.16. In fact, in some sense, Theorems 3.15 and 3.16 are fairly natural reformulations of Theorem 3.2 in terms of, respectively, Ψ and ψ -approximable points. In essentially the same way that Theorem 3.2 may be used to prove Theorem 3.15 a more general statement can also be obtained. Namely, suppose we are now given a function $\Psi : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+$ which can depend upon both \mathbf{p} and \mathbf{q} . Furthermore, suppose we are also given fixed $\Phi \in \mathbb{I}^{mm}$ and $\mathbf{y} \in \mathbb{I}^m$. We denote by $\mathcal{M}_{n,m}^{\mathbf{y},\Phi}(\Psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ for which

$$|\mathbf{q}\mathbf{x} + \mathbf{p}\Phi - \mathbf{y}| < \Psi(\mathbf{p}, \mathbf{q})$$

holds for $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $|\mathbf{q}|$ arbitrarily large.

The following statement, which actually includes Theorems 3.15 and 3.16, can be made.

THEOREM 3.17 (Allen – Beresnevich [2]). *Let $\Psi : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+$ be such that*

$$\lim_{|\mathbf{q}| \rightarrow \infty} \sup_{\mathbf{p} \in \mathbb{Z}^m} \frac{\Psi(\mathbf{p}, \mathbf{q})}{|\mathbf{q}|} = 0,$$

and let $\mathbf{y} \in \mathbb{I}^m$ and $\Phi \in \mathbb{I}^{m,m}$ be fixed. Let f and $g : r \mapsto g(r) = r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Let

$$\Theta : \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+ \quad \text{be defined by} \quad \Theta(\mathbf{p}, \mathbf{q}) = |\mathbf{q}| g \left(\frac{\Psi(\mathbf{p}, \mathbf{q})}{|\mathbf{q}|} \right)^{\frac{1}{m}}.$$

Then

$$|\mathcal{M}_{n,m}^{\mathbf{y}, \Phi}(\Theta)| = 1 \quad \text{implies} \quad \mathcal{H}^f(\mathcal{M}_{n,m}^{\mathbf{y}, \Phi}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{m,m}).$$

The above theorem not only allows us to consider the usual homogeneous and inhomogeneous settings of Diophantine approximation for systems of linear forms (see [5]) but also allows us to consider Hausdorff measure statements where we may have some restrictions on our “approximating points” (\mathbf{p}, \mathbf{q}) . As an example, recently Dani, Laurent and Nogueira have established Lebesgue measure “Khinchine–Groshev” type statements for sets of ψ -approximable points where they have imposed certain primitivity conditions on their “approximating points” [14]. In [2], Theorem 3.17 has been used to establish Hausdorff measure versions of these results.

4. Extension to rectangles

Another very natural situation, not covered by the setting of systems of linear forms, for which we might hope for some kind of mass transference principle is when our lim sup sets of interest are defined by sequences of rectangles. Recently some progress has been made in this direction by Wang, Wu and Xu [64]. For example, this is of interest when we consider weighted simultaneous approximation.

4.1. A mass transference principle from balls to rectangles. Throughout this section let $k \in \mathbb{N}$ and, as usual, denote by \mathbb{I}^k the unit cube $[0, 1]^k$ in \mathbb{R}^k . Given a ball $B = B(\mathbf{x}, r)$ in \mathbb{R}^k of radius r centred at \mathbf{x} and a k -dimensional real vector $\mathbf{a} = (a_1, a_2, \dots, a_k)$ we will denote by $B^{\mathbf{a}}$ the rectangle with centre \mathbf{x} and side-lengths $(r^{a_1}, r^{a_2}, \dots, r^{a_k})$. Given a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ of points in \mathbb{I}^k and a sequence $(r_n)_{n \in \mathbb{N}}$ of positive real numbers such that $r_n \rightarrow 0$ as $n \rightarrow \infty$ we define

$$W_0 = \{\mathbf{x} \in \mathbb{I}^k : \mathbf{x} \in B_n = B(\mathbf{x}_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

For any $\mathbf{a} \in \mathbb{R}^k$ we will also write

$$W_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{I}^k : \mathbf{x} \in B_n^{\mathbf{a}} \text{ for infinitely many } n \in \mathbb{N}\}.$$

In [64], Wang, Wu and Xu established the following mass transference principle.

THEOREM 4.1 (Wang – Wu – Xu [64]). *Let $(\mathbf{x}_n)_{n \in \mathbb{N}}$ be a sequence of points in \mathbb{I}^k and let $(r_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that $r_n \rightarrow 0$ as*

$n \rightarrow \infty$. Let $\mathbf{a} = (a_1, a_2, \dots, a_k) \in \mathbb{R}^k$ be such that $1 \leq a_1 \leq a_2 \leq \dots \leq a_k$. Suppose that $|W_0| = 1$. Then,

$$\dim_{\mathbb{H}} W_{\mathbf{a}} \geq \min_{1 \leq j \leq k} \left\{ \frac{k + ja_j - \sum_{i=1}^j a_i}{a_j} \right\}.$$

Furthermore, if we have the additional constraint $a_d > 1$, Wang, Wu and Xu are also able to say something about the Hausdorff measure of $W_{\mathbf{a}}$ at the critical value

$$(4.1) \quad s := \min_{1 \leq j \leq k} \left\{ \frac{k + ja_j - \sum_{i=1}^j a_i}{a_j} \right\}.$$

THEOREM 4.2 (Wang – Wu – Xu [64]). *Assume the same conditions as in Theorem 4.1. If the additional constraint that $a_d > 1$ holds, then*

$$\mathcal{H}^s(W_{\mathbf{a}}) = \infty.$$

Essentially, the results of Wang, Wu and Xu allow us to pass from a full Lebesgue measure statement for a limsup set defined by a sequence of balls to a Hausdorff measure statement for a limsup set defined by an associated sequence of rectangles. As an application, Wang, Wu and Xu demonstrate how Theorem 4.1 may be applied to obtain the Hausdorff dimension of the following set of weighted simultaneously well-approximable points. Let $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_k) \in \mathbb{R}^k$ be such that $\tau_i > 0$ for $1 \leq i \leq k$ and denote by $W_k(\boldsymbol{\tau})$ the set of points $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{I}^k$ such that

$$(4.2) \quad |qx_i + p_i| < q^{-\tau_i}, \quad 1 \leq i \leq k,$$

for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^k \times \mathbb{N}$. The following is derived in [64] as a corollary to Theorem 4.1.

COROLLARY 4.3 (Wang – Wu – Xu [64]). *Let $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_k) \in \mathbb{R}^k$ be such that $\frac{1}{k} \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_k$, then*

$$\dim_{\mathbb{H}}(W_k(\boldsymbol{\tau})) = \min_{1 \leq j \leq k} \left\{ \frac{k + 1 + j\tau_j - \sum_{i=1}^j \tau_i}{1 + \tau_j} \right\}.$$

While the proof of Corollary 4.3 given in [64] is novel, and is a neat application of Theorem 4.1, the result itself was already previously known. In fact, Corollary 4.3 is a special case of an earlier more general theorem due to Rynne [58] which we now state.

Suppose Q is an arbitrary infinite set of natural numbers and, given $\boldsymbol{\tau} \in \mathbb{R}^k$, let $W_k^Q(\boldsymbol{\tau})$ denote the set of points $\mathbf{x} \in \mathbb{I}^k$ for which the inequalities in (4.2) hold for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^k \times Q$, hence $W_k^{\mathbb{N}}(\boldsymbol{\tau}) = W_k(\boldsymbol{\tau})$. Define

$$\nu(Q) = \inf \left\{ \nu \in \mathbb{R} : \sum_{q \in Q} q^{-\nu} < \infty \right\}$$

and let $\sigma(\boldsymbol{\tau}) = \sum_{i=1}^k \tau_i$.

THEOREM 4.4 (Rynne [58]). *Let $\tau = (\tau_1, \tau_2, \dots, \tau_k) \in \mathbb{R}^k$ be such that $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_k$. Let Q be an arbitrary infinite subset of \mathbb{N} and suppose that $\sigma(\tau) \geq \nu(Q)$. Then,*

$$\dim_{\mathbb{H}} W_k^Q(\tau) = \min_{1 \leq j \leq k} \left\{ \frac{k + \nu(Q) + j\tau_j - \sum_{i=1}^j \tau_i}{1 + \tau_j} \right\}.$$

We may easily recover Corollary 4.3 by taking $Q = \mathbb{N}$ in Theorem 4.4 and noting that $\nu(\mathbb{N}) = 1$. Since the hypotheses of Corollary 4.3 demand that $\tau_i \geq \frac{1}{k}$ for all $1 \leq i \leq k$, we see that the condition $\sigma(\tau) \geq \nu(Q)$ in Theorem 4.4 is also satisfied.

Sets such as $W_k(\tau)$ and variations on $W_k^Q(\tau)$ have been studied in some depth, with particular attention paid to the question of determining their Hausdorff dimension, even before the work of Rynne [58]. For example, consider $\tau \in \mathbb{R}$ for some $\tau > 1$. Then the set $W_1^{\mathbb{N}}(\tau) = W_1(\tau)$ coincides precisely with the set $\mathcal{A}(\tau)$ considered in the Jarník–Besicovitch Theorem (Theorem 2.8). For an overview of some other earlier work in this direction we direct the reader to the discussion given in [58] and references therein.

4.2. Rectangles to rectangles. The original Mass Transference Principle (Theorem 2.5) allows us to transition from Lebesgue to Hausdorff measure statements when our original and “transformed” lim sup sets are defined by sequences of balls, *i.e.* it allows us to go from “balls to balls”. Theorem 4.1 allows us to go from “balls to rectangles”. Another goal which we might like to achieve, which is not covered by any of the frameworks mentioned so far, would be to prove a similar mass transference principle where we both start and finish with lim sup sets arising from sequences of rectangles, *i.e.* from “rectangles to rectangles”.

PROBLEM 4.5. *Does there exist a mass transference principle, similar to Theorem 2.5 or Theorem 4.1, where both the original and transformed lim sup sets are defined by sequences of rectangles?*

Although in the most general settings this problem remains open, we survey what can be said in a few special cases.

In [9] Beresnevich and Velani employ a “slicing” technique, which uses a combination of a slicing lemma and the original Mass Transference Principle, to prove Theorem 3.1. We show how an appropriate combination of these two results can also be applied to considering the problem of proving a mass transference principle for rectangles. We proceed by stating the “Slicing Lemma” as given by Beresnevich and Velani in [9].

LEMMA 4.6 (Slicing Lemma [9]). *Let $l, k \in \mathbb{N}$ be such that $l \leq k$ and let f and $g : r \mapsto r^{-l}f(r)$ be dimension functions. Let $A \subset \mathbb{R}^k$ be a Borel set and let V be a $(k - l)$ -dimensional linear subspace of \mathbb{R}^k . If for a subset S of V^\perp of positive \mathcal{H}^l -measure*

$$\mathcal{H}^g(A \cap (V + b)) = \infty \quad \text{for all } b \in S,$$

then $\mathcal{H}^f(A) = \infty$.

Suppose that $(\mathbf{x}_n)_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k})_n$ is a sequence of points in $[0, 1]^k$. Let $(r_n^1)_n, (r_n^2)_n, \dots, (r_n^k)_n$ be sequences of positive real numbers and suppose that

$r_n^1 \rightarrow 0$ as $n \rightarrow \infty$. Let

$$H_n = \prod_{i=1}^k B(x_{n,i}, r_n^i)$$

be a sequence of rectangles in $[0, 1]^k$, where $\prod_{i=1}^k A_i = A_1 \times A_2 \times \dots \times A_k$ is the Cartesian product of subsets A_i of \mathbb{R}^k . Let $\alpha > 1$ be a real number and define another sequence of rectangles by

$$h_n = B(x_{n,1}, (r_n^1)^\alpha) \times \prod_{i=2}^k B(x_{n,i}, r_n^i).$$

So, h_n is essentially a “shrunk” rectangle corresponding to H_n from the original sequence. Note that in this case we only allow shrinking of the original rectangle in one direction. Then, we are able to establish the following.

PROPOSITION 4.7. *Let the sequences H_n and h_n be as given above and further suppose that $|\limsup_{n \rightarrow \infty} H_n| = 1$. Then,*

$$\dim_H \left(\limsup_{n \rightarrow \infty} h_n \right) \geq \frac{1}{\alpha} + k - 1.$$

PROOF. Let $V = \{\mathbf{x} = (x_1, \dots, x_k) \in [0, 1]^k : x_i = 0 \text{ for all } i \neq 1\}$. Since $|\limsup_{n \rightarrow \infty} H_n| = 1$, for Lebesgue almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, \dots, x_k) \in [0, 1]^k : x_1 = 0\}$$

we have

$$|(V + \mathbf{b}) \cap \limsup_{n \rightarrow \infty} H_n| = 1.$$

Let us fix a \mathbf{b} for which this holds and let $W = V + \mathbf{b}$. Now, $\limsup_{n \rightarrow \infty} H_n \cap W$ can be written as the lim sup set of a sequence of balls $B_j = B(x_{n_j,1}, r_{n_j}^1)$ with radii $r_{n_j}^1$. Note that $|\limsup_{j \rightarrow \infty} B_j \cap W| = 1$. For each j also let $b_j = B(x_{n_j,1}, (r_{n_j}^1)^\alpha)$ and note that

$$\limsup_{j \rightarrow \infty} b_j \cap W = \limsup_{n \rightarrow \infty} h_n \cap W.$$

In accordance with our earlier notation, $b_j^s = B(x_{n_j,1}, (r_{n_j}^1)^{\alpha s})$. Therefore, if $s \leq \frac{1}{\alpha}$ then $(r_{n_j}^1)^{\alpha s} \geq r_{n_j}^1$ for sufficiently large j and so

$$b_j^s \supseteq B_j \quad \text{and} \quad |\limsup_{j \rightarrow \infty} b_j^s \cap W| = 1.$$

Thus, for any $s \leq \frac{1}{\alpha}$ we may use the Mass Transference Principle to conclude that for any ball $B \subseteq W$ we have

$$\mathcal{H}^s(\limsup_{j \rightarrow \infty} b_j \cap B) = \mathcal{H}^s(B).$$

In particular, since $s \leq \frac{1}{\alpha} < 1$, this means

$$\mathcal{H}^s(\limsup_{n \rightarrow \infty} h_n \cap W) = \mathcal{H}^s(W) = \infty.$$

Since this is the case for Lebesgue almost every $\mathbf{b} \in \{\mathbf{x} = (x_1, \dots, x_k) : x_1 = 0\}$, we can use the Slicing Lemma (Lemma 4.6) to conclude that

$$\mathcal{H}^{s'}(\limsup_{n \rightarrow \infty} h_n) = \infty$$

for all $s' \leq \frac{1}{\alpha} + k - 1$. Therefore, it follows that

$$\dim_{\mathbb{H}} \left(\limsup_{n \rightarrow \infty} h_n \right) \geq \frac{1}{\alpha} + k - 1.$$

□

Using Theorem 4.2 in place of Theorem 2.5, we are actually able to extend this argument a little further. Again, let $(\mathbf{x}_n)_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k})_n$ be a sequence of points in $[0, 1]^k$ and let $(r_n^1)_n, (r_n^2)_n, \dots, (r_n^k)_n$ be sequences of positive real numbers. Suppose that for some $1 \leq k_0 \leq k$ we have $r_n^1 = r_n^2 = \dots = r_n^{k_0}$ for all $n \in \mathbb{N}$ and also that $r_n^1 \rightarrow 0$ as $n \rightarrow \infty$. Let

$$H_n = \prod_{i=1}^k B(x_{n,i}, r_n^i)$$

be a sequence of rectangles in $[0, 1]^k$. Next, let $1 \leq a_1 \leq a_2 \leq \dots \leq a_{k_0}$ be real numbers and suppose $a_{k_0} > 1$. For each rectangle H_n in our original sequence we define a corresponding “shrunk” rectangle

$$h_n = \prod_{i=1}^{k_0} B(x_{n,i}, (r_n^i)^{a_i}) \times \prod_{i=k_0+1}^k B(x_{n,i}, r_n^i).$$

In this case we are able to prove the following.

PROPOSITION 4.8. *Let the sequences of rectangles H_n and h_n be as given above and further suppose that $|\limsup_{n \rightarrow \infty} H_n| = 1$. Then,*

$$\dim_{\mathbb{H}} \left(\limsup_{n \rightarrow \infty} h_n \right) \geq \min_{1 \leq j \leq k_0} \left\{ \frac{k_0 + j a_j - \sum_{i=1}^j a_i}{a_j} + k - k_0 \right\}.$$

PROOF. Let $V = \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in [0, 1]^k : x_i = 0 \text{ for all } i \geq k_0 + 1\}$. Since $|\limsup_{n \rightarrow \infty} H_n| = 1$, for almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in [0, 1]^k : x_i = 0 \text{ for all } i \leq k_0\}$$

we have

$$|(V + \mathbf{b}) \cap \limsup_{n \rightarrow \infty} H_n| = 1.$$

Let us fix a \mathbf{b} for which this holds and let $W = V + \mathbf{b}$. As before, $\limsup_{n \rightarrow \infty} H_n \cap W$ can be written as a sequence of k_0 -dimensional balls $B_j = B(\mathbf{x}_{n_j}^{k_0}, r_{n_j}^1)$ with radii $r_{n_j}^1 (= r_{n_j}^2 = \dots = r_{n_j}^{k_0})$ and centres $\mathbf{x}_{n_j}^{k_0} = (x_{n_j,1}, x_{n_j,2}, \dots, x_{n_j,k_0})$. Note that $|\limsup_{j \rightarrow \infty} B_j \cap W| = 1$.

This time, for each j let

$$b_j = \prod_{i=1}^{k_0} B(x_{n_j,i}, (r_{n_j}^i)^{a_i})$$

and note that

$$\limsup_{j \rightarrow \infty} b_j \cap W = \limsup_{n \rightarrow \infty} h_n \cap W.$$

By Theorem 4.2 it follows that

$$\mathcal{H}^s(\limsup_{n \rightarrow \infty} h_n \cap W) = \infty$$

where

$$s := \min_{1 \leq j \leq k_0} \left\{ \frac{k_0 + ja_j - \sum_{i=1}^j a_i}{a_j} \right\}.$$

Since this is the case for almost every

$$\mathbf{b} \in \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in [0, 1]^k : x_i = 0 \text{ for all } i \leq k_0\}$$

we may use Lemma 4.6 (with $l = k - k_0$) to conclude that

$$\mathcal{H}^{s'}(\limsup_{n \rightarrow \infty} h_n) = \infty$$

where

$$s' := \min_{1 \leq j \leq k_0} \left\{ \frac{k_0 + ja_j - \sum_{i=1}^j a_i}{a_j} + k - k_0 \right\}.$$

Hence

$$\dim_{\mathbb{H}} \left(\limsup_{n \rightarrow \infty} h_n \right) \geq s',$$

as required. □

A disadvantage of using the “slicing” arguments above is that we have to impose quite strict conditions on both the original and transformed rectangles. Namely, the sides of the original rectangle which are permitted to “shrink” have to be of the same initial length (but can shrink at different rates). Meanwhile, the rest of the sides of the original rectangle are not allowed to “shrink” at all when passing to the corresponding transformed rectangle. We conclude this section by considering one more situation where all sides of the original rectangles may have different lengths and are all allowed to “shrink” in a specified manner. Let

$$H_n = \prod_{i=1}^k B(\mathbf{x}_{n,i}, r_n^{t_i})$$

be a sequence of rectangles in $[0, 1]^k$ with $1 \leq t_i$ for $1 \leq i \leq k$ and $r_n \rightarrow 0$.

Let the corresponding “shrunk” rectangles be defined as

$$h_n = \prod_{i=1}^k B(\mathbf{x}_{n,i}, r_n^{a_i t_i}),$$

where $1 \leq a_i$ for $1 \leq i \leq k$. Suppose without loss of generality that $1 \leq a_1 t_1 \leq a_2 t_2 \leq \dots \leq a_k t_k$. Furthermore, suppose that

$$D := \inf \left\{ d \in \mathbb{R} : \sum_{n=1}^{\infty} r_n^d < \infty \right\}.$$

By using the “natural” covers of $\limsup_{n \rightarrow \infty} h_n$ we can get an upper bound for the Hausdorff dimension of this lim sup set; namely, we see that

$$(4.3) \quad \dim_{\mathbb{H}} \left(\limsup_{n \rightarrow \infty} h_n \right) \leq \min_{1 \leq j \leq k} \left\{ \frac{D + ja_j t_j - \sum_{i=1}^j a_i t_i}{a_j t_j} \right\}.$$

PROBLEM 4.9. *Under what conditions do we get a lower bound which coincides with the upper bound given above?*

REMARK. Throughout this section we have only considered lim sup sets of rectangles which are all aligned. It would also be natural to consider situations where this is not necessarily the case.

5. Random Mass Transference Principles

It is a well known phenomenon that introducing randomness to a construction can simplify results by “smoothing” out almost impossible values in the probability space that cause problems in deterministic settings. In this section we will summarise recent progress on random analogues of the statements presented in the preceding sections. We note that the assumptions required are much weaker but with the caveat that randomness has to be introduced somewhere and precise number theoretic results cannot be recovered. The random covering sets that we will mention, as well as the random and deterministic sets we will relate to lim sup sets, have a long history of their own. While we highlight their connection to the lim sup sets mentioned in the previous sections and focus on their similarities, we note that the methods used in their proofs differ quite substantially.

We first consider a problem known as the (random) moving target problem. Let (X, μ) be a probability space, where X is a complete metric space. Let $\{B_i\}_{i \in \mathbb{N}} = \{B(x, r_i)\}_{i \in \mathbb{N}}$ be a sequence of balls centred at $x \in X$ such that $r_i \rightarrow 0$ as $i \rightarrow \infty$. We are interested in the following question.

PROBLEM 5.1. Let $\{\tilde{B}_i\}_{i \in \mathbb{N}} = \{B(x + a_i, r_i)\}_{i \in \mathbb{N}}$ be a sequence of balls with random centres $x + a_i$, where $a_i \in X$ are chosen independently according to the probability measure μ . Under what conditions can we deduce measure statements for the lim sup set $E(B_i) = \limsup_{i \rightarrow \infty} \tilde{B}_i$?

If $X = \mathbb{T}^1$ is the circle and μ is the uniform measure, one answer to that question should be familiar. It is the Borel–Cantelli Lemma.

LEMMA 5.2 (Borel–Cantelli Lemma). Let $X = \mathbb{T}^1$ and let $\{B_i\}_{i \in \mathbb{N}} = \{B(x, r_i)\}_{i \in \mathbb{N}}$ be a sequence of balls centred at $x \in X$ such that $r_i \rightarrow 0$ as $i \rightarrow \infty$. Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of random translations chosen independently according to the uniform measure μ . Then, we again consider $E(B_i) = \limsup_{i \rightarrow \infty} \tilde{B}_i$ and, for almost every choice of sequence $(a_i)_{i \in \mathbb{N}}$ with respect to the product measure $\mu^{\mathbb{N}}$, we have

$$|E(B_i)| = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} r_i < \infty, \\ 1 & \text{if } \sum_{i=0}^{\infty} r_i = \infty. \end{cases}$$

Note that the first implication, *i.e.* that the sum being finite implies zero Lebesgue measure, holds surely for any arbitrary sequence $(a_i)_{i \in \mathbb{N}}$. In particular, the a_i do not have to be chosen randomly. Using randomness though, we can make a more precise statement about the Hausdorff dimension when the lim sup set is Lebesgue null.

THEOREM 5.3 (Fan – Wu [27], Durand [19]). Let $X = \mathbb{T}^1$ and let $\{B_i\}_{i \in \mathbb{N}}$ be a sequence of balls in X with radii r_i such that $r_i \rightarrow 0$ as $i \rightarrow \infty$. Given this sequence of radii, assume that $|E(B_i)| = 0$ for almost every sequence of uniformly chosen translations $(a_i)_{i \in \mathbb{N}} \subset \mathbb{T}^1$. Then, for almost all sequences of random translations,

$$\dim_{\mathbb{H}} E(B_i) = \min\{1, s_0\},$$

where

$$(5.1) \quad s_0 = \inf \left\{ s > 0 : \sum_{i=0}^{\infty} r_i^s < \infty \right\}.$$

Related to such statements are results in fractal geometry. We write $\mathcal{I}_0 = \{T_1, T_2, \dots, T_N\}$ for a finite collection of contracting similarity maps on \mathbb{R}^k , *i.e.*

$$\|T_i(x) - T_i(y)\| = c_i \|x - y\|$$

for some $0 < c_i < 1$ for each $1 \leq i \leq N$ for all $x, y \in \mathbb{R}^k$, where $\|\cdot\|$ is the Euclidean norm. The “best guess” for the Hausdorff dimension of the unique compact invariant attractor $F \subset \mathbb{R}^k$ satisfying $F = \bigcup_i T_i(F)$ is the *similarity dimension*. The similarity dimension is the minimum of the dimension of the ambient space and the unique exponent, s_0 , satisfying the Hutchinson–Moran formula

$$(5.2) \quad \sum_{i=1}^N c_i^{s_0} = 1,$$

see [35, 52]. Its relation to (5.1) can be seen by writing F as a lim sup set

$$F = \bigcap_{l=1}^{\infty} \bigcup_{i=l}^{\infty} \bigcup_{j_1, j_2, \dots, j_i \in \{1, \dots, N\}} T_{j_1} \circ T_{j_2} \circ \dots \circ T_{j_i}(\Delta),$$

where $\Delta = [-c, c]^k$ for some large enough $c \in \mathbb{R}$ such that $F \subseteq \Delta$. Notice that $\sum_j c_j^{s_0 + \delta} < 1$ for any $\delta > 0$ and that $\text{diam}(T_{j_1} \circ \dots \circ T_{j_i}(\Delta)) = c_{j_1} \dots c_{j_i} c$. So,

$$(5.3) \quad \begin{aligned} \sum_{i=1}^{\infty} \sum_{j_1, j_2, \dots, j_i \in \{1, \dots, N\}} \text{diam}(T_{j_1} \circ T_{j_2} \circ \dots \circ T_{j_i}(\Delta))^{s_0 + \delta} \\ = c^{s_0 + \delta} \sum_{i=1}^{\infty} \sum_{j_1, j_2, \dots, j_i \in \{1, \dots, N\}} c_{j_1}^{s_0 + \delta} c_{j_2}^{s_0 + \delta} \dots c_{j_i}^{s_0 + \delta} \\ = c^{s_0 + \delta} \sum_{i=1}^{\infty} \left(\sum_j c_j^{s_0 + \delta} \right)^i < \infty \end{aligned}$$

using multiplicativity. Similarly, if $\delta < 0$ the sum above diverges and the similarity dimension s_0 in (5.3) coincides with the expression in (5.1). We would typically expect the similarity dimension to coincide with the Hausdorff dimension for these sets. However, this is not true in general in the deterministic setting and randomisation is one mechanism by which one can get an almost sure equality. We refer the reader to the wide literature on dimension theory of random and deterministic attractors [23, 24, 51], see also [62] for an overview of self-similar random sets.

Naturally, one is interested in higher dimensional analogues and relaxing the conditions on the covering set $E(B_i)$. Let $X = \mathbb{T}^k$ and let $\Delta \subset [0, 1]^k$ have non-empty interior. Let $T_i : \mathbb{R}^k \rightarrow \mathbb{T}^k$ be a linear contraction with singular values $\sigma_1(T_i) \geq \sigma_2(T_i) \geq \dots \geq \sigma_k(T_i)$. Recall that $\sigma_j(T_i)$ is the length of the j^{th} longest principal semi-axis of the ellipsoid $T_i(B(\mathbf{0}, 1))$. We define the *singular value function*

$\Phi^t(T_i)$ by

$$\Phi^t(T_i) = \begin{cases} \sigma_1(T_i)\sigma_2(T_i)\dots\sigma_n(T_i)^{t-n+1} & \text{for } n \leq t+1 < n+1 \text{ and } t < k, \\ \sigma_1(T_i)\sigma_2(T_i)\dots\sigma_k(T_i)^t & \text{for } t \geq k. \end{cases}$$

The Hausdorff dimension of the natural lim sup set appearing in this setting is related to the behaviour of the singular value function.

THEOREM 5.4 (Järvenpää – Järvenpää – Koivusalo – Li – Suomala [40]). *Let $(T_i)_{i \in \mathbb{N}}$ be a sequence of maps as above with $\sigma_j(T_i) \rightarrow 0$ as $i \rightarrow \infty$ for all j . Set*

$$E(T_i) := \limsup_i (T_i(\Delta) + a_i),$$

where $a_i \in \mathbb{T}^k$ is a translation chosen independently according to the Lebesgue measure on \mathbb{T}^k . Then, almost surely,

$$(5.4) \quad \dim_{\text{H}} E(T_i) = \inf \left\{ 0 < t \leq k : \sum_{i=1}^{\infty} \Phi^t(T_i) < \infty \right\}.$$

In particular, the sets can now be chosen to be rectangles, as opposed to balls. Indeed, even in the deterministic setting considered by Wang, Wu and Xu [64] the expression they obtain, namely (4.1), coincides with (5.4). We see this expression appearing yet again in the upper bound (4.3).

The singular value function was first used by Falconer in determining the Hausdorff dimension of self-affine sets [22]. Recall that a map is affine if it can be written as $\mathbf{M}x + \mathbf{v}$ for some non-singular matrix $\mathbf{M} \in \mathbb{R}^{k \times k}$ and some vector $\mathbf{v} \in \mathbb{R}^k$. Analogously to the self-similar case, if one considers the unique compact attractor F of a finite collection \mathcal{I} of affine contractions, the “best guess” for the Hausdorff dimension of F is the *affinity dimension* given by the unique value $s \geq 0$ such that

$$\sum_{T \in \mathcal{I}} \Phi^s(T) = 1.$$

In the case where we are given fixed maps and randomly chosen translation vectors the Hausdorff dimension and affinity dimension coincide, see Falconer [22]. More recently, it was shown by Bárány, Käenmäki and Koivusalo [3] that one could alternatively randomise the matrices defining the maps while keeping the translation vectors fixed. The problem of determining exact conditions under which self-affine sets have Hausdorff dimension equal to affinity dimension is still open and much progress has been made towards resolving it; see a recent survey by Falconer [25] and [26, 44, 54] (and references within) for the deterministic setting, and [30, 32, 34, 41–43, 45, 63] for the random setting.

Dropping the linearity of the maps, T_i , Persson [55] proved a lower bound for the Hausdorff dimension of lim sup sets of open sets.

THEOREM 5.5 (Persson [55]). *Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of open sets in \mathbb{T}^k . Let V be the Riemannian volume on \mathbb{T}^k and let*

$$g_s(A_i) = \frac{|A_i|^2}{\mathcal{E}^s(A_i)}, \quad \text{where} \quad \mathcal{E}^s(A_i) = \iint_{A_i \times A_i} \frac{dV(x) dV(y)}{|x - y|^s}$$

is the s -energy of A_i . Then, for the lim sup set $E(A_i)$ we obtain,

$$\dim_{\mathbb{H}} E(A_i) \geq \inf \left\{ 0 < s \leq k : \sum_{i=1}^{\infty} g_s(A_i) < \infty \right\}.$$

Now consider the following general set up. Let U and V be open subsets of \mathbb{R}^k and let $T : U \times V \rightarrow \mathbb{R}^k$ be a C^1 map such that $T(\cdot, y) : U \rightarrow \mathbb{R}^k$ and $T(x, \cdot) : V \rightarrow \mathbb{R}^k$ are diffeomorphisms for all $x \in U$ and $y \in V$. Let D_1T and D_2T be the derivatives of $T(\cdot, y)$ and $T(x, \cdot)$, respectively. Assume that

$$(5.5) \quad \|D_iT(x, y)\| \leq C_u \quad \text{and} \quad \|(D_iT(x, y))^{-1}\| \leq C_u$$

for some uniform $C_u > 0$ and all $i \in \{1, 2\}$. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of subsets of V and let $(a_i)_{i \in \mathbb{N}}$ be a sequence of points in U . The function T defines an interaction between a “generalised translation” a_i and a set A_i and embeds them without “too much distortion” into \mathbb{R}^k . Let $E(T, a_i, A_i) = \limsup_{i \rightarrow \infty} T(a_i, A_i)$. Note that for $T(a_i, y) = x + a_i + y$ this is equivalent to the translates setting considered above. Feng et al. [29] proved a (random) mass transference type statement in this general set up.

THEOREM 5.6 (Feng – Järvenpää – Järvenpää – Suomala [29]). *Let f be a dimension function and for each $i \in \mathbb{N}$ let $a_i \in U$ and let $A_i \subset \Delta \subset V$, where Δ is compact. Then*

$$\sum_{i=1}^{\infty} \mathcal{H}_{\infty}^f(A_i) < \infty \quad \text{implies} \quad \mathcal{H}^f(E(T, a_i, A_i)) = 0.$$

Let μ be a measure on U that is not entirely singular with respect to the Lebesgue measure (see [51] for a definition). We denote the natural product measure on all sequences with entries in U by $\mathbb{P} = \mu^{\mathbb{N}}$ and now choose a sequence $(a_i)_{i \in \mathbb{N}}$ according to \mathbb{P} . Let

$$G_f(F) = \sup\{g_f(L) : L \subset F \text{ and } L \text{ is Lebesgue measurable with } |L| > 0\},$$

where g_f is the natural extension of g_s to dimension functions f ,

$$g_f(A_i) = \frac{|A_i|^2}{\mathcal{E}^f(A_i)}, \quad \text{where} \quad \mathcal{E}^f(A_i) = \iint_{A_i \times A_i} \frac{dV(x) dV(y)}{f(|x - y|)}.$$

THEOREM 5.7 (Feng – Järvenpää – Järvenpää – Suomala [29]). *Suppose the same assumptions as in Theorem 5.6. Provided that $\mathcal{E}^f(B(0, R)) < \infty$ for all $R > 0$ and the A_i are Lebesgue measurable, then*

$$\sum_{i=1}^{\infty} G_f(A_i) = \infty \quad \text{implies} \quad \mathcal{H}^f(E(T, a_i, A_i)) = \infty \quad \text{for } \mathbb{P}\text{-a.e. } (a_i)_{i \in \mathbb{N}} \in U^{\mathbb{N}}.$$

Finally, a set $L \subset \mathbb{R}^k$ has positive Lebesgue density if

$$\liminf_{r \rightarrow 0} \frac{|L \cap B(x, r)|}{|B(x, r)|} > 0$$

for all $x \in L$.

THEOREM 5.8 (Feng – Järvenpää – Järvenpää – Suomala [29]). *Let f be a dimension function and recall that $V \subset \mathbb{R}^k$. Assume that $r^{-k+\varepsilon}f(r)$ is decreasing in r for some $\varepsilon > 0$. Let h be a dimension function such that $h(r) \leq f(r)^{1+\delta}$ for some $\delta > 0$ and all $r > 0$. Under the same assumptions as in Theorem 5.6, and*

provided that the A_i are Lebesgue measurable with positive Lebesgue density, we obtain

$$\sum_{i=1}^{\infty} G_f(A_i) < \infty \quad \text{implies} \quad \sum_{i=1}^{\infty} \mathcal{H}_{\infty}^h(A_i) < \infty.$$

As one can readily see, these latter results hold for lim sup sets of very general subsets. However, we still require positive Lebesgue density and a “nice” measure that is not singular with respect to the Lebesgue measure. Recently, Seuret [60] has made advances in relaxing these conditions on the measures by considering random lim sup sets with random centres chosen according to an arbitrary Borel measure and formulating their results in terms of multifractal formalism. This, in turn, was more recently extended by Ekström and Persson [21].

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