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# Introduction to Abelian Varieties

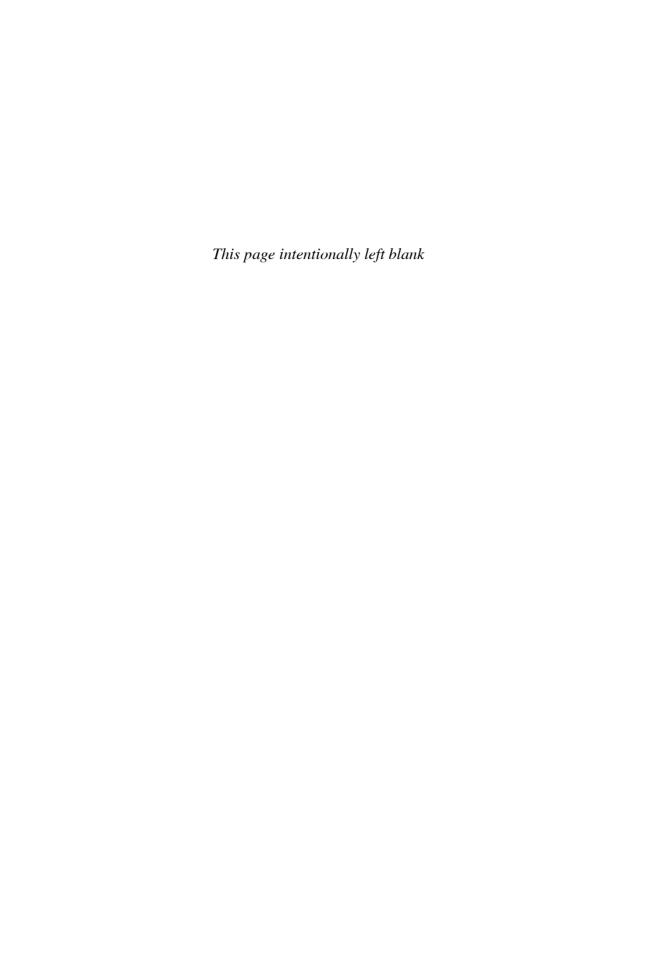
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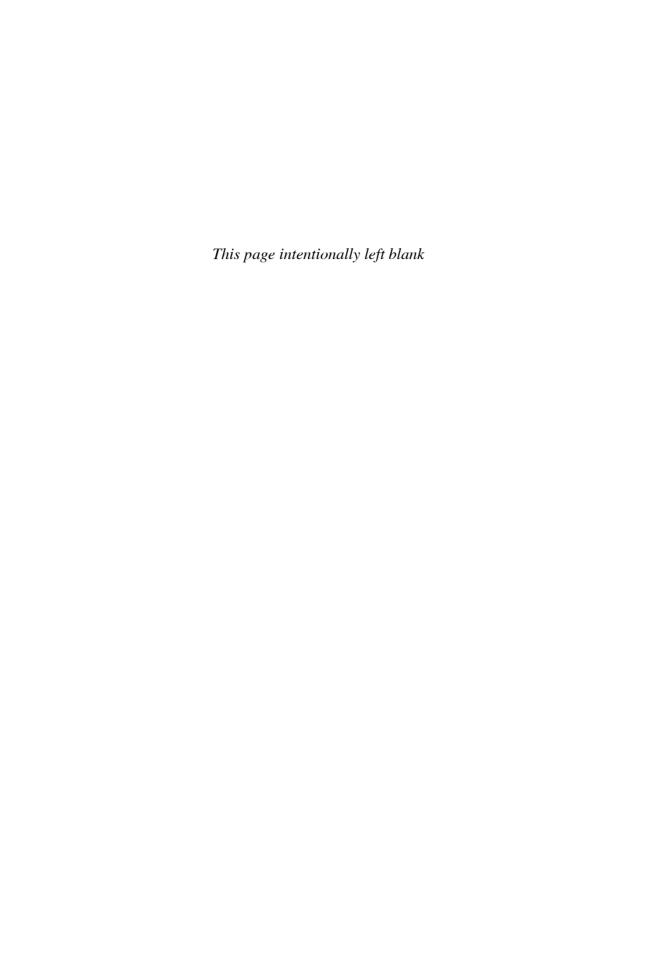
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V. Kumar Murty

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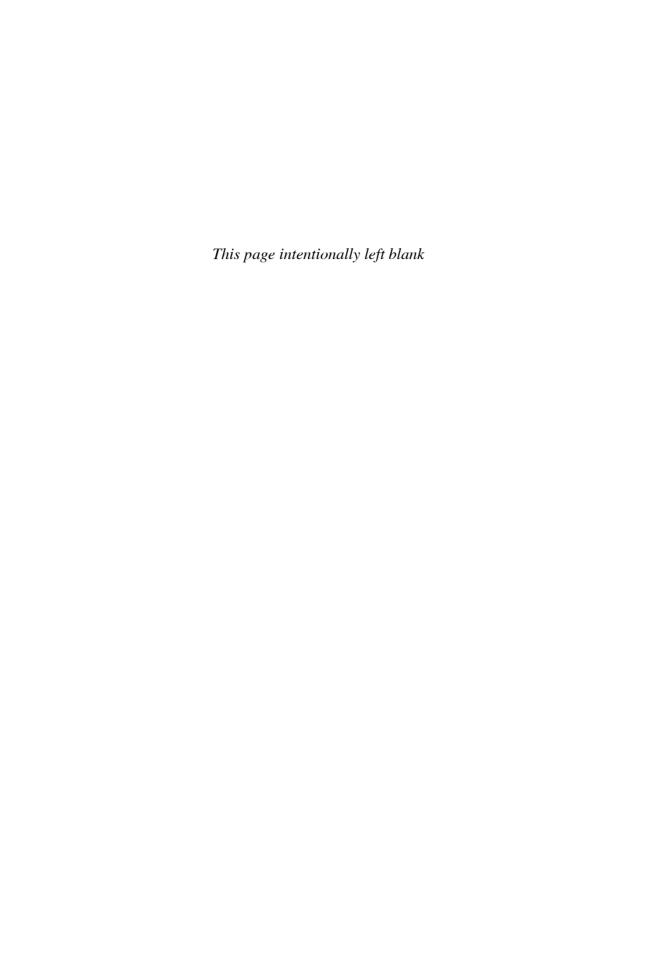
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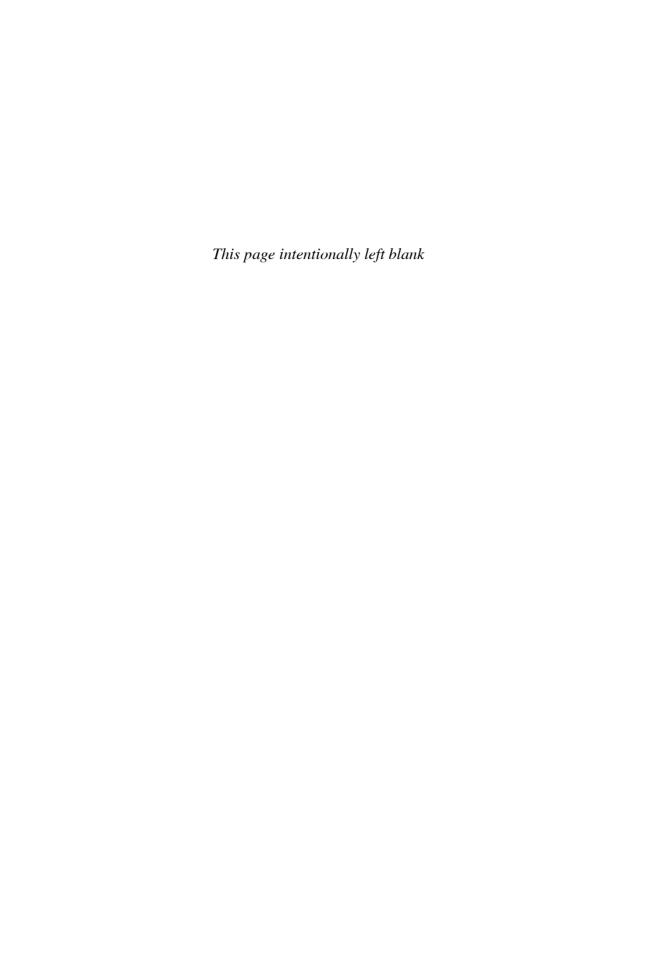
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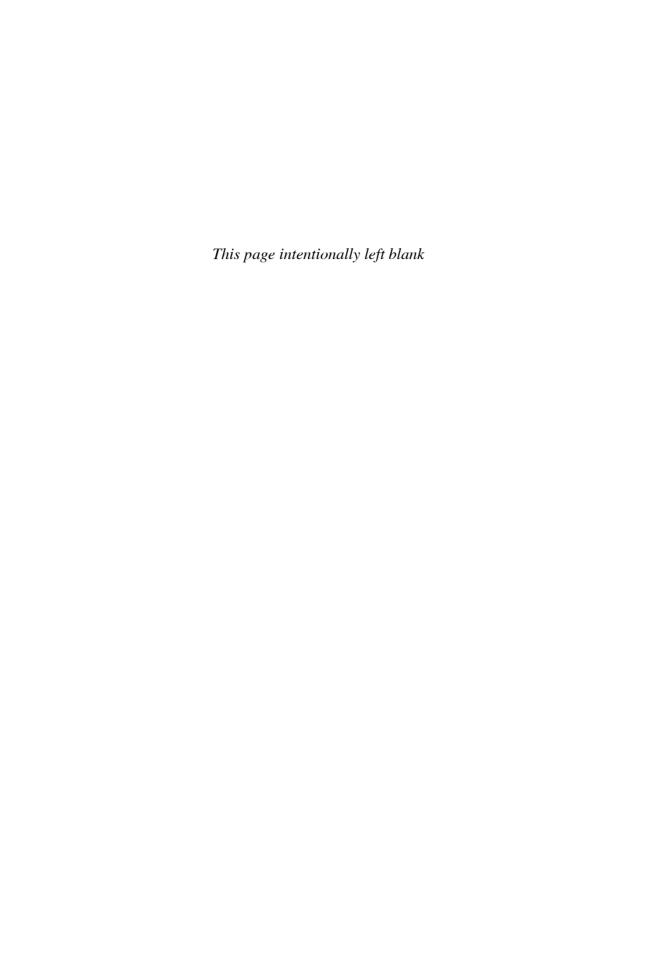
### Contents

List of Figures		xi
Preface		xiii
Introduction		1
Chapter 1.	Riemann Surfaces	5
Chapter 2.	Riemann-Roch Theorem	11
Chapter 3.	Abel–Jacobi Theorem and Period Relations	21
Chapter 4.	Divisors and Theta Functions	31
Chapter 5.	Dimension of the Space of Theta Functions	39
Chapter 6.	Projective Embedding and Theta Functions	47
Chapter 7.	Elliptic Curves as the Intersection of Two Quadrics	59
Chapter 8.	The Fermat Curve	67
Chapter 9.	Discrete Subgroups of $\mathbf{SL_2}(\mathbb{R})$	75
Chapter 10.	Riemann Surface Structure of $\Gamma \backslash \mathcal{H}^*$	79
Chapter 11.	The Modular Curve $\boldsymbol{X(N)}$	85
Chapter 12.	Generalities on Abelian Varieties	91
Chapter 13.	The Conjecture of Tate	95
Chapter 14.	Finiteness of Isogeny Classes	101
Chapter 15.	Mordell's Conjecture	105
Bibliography		109
Index		111



# List of Figures

1.1	Coordinate neighbourhoods.	6
1.2	The choice of $R_i$ and $S_i^*$ .	8
2.1	The cover $M \to \mathbb{P}^1$ .	12
3.1	The polygon $\mathcal{P}$ .	21
3.2	The polygon $\mathcal{P}^0$ .	23
7.1	A fundamental parallelogram $P$ .	62
11.1	The natural map between modular curves.	86
12.1	Complete reducibility.	94
13.1	The inverse isogeny.	95
15.1	Kodaira-Parshin construction.	107



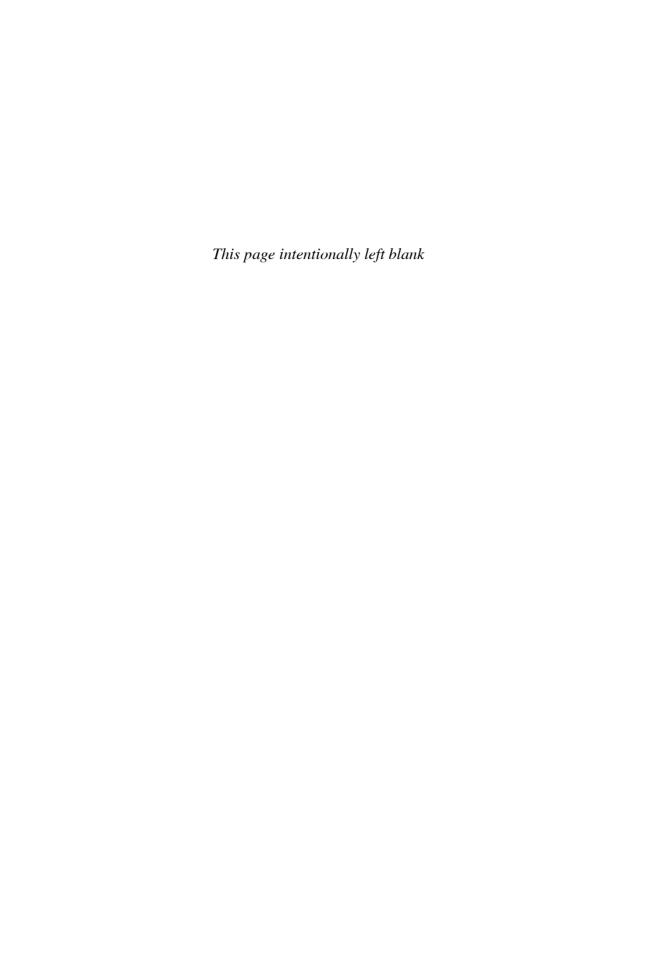
#### **Preface**

This work is based on the notes of an elementary course on *Abelian Varieties* given at Concordia University from January to May 1986. In the last few years, the notes have been distributed informally to students wishing to have an introduction to the subject. The positive response from them and from colleagues has been an encouragement to make them available on a wider basis.

Throughout, we have tried to preserve the informal style of the lectures. The aim has been to present the material in a form suitable for independent study by graduate students and by researchers in other fields who may wish an introduction to the subject.

I would like to thank Clifton Cunningham and Damien Roy for a careful reading of the manuscript and for offering many helpful suggestions and comments. I would also like to thank Abid Zaidi for typing the notes and Emile LeBlanc for help with IATEX and other computer related problems. Finally, I thank Francis Clarke for the invitation to publish in the CRM Monograph Series, and the editorial staff of CRM for their helpfulness and efficiency.

Vijaya Kumar Toronto September 1992



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## Index

Abel-Jacobi theorem, 21	of $\mathrm{SL}_2(\mathbb{Z})$ , 85
Abelian function, 47	exponential map, 92
Abelian variety, 1, 2, 91	
L-function of, 99	Faltings, 2, 98, 106
complete reducibility of, 93	Fermat, 2
dual, 48	Fermat curve, 2, 67
endomorphisms of, 50	quotients of, 72
isogeny, 95	Fermat's last theorem, 89
simple, 91	Frobenius decomposition, 43
Tate module of an, 96	Probenius decomposition, 45
with real multiplication, 55, 89	• •
Arakelov, 106	genus, 14
	of smooth plane curves, 67
Belyi, 20	Hermite-Minkowski theorem, 101
Bezout's theorem, 65, 67	holomorphic form, 7
	on the Fermat curve, 69
canonical class, 9	hyperbolic element, 76
Chebotarev density theorem, 101	ny per bone element, 10
Chow's Theorem, 31, 95, 96	
congruence subgroup, 86	Jacobian, 21
principal, 86	
cusps, 76	l-adic representations, 98, 99
number of inequivalent, 82	Lefschetz's theorem, 31, 33, 47, 51
of commensurable subgroups, 82	
or commensurable subgroups, 02	meromorphic form, 7
1 D 1: 10 100	modular curve, 83
de Franchis, 19, 108	analytic structure, 83
differential	genus of, 86, 88
of the first kind, 69	Mordell, 2
of the second kind, 69, 71	Mordell's conjecture, 2, 98, 105
of the third kind, 24	
divisor	Néron-Severi group, 48
algebraic equivalence, 48	Title Bevoll Broup, 10
Cartier, 33	narabalia alament 76
linear equivalence, 9	parabolic element, 76
non-degenerate, 51	Parshin, 2, 101, 105, 107
non-degenerate, 47	period matrix, 29
positive, 32, 48	polarization, 50
principal, 9	degree of the, 50
very ample, 33	principal, 49, 55
Weil, 32	polarization, 42
divisor class, 9	projective embedding, 31, 51, 54
elliptic curve, 1, 14, 59	rational point, 2
elliptic element, 76	Riemann, 1, 5, 69
elliptic point, 76	theta relations of, 63, 65
empire point, 10	theta relations of, 00, 00

112 INDEX

Riemann form, 42 Riemann surface, 5 automorphisms of, 18 periods, 21, 56 Riemann-Hurwitz formula, 15, 68 Riemann-Roch theorem, 14 Riemann-Hurwitz formula, 88 Rosati involution, 51

Shafarevitch, 2, 99, 101, 105, 107, 108

Taniyama conjecture, 89 Tate, 2, 3, 95, 97, 98, 101, 103, 105 Tate's conjecture, 98 theta characteristics, 64 theta function, 39 algebraically equivalent, 49 non-degenerate, 47 normalized, 42 trivial, 41 Torelli's theorem, 55, 105, 107

upper half plane, 59 quotients by discrete groups, 76

Weierstrass points, 15

Zarhin, 2, 105

The book represents an introduction to the theory of abelian varieties with a view to arithmetic. The aim is to introduce some of the basics of the theory as well as some recent arithmetic applications to graduate students and researchers in other fields. The first part contains proofs of the Abel-Jacobi theorem, Riemann's relations and the Lefschetz theorem on projective embeddings over the complex numbers in the spirit of S. Lang's book *Introduction to algebraic and abelian functions*. Then the Jacobians of Fermat curves as well as some modular curves are discussed. Finally, as an application, Faltings' proof of the Mordell conjecture and its intermediate steps, the Tate conjecture and the Shafarevich conjecture, are sketched.

— H. Lange for MathSciNet



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