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V. Kumar Murty



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Introduction to Abelian Varieties

V. Kumar Murty

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Preface

This work is based on the notes of an elementary course on *Abelian Varieties* given at Concordia University from January to May 1986. In the last few years, the notes have been distributed informally to students wishing to have an introduction to the subject. The positive response from them and from colleagues has been an encouragement to make them available on a wider basis.

Throughout, we have tried to preserve the informal style of the lectures. The aim has been to present the material in a form suitable for independent study by graduate students and by researchers in other fields who may wish an introduction to the subject.

I would like to thank Clifton Cunningham and Damien Roy for a careful reading of the manuscript and for offering many helpful suggestions and comments. I would also like to thank Abid Zaidi for typing the notes and Emile LeBlanc for help with L^AT_EX and other computer related problems. Finally, I thank Francis Clarke for the invitation to publish in the CRM Monograph Series, and the editorial staff of CRM for their helpfulness and efficiency.

Vijaya Kumar
Toronto
September 1992

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Bibliography

- [ACGH] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves I*, Springer-Verlag, New York, 1985.
- [Be] G. V. Belyi, *On Galois extensions of a maximal cyclotomic field*, Math. USSR-Izv. **14** (1980), 247–256.
- [CS] G. Cornell and J. Silverman (eds.), *Arithmetic geometry*, Springer-Verlag, New York, 1986.
- [D] J. Dieudonné, *History of algebraic geometry*, Wadsworth, Belmont, 1985.
- [FW] G. Faltings and G. Wüstholz, *Rational points*, Aspects of Mathematics, Vieweg.
- [Ha] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
- [HS] A. Howard and A. Sommese, *On a theorem of de Franchis*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 429–436.
- [Ka] E. Kani, *Bounds on the number of non-rational subfields of a function field*, Invent. Math. **85** (1986), 185–198.
- [KR] E. Kani and M. Rosen, *Idempotent relations and factors of Jacobians*, Math. Ann. **284** (1989), 307–327.
- [K] G. Kempf, *Complex Abelian varieties and theta functions*, Springer-Verlag, Berlin, 1991.
- [La1] S. Lang, *Introduction to Algebraic and Abelian functions*, Springer-Verlag, New York, 1982,
- [La2] S. Lang, *Abelian varieties*, Springer-Verlag, New York, 1983.
- [Lan] H. Lange, *Abelian varieties with several principal polarizations*, Duke Math. J. **55** (1987), 617–628.
- [Ma] B. Mazur, *Arithmetic on curves*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 207–259.
- [Mu1] D. Mumford, *Abelian varieties*, Tata Studies in Mathematics, Oxford, Bombay, 1970.
- [Mu2] D. Mumford, *Tata lectures on theta I*, Birkhauser, Boston, 1983; II 1984; III, 1991.
- [NN] M. S. Narasimhan and M. V. Nori, *Polarizations on an Abelian variety*, Proc. Indian Acad. Sci. Math. Sci. **90** (1981), 125–128.
- [NSNS] M. S. Narasimhan, R. R. Simha, R. Narasimhan, and C. S. Seshadri, *Riemann surfaces*, Mathematical Pamphlets Number 1, Tata Institute of Fundamental Research, Bombay, 1963.
- [Pa1] A. N. Parshin, *Quelques conjectures de finitude en géométrie diophantienne*, Proc. ICM I (Nice 1970), Gauthier-Villars, Paris, 1971, pp. 467–471.
- [Pa2] A. N. Parshin, *Algebraic curves over function fields*, Math. USSR-Izv. **2** (1968), 1145–1170.
- [R] K. Ribet, *On modular representations of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms*, Invent. Math. **100** (1990), 431–476.
- [Sa] P. Samuel, *Lectures on old and new results on algebraic curves*, Tata Institute of Fundamental Research Lecture Notes, Tata Institute of Fundamental Research, Bombay, 1966.
- [ST] J.-P. Serre and J. Tate, *Good reduction of Abelian varieties*, Ann. of Math. (2), **88** (1968), 492–517.
- [Sha1] I. Shafarevitch, *Basic algebraic geometry*, Springer-Verlag, Berlin, 1974.
- [Sha2] I. Shafarevitch, *Algebraic number fields*, Proc. ICM (Stockholm 1962), Transl. Amer. Math. Soc. **31** (1963), 25–39; Collected Mathematical Papers, Springer-Verlag, Berlin, 1989, pp. 283–294.
- [Shi] G. Shimura, *Arithmetic theory of automorphic functions*, Iwanami-Shoten & Princeton, 1971.

- [SwD] P. Swinnerton-Dyer, *Analytic theory of Abelian varieties*, Cambridge University Press, Cambridge, 1974.
- [Sz] L. Szpiro (ed.), *Séminaire sur les pincesaux arithmétiques : la conjecture de Mordell*, Astérisque **127**, Soc. Math. France, Paris, 1985.
- [Ta1] J. Tate, *Algebraic cycles and poles of L-functions*, Arithmetic Algebraic Geometry, O. Schilling ed., Harper and Row, New York, 1966, pp. 93–110.
- [Ta2] J. Tate, *Endomorphisms of Abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.
- [W] A. Weil, *Courbes algébriques et variétés Abéliennes*, Hermann, Paris, 1971.
- [ZP] Yu. G. Zarhin and A. N. Parshin, *Finiteness problems in Diophantine geometry*, Transl. Amer. Math. Soc. **143** (1989), 35–102.

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The book represents an introduction to the theory of abelian varieties with a view to arithmetic. The aim is to introduce some of the basics of the theory as well as some recent arithmetic applications to graduate students and researchers in other fields. The first part contains proofs of the Abel-Jacobi theorem, Riemann's relations and the Lefschetz theorem on projective embeddings over the complex numbers in the spirit of S. Lang's book *Introduction to algebraic and abelian functions*. Then the Jacobians of Fermat curves as well as some modular curves are discussed. Finally, as an application, Faltings' proof of the Mordell conjecture and its intermediate steps, the Tate conjecture and the Shafarevich conjecture, are sketched.

— H. Lange for MathSciNet

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