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Eyal Z. Goren



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Eyal Z. Goren

with the assistance of Marc-Hubert Nicole

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Lectures on Hilbert Modular Varieties and Modular Forms

Eyal Z. Goren

This book is devoted to certain aspects of the theory of p -adic Hilbert modular forms and moduli spaces of abelian varieties with real multiplication.

The theory of p -adic modular forms is presented first in the elliptic case, introducing the reader to key ideas of N. M. Katz and J.-P. Serre. It is re-interpreted from a geometric point of view, which is developed to present the rudiments of a similar theory for Hilbert modular forms.

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