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Centre de Recherches Mathématiques
Université de Montréal

Fermionic Functional Integrals and the Renormalization Group

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American Mathematical Society

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ABSTRACT. The Renormalization Group is the name given to a technique for analyzing the qualitative behaviour of a class of physical systems by iterating a map on the vector space of interactions for the class. In a typical nonrigorous application of this technique one assumes, based on one's physical intuition, that only a certain finite dimensional subspace (usually of dimension three or less) is important. These notes concern a technique for justifying this approximation in a broad class of fermionic models used in condensed matter and high energy physics.

These notes expand upon the Aisenstadt Lectures given by Joel Feldman at the Centre de recherches mathématiques, Université de Montréal in August 1999.

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Preface

The Renormalization Group is the name given to a technique for analyzing the qualitative behavior of a class of physical systems by iterating a map on the vector space of interactions for the class. In a typical non-rigorous application of this technique one assumes, based on one's physical intuition, that only a certain finite dimensional subspace (usually of dimension three or less) is important. These notes concern a technique for justifying this approximation in a broad class of fermionic models used in condensed matter and high energy physics.

The first chapter provides the necessary mathematical background. Most of it is easy algebra—primarily the definition of Grassmann algebra and the definition and basic properties of a family of linear functionals on Grassmann algebras known as Grassmann Gaussian integrals. To make Section 1.1 really trivial, we consider only finite dimensional Grassmann algebras. A simple-minded method for handling the infinite dimensional case is presented in Appendix A. There is also one piece of analysis in Section 1.1—the Gram bound on Grassmann Gaussian integrals—and a brief discussion of how Grassmann integrals arise in quantum field theories.

The second chapter introduces an expansion that can be used to establish analytic control over the Grassmann integrals used in fermionic quantum field theory models, when the covariance (propagator) is "really nice." It is also used as one ingredient in a renormalization group procedure that controls the Grassmann integrals when the covariance is not so nice. To illustrate the latter, we look at the Gross-Neveu$_2$ model and at many-fermion models in two space dimensions.

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The "abstract" fermionic expansion discussed in Chapter 2 is similar to that in

[FKT] J. Feldman, H. Knörrer, and E. Trubowitz, *A nonperturbative representation for fermionic correlation functions*, Comm. Math. Phys., **195** (1998), 465–493.

[FMRT] J. Feldman, J. Magnen, V. Rivasseau, and E. Trubowitz, *An infinite volume expansion for many fermion Green's functions*, Helv. Phys. Acta **65** (1992), 679–721.

The latter also contains a discussion of sectors. Both are available over the web at `http://www.math.ubc.ca/~feldman/research.html`.

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Fermionic Functional Integrals and the Renormalization Group

Joel Feldman, Horst Knörrer, and Eugene Trubowitz

This book, written by well-known experts in the field, offers a concise summary of one of the latest and most significant developments in the theoretical analysis of quantum field theory.

The renormalization group is the name given to a technique for analyzing the qualitative behavior of a class of physical systems by iterating a map on the vector space of interactions for the class. In a typical nonrigorous application of this technique, one assumes, based on one's physical intuition, that only a certain finite dimensional subspace (usually of dimension three or less) is important. The material in this book concerns a technique for justifying this approximation in a broad class of fermionic models used in condensed matter and high energy physics.

This volume is based on the Aisenstadt Lectures given by Joel Feldman at the Centre de Recherches Mathématiques (Montréal, Canada). It is suitable for graduate students and research mathematicians interested in mathematical physics. Included are many problems and solutions.

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