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# Hecke Algebras with Unequal Parameters

G. Lusztig



American Mathematical Society

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G. Lusztig

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## Bibliography

- [Bé] R. Bédard, *Cells in two Coxeter groups*, Comm. Algebra **14** (1986), 1253–1286.
- [Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Chapters 4, 5, 6, Actualités Sci. Indust., vol. 1337, Hermann, Paris, 1968.
- [Br] K. Bremke, *On generalized cells in affine Weyl groups*, J. Algebra **191** (1997), 149–173.
- [BM] K. Bremke and G. Malle, *Reduced words and a length function for  $G(e, 1, n)$* , Indag. Math. (N.S.) **8** (1997), 453–469.
- [DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), 103–161.
- [Ge] M. Geck, *Constructible characters, leading coefficients and left cells for finite Coxeter groups with unequal parameters*, Represent. Theory **6** (2002), 1–30.
- [GP] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Math. Soc. Monogr. (N.S.), vol. 21, Clarendon Press, New York, 2000.
- [Ho] P. N. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pairs of classical type*, Ph.D. Thesis, Univ. of British Columbia, Vancouver, BC, 1974.
- [Iw] N. Iwahori, *On the structure of the Hecke ring of a Chevalley group over a finite field*, J. Fac. Sci. Univ. Tokyo Sect. I **10** (1964), 215–236.
- [IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke ring of  $p$ -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48.
- [KL1] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
- [KL2] ———, *Schubert varieties and Poincaré duality*, Geometry of the Laplace Operator (Honolulu, 1979), Proc. Symp. Pure Math., vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.
- [Lu1] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **28** (1976), 101–159.
- [Lu2] ———, *Irreducible representations of finite classical groups*, Invent. Math. **43** (1977), 125–175.
- [Lu3] ———, *Left cells in Weyl groups*, Lie Group Representations. I, Lecture Notes in Math., vol. 1024, Springer, 1983, pp. 99–111.
- [Lu4] ———, *Some examples of square integrable representations of semisimple  $p$ -adic groups*, Trans. Amer. Math. Soc. **227** (1983), 623–653.
- [Lu5] ———, *Characters of reductive groups over a finite field*, Ann. of Math. Stud., vol. 107, Princeton Univ. Press, Princeton, NJ, 1984.
- [Lu6] ———, *Cells in affine Weyl groups*, Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 255–287.
- [Lu7] ———, *Sur les cellules gauches des groupes de Weyl*, C. R. Acad. Sci. Paris Sér. I Math. **302** (1986), 5–8.
- [Lu8] ———, *Cells in affine Weyl groups. II*, J. Algebra **109** (1987), 536–548.
- [Lu9] ———, *Introduction to character sheaves*, The Arcata Conference on Representations of Finite Groups (Arcata, CA, 1986), Proc. Symp. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 165–180.
- [Lu10] ———, *Intersection cohomology methods in representation theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 155–174.
- [Lu11] ———, *Classification of unipotent representations of simple  $p$ -adic groups*, Internat. Math. Res. Notices (1995), 517–589.



- [Lu12] ———, *Lectures on Hecke algebras with unequal parameters*, MIT Lectures, 1999; math.RT/0108172.
- [Lu13] ———, *Classification of unipotent representations of simple  $p$ -adic groups*. II, *Represent. Theory* **6** (2002), 243–289; math.RT/0111248.
- [Xi] N. Xi, *Representations of affine Hecke algebras*, *Lecture Notes in Math.* vol. 1587, Springer, Berlin, 1994.

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