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Centre de Recherches Mathématiques
Université de Montréal

Quantization, Classical
and Quantum Field Theory
and Theta Functions

Andrei Tyurin



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Quantization, Classical and Quantum Field Theory and Theta Functions

Andrei Tyurin

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*To Igor Rostislavovich Shafarevich
on his 80th birthday*

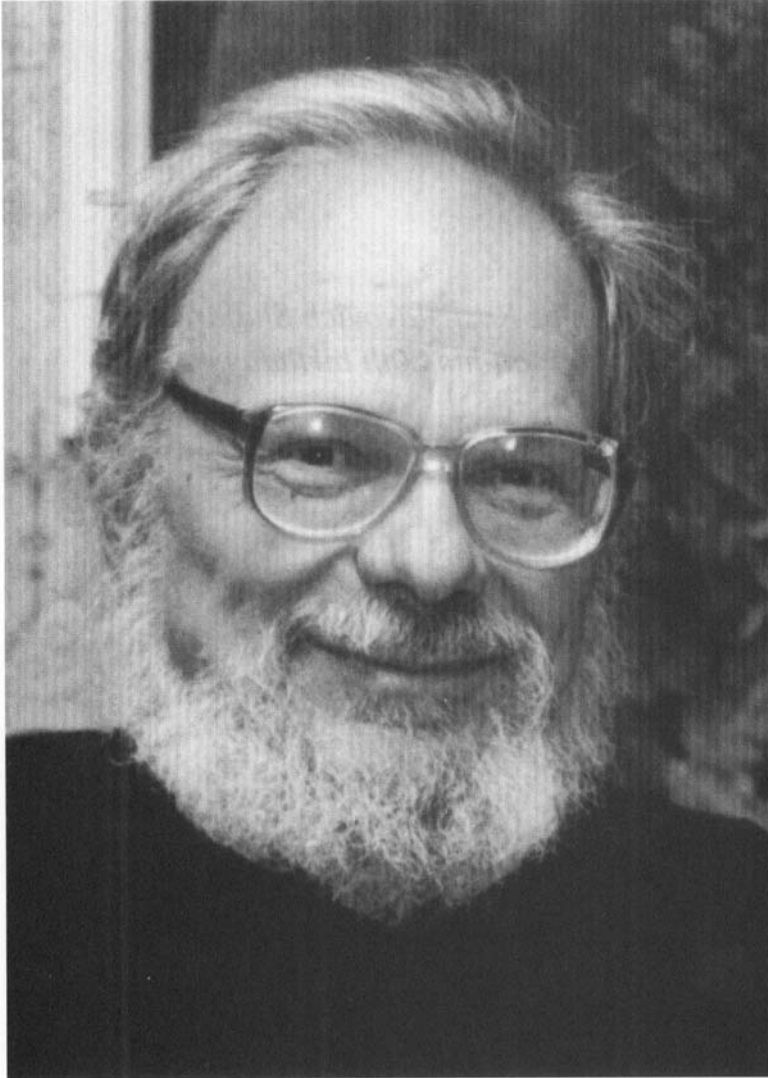


Photo courtesy of Ilya Blokhintsev

Andrei Nikolaevich Tyurin

Foreword

In October 2001, exactly a year before his fatal heart attack in Bonn, Andrei Nikolaevich Tyurin was in Montreal giving the wonderful series of lectures on which this book is based. He eagerly awaited the book's publication; having dedicated it to his old teacher Igor Rostislavovich Shafarevich, he wanted to personally present it to him at home in Moscow.

Sadly, this was not to be.

During the last year of Tyurin's life, I was fortunate enough to enjoy two entire months of lively, continuous conversations with him in the fall in Montreal and in the spring in Cambridge. This is why his relatives have asked me to write this brief foreword, sharing my very warm memories of this unique man.

Remembering the last meetings with someone who has passed away, especially if nothing has prepared us for his death, we try to reach into the past for some subtle hint, some strange shadow that might have slid by us, maybe some momentary flicker of obscure light, which allows us to see something of the essence of a person, to understand him more fully, without having to stumble through his endless complexity.

Tyurin often happily engaged in literary discussions. His taste was impeccable, his opinion strong and to the point. Once, in Montreal—I don't know why—our conversation turned to Pierre Bezuhov. (It has long been said that for many Russians Tolstoy's characters are like family.) Tyurin immediately recalled an episode when, after a fight with his wife, Pierre spends the evening with a beautiful Polish girl on his house balcony. In my arrogance and stupidity, I indignantly disagreed—this couldn't have happened! But Tyurin insisted that this was so, though he was unable to point out the exact spot in the book. I lived nearby, so I brought a copy of *War and Peace*, and Tyurin began to search.

He soon complained to me that he was unable to prepare his lectures—because as soon as he opened Tolstoy to search, he was dragged into the book and could not tear himself away. (This confession warmed my heart . . .) But by the end of the week the disputed place still hadn't been found. Upon leaving, Tyurin was somewhat flustered—never before had his amazing memory failed him.

Not too long ago, I picked up the book and finally opened it to the right page. It was astounding; it spoke of death.

Pierre, who hasn't responded to Plato's last silent call, hears the shot that killed his friend, but he still cannot comprehend that Plato is dead. He sees Plato's dog and ought to think of its master as a dead man. Instead, a memory—mortifying in its uselessness—appears to him; it is the memory of the evening spent with the Polish girl.

In a sudden flash of light, an unfathomable distance becomes unbelievably close. Tolstoy writes that Plato forever remains in Pierre's heart as his strongest

and dearest memory, the personification of everything *Russian, good and round*. These words also apply to Andrei Tyurin. (His expression on the frontispiece, momentarily caught by a lucky photographer,¹ is quite characteristic of him. This is who he was.)

This was the source of his memorable character traits and actions. I will not speak here of what I did not witness, but I will say that in Russia, the name Andrei Nikolaevich Tyurin should not be familiar to mathematicians only. Anyone who has read Alexander Solzhenitsyn's memoirs remembers how the famous Russian writer spoke of Tyurin.

Above all other traits, he valued broad-mindedness and variety in life. His spirit could never find enough room. He was an avid and experienced mountaineer. Last spring, in Cambridge, he easily trekked eighteen miles through swamps and gullies from Cambridge to Ely—and at the end of the trip, his son Kolya and I looked far more exhausted than he. He loved music passionately, especially Schubert (of course, they were of similar natures!). He would listen in stillness to the lengthy sonata in B major and invariably ask to repeat the second, slow, movement. He read avidly and rigorously—I recall how amazed I was to find that he had *memorized* a wonderful story by the Russian writer Bogomolov, which has appeared in print only once, about 15 years ago, and passed away almost unnoticed.

And, of course, he was a brilliant mathematician.

Hardy, at the end of his well-known *A mathematician's apology*, writes proudly and seemingly not completely truthfully, that he prefers the impersonal beauty of a mathematical result to the inimitable human imprint on a work of art.

This is a terrible but also, fortunately, incorrect statement. Andrei Tyurin's beautiful, deep mathematics is not impersonal—it will forever remind us of the big, bright person behind it. This book should serve as proof.

Alexei Kokotov
Concordia University
Montréal, January 2003

Editor's note

The untimely death of Andrei Tyurin prevented him from supervising the final edition of his manuscript. This lengthy task was carried on by S. Twareque Ali (Concordia University) and Arthur Greenspoon (Mathematical Reviews). Special points were also clarified by Brian Hall (University of Notre Dame), Carlos Florentino, José Mourão and João P. Nunes (Instituto Superior Técnico), and Nikolai Tyurin (BLTP JINR). To all of them, our warmest thanks.

Jean LeTourneux
Deputy director responsible for publications
Centre de recherches mathématiques
Montréal, July 2003

¹Ilya Blokhintsev

Introduction

Arnaud Beauville's survey *Vector bundles on curves and generalized theta functions: recent results and open problems* [Bea] appeared 10 years ago. This elegant survey is short (16 pages) but provides a complete introduction to a specific part of algebraic geometry. To emulate his success now, we need many more pages, even though we assume that the reader is already acquainted with the material presented there. Moreover, in Beauville's survey the relation between generalized theta functions and conformal field theories (classical and quantum) was already presented.

Following Beauville's strategy, we do not provide any proofs or motivation. But we would like to present *all constructions* of this large domain of mathematics in such a way that the proofs can be guessed from the geometric picture. Thus this text is not quite a mathematical monograph but rather a digest of a field of mathematical investigations.

In the abelian case (the subject of several beautiful classical books ([Ba, C, Wi, Fa1] and many others) by fixing a combinatorial structure (a so-called theta structure of order k), one obtains a special basis in the space of sections of powers of the canonical polarization on Jacobians. These sections can be presented as holomorphic functions on the "abelian Schottky" space $(\mathbb{C}^*)^g$. This fact provides various applications of these concrete analytic formulas to integrable systems, classical mechanics and PDE's (see the references in [DKN]).

Our practical goal is to do the same in the non-abelian case, that is, to give the answer to the final question of Beauville's survey (Question 9 in [Bea]).

It has been observed many times that the *construction* of theta functions with characteristics is intricately related to the paradigm of the quantization procedure (which is a quantum field theory in dimension 1). New features came from Conformal Field Theory (which is a field theory in dimension $2 = 1 + 1$). This new pathway brings the standard physical paradigm of "symmetries, fields, equations, etc., and gluing properties corresponding to local Lagrangians." New CFT methods provided powerful computational tools while "algebraic geometers would have never dreamed of being able to perform such computations" (A. Beauville).

In the future we hope to extend this digest to a mathematical monograph with the title "VBAC."

Acknowledgments. These notes were written for a series of lectures given at the Centre de recherches mathématiques at the Université de Montréal in September 2002. I am grateful to J. McKay, J. Hurtubise, J. Harnad, D. Korotkin and A. Kokotov who made my stay in Montreal very agreeable and productive. I would like to express my gratitude to my collaborators C. Florentino, J. Mourão, J.P. Nunes and participants of the Quantum Gravity seminar in IST (Lisboa, 2000).

I would like to thank Jean LeTourneux and André Montpetit for their assistance in the preparation of these notes. Special thanks go to my daughter, Yulia Tiourina, for catching numerous mistakes and misprints in this huge text.

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Ordinary (abelian) theta functions describe the properties of moduli spaces of one-dimensional vector bundles on algebraic curves. Non-abelian theta functions, which are the main topic of this book, play a similar role in the study of higher-dimensional vector bundles. The book describes various aspects of the theory of non-abelian theta functions and the moduli spaces of vector bundles, including their applications to problems of quantization and to classical and quantum conformal field theories.

The book is an important source of information for specialists in algebraic geometry and its application to mathematical aspects of quantum field theory.



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